## Pointed Drawings of Planar Graphs<sup> $\ddagger$ </sup>

Oswin Aichholzer<sup>a</sup>, Günter Rote<sup>b</sup>, André Schulz<sup>c</sup>, Birgit Vogtenhuber<sup>d</sup>,

<sup>a</sup>Institute for Software Technology, Graz University of Technology, Austria oaich@ist.tugraz.at <sup>b</sup>Institut für Informatik, Freie Universität Berlin, Germany rote@inf.fu-berlin.de <sup>c</sup>MIT, Computer Science and Artificial Intelligence Laboratory, Cambridge, USA schulz@csail.mit.edu

<sup>d</sup>Institute for Software Technology, Graz University of Technology, Austria bvogt@ist.tugraz.at

#### Abstract

We study the problem how to draw a planar graph crossing-free such that every vertex is incident to an angle greater than  $\pi$ . In general a plane straight-line drawing cannot guarantee this property. We present algorithms which construct such drawings with either tangent-continuous biarcs or quadratic Bézier curves (parabolic arcs), even if the positions of the vertices are predefined by a given plane straight-line drawing of the graph. Moreover, the graph can be drawn with circular arcs if the vertices can be placed arbitrarily. The topic is related to non-crossing drawings of multigraphs and vertex labeling.

## 1. Introduction

Throughout this paper, let G = (V, E) be a simple planar graph without loops, with finite vertex set V and thus a finite set of edges E. We use the natural understanding of a drawing of a graph. Vertices are represented as points in the plane and edges as continuous and (at least piecewise) differentiable curves connecting the points of adjacent vertices. A drawing is called *non-crossing* or *plane*, if the drawn edges do not intersect in their interior. If we consider only topological properties, that is, the order of the edges and consequently of the faces, we refer to this as *combinatorial embedding*. Given a (combinatorial) embedding of a graph G, the faces of G are defined as usual.

For a drawing  $\mathcal{F}(G)$  of G we denote the placement of a vertex  $v \in V$  by  $\mathcal{F}(v)$ , and the drawing of an edge  $e \in E$  by  $\mathcal{F}(e)$ . Note that we consider embedded edges to be open, i.e., to not contain their endpoints. For simplicity, and as there is no risk of confusion, in the figures we will denote embedded vertices just by v instead of  $\mathcal{F}(v)$ .

The tangent of an edge  $\mathcal{F}(e)$  at a vertex  $\mathcal{F}(v)$  is the limit of the tangents to  $\mathcal{F}(e)$ when approaching  $\mathcal{F}(v)$  along  $\mathcal{F}(e)$ . The tangent ray of  $\mathcal{F}(e)$  at  $\mathcal{F}(v)$  is the open ray along the tangent to  $\mathcal{F}(e)$  at  $\mathcal{F}(v)$  from  $\mathcal{F}(v)$  towards  $\mathcal{F}(e)$ . A drawing gives us a cyclic order of incident edges around each vertex. The angle between two consecutive edges incident to a vertex  $\mathcal{F}(v)$  is defined as the angle between the corresponding tangent rays at  $\mathcal{F}(v)$  that does not contain the tangent ray of any other incident edge. We say that

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this angle is incident to  $\mathcal{F}(v)$  (and vice versa). In the case of a degree two vertex there are two such angles between the two incident edges. If a vertex has degree at most one, we say that it is incident to one angle (having value  $2\pi$ ).

**Definition 1 (Pointedness).** A vertex in a drawing  $\mathcal{F}(G)$  is called *pointed* if it is incident to an angle greater than  $\pi$  (see Figure 1). We say that a vertex is pointed to a face if its large angle lies in this face. If all vertices in a drawing are pointed we call the drawing pointed.



Figure 1: Drawing with a non-pointed vertex  $v_1$  and a pointed vertex  $v_2$ .

For the special case of straight-line drawings, this definition is identical to the classic definition of pointedness, a term which stems from the field of pseudo-triangulations. A *pseudo-triangle* is a simple polygon with exactly three vertices with interior angle smaller than  $\pi$ . A *pseudo-triangulation* is a plane straight-line graph where every interior face is a pseudo-triangle and the outer face is convex. Pseudo-triangulations have rich applications and are an important geometric data structure, see for example [16, 17, 20], and [18] for a survey.

A graph is called *generically rigid*, if its straight-line realization on a generic point set induces a rigid framework (edges represent fixed length rods and vertices represent joints). In two dimensions, there exists an easy combinatorial characterization of generically rigid graphs that become non-rigid after removing an arbitrary edge [11]. These graphs are called *Laman graphs*. Due to Streinu [21], a graph of a pointed pseudo-triangulation is a Laman graph. Conversely, as observed by Haas et al. [8], every planar Laman graph can be realized as pointed pseudo-triangulation. As a consequence, subsets of plane Laman graphs are exactly the graphs that admit a pointed non-crossing straight-line drawing. A simple example of a planar graph that has no pointed drawing without crossings is the complete graph with four vertices.

We consider various incarnations of the problem how to draw a planar graph pointed, using different kinds of edge shapes. With smooth curves or polygonal chains, the task of constructing a pointed drawing of a given planar graph is trivial. As natural, but still quite simple edge shapes, we study circular arcs, tangent continuous biarcs, and quadratic Bézier curves. Let us briefly review the definition and basic properties of these curves. A tangent continuous biarc consists of two circular arcs that are concatenated in a way that together they form a  $C^1$  continuous curve. A quadratic Bézier curve b spanned by three points  $p_1$ ,  $p_m$  and  $p_2$  is defined by the equation

$$b(t) = (1-t)^2 p_1 + 2(1-t)p_m + t^2 p_2, \quad t \in [0,1].$$

It lies completely inside the triangle  $p_1p_mp_2$  (which is also called *control polygon* of b), has  $p_1$  and  $p_2$  as endpoints, and is tangent to  $p_1p_m$  at  $p_1$  and to  $p_2p_m$  at  $p_2$ .

We also issue the "extent of pointedness". For example, can we guarantee a free angular space around each vertex bigger than a given fixed angle larger than  $\pi$ ? For this stronger pointedness criterion we define the term  $\varepsilon$ -pointedness.

**Definition 2** ( $\varepsilon$ -Pointedness). Let  $\varepsilon > 0$  be a real number. A vertex in the drawing  $\mathcal{F}(G)$  is called  $\varepsilon$ -pointed, if it is incident to an angle greater than  $2\pi - \varepsilon$ . We call a drawing  $\varepsilon$ -pointed if every vertex is  $\varepsilon$ -pointed.

Further, we propose a stronger version of the pointed drawing problem: Given a plane straight-line drawing  $\mathcal{F}_s(G)$ . Can we construct a plane pointed drawing with a certain family of edge shapes *without* changing the placement of the vertices? We call a drawing with this property a *pointed redrawing*. The advantage of a pointed redrawing algorithm is clear, we can profit form the given drawing and keep its advantages (e.g., all vertices are placed on an integer grid or fulfill other optimality criteria).

#### Results

In Section 2, we consider the problem of *pointed redrawings*. We show that every plane straight-line drawing  $\mathcal{F}_s(G)$  can be redrawn pointed and plane with Bézier curves as well as with tangent continuous biarcs. We also disprove that this is always possible by using circular arcs as edges.

Section 3 then deals with *pointed drawings* of (abstract) planar graphs. We prove that every planar graph can be drawn  $\varepsilon$ -pointed with Bézier curves, for arbitrary small  $\varepsilon > 0$ . We show that by using biarcs as edges, every planar graph can be drawn such that for all vertices v, all incident edges share a common tangent ray at v. This is maybe one of the most beautiful results in this paper from an aesthetic point of view. We further prove that every planar graph can be drawn pointed and plane with circular arcs as edges. For pointed drawings with biarcs, Bézier curves, or polygonal chains of length two, we give an explicit tight bound for the number of edges that cannot be drawn as straight-line segments.

We summarize the results presented in this paper in Table 1. Note that all obtained drawings can easily be constructed, with the exception of the method described in the proof of Theorem 3.2, which has to compute a disk packing of the planar graph in a preprocessing step.

edge shape	problem instance	obtained result
circular arcs	pointed drawing	possible, Theorem 3.5
	pointed redrawing	not possible, Theorem $2.3$
tangent continuous biarcs	$\varepsilon$ -pointed drawing	possible, Theorem 3.2
	pointed redrawing	possible, Theorem 2.2
quadratic Bézier curves	$\varepsilon$ -pointed drawing	possible, Theorem 3.1
	pointed redrawing	possible, Theorem 2.1

Table 1: Results presented in this paper.

#### Related Work and Applications

Traditionally, graph drawing is mainly concerned with using the simplest class of curves for the edges: straight-line segments. According to Fáry's theorem [7], every (simple) planar graph has a plane straight-line drawing in the Euclidean plane. There is a vast literature dealing with the question of efficiently finding plane straight-line drawings that fulfill certain (optimality) criteria (see [2, 13] for an overview). Improving work of De Fraysseix, Pach and Pollack [6], Schnyder [19] proved that every planar graph with n vertices has a plane straight-line drawing where the vertices lie on a grid of size  $n-2 \times n-2$ . The famous Koebe-Andreev-Thurston circle packing theorem [1, 10, 22] states that every planar graph can be embedded with straight-line edges in a way such that its vertices correspond to interior disjoint disks, which touch if and only if the corresponding vertices are connected with an edge, see also [15, 4].

If we relax the condition that the given planar graph has to be simple, Fáry's theorem does not hold. The reason is that straight-line drawings are not well defined for loops, or multiple edges between two vertices. However, one can ask how to draw planar multigraphs with loops crossing-free, allowing more complex edge shapes. A natural approach is to use circular arcs. Drawing multiple edges as circular arcs is no problem, as an edge in a straight-line drawing can be perturbed to any number of close-by circular arcs. Loops, however, require more space. The only circular arc between a vertex and itself is a full circle through this vertex. Thus, the vertex has to be incident to an angle of at least  $\pi$ , which then is sufficient for any number of loops at this vertex. This naturally leads to the question of pointed drawings of simple graphs without loops. Thus, as a consequence of Theorem 3.5 we obtain:

**Corollary 1.1.** Every planar multigraph admits a planar drawing with circular arcs on the  $O(n) \times O(n^2)$  grid.

Another potential application for constructing pointed drawings of graphs comes from drawing vertex labels. If the edges incident to a vertex point in all directions, it might be hard to place a label close to its vertex. Thus it is good to have some angular space without incident edges.

#### 2. Pointed Redrawings

We start with the redrawing problem setting. Throughout this section we consider a plane straight-line drawing as input of our problem instance. Let this drawing be  $\mathcal{F}_s(G)$ .

**Theorem 2.1.** For every plane straight-line drawing  $\mathcal{F}_s(G)$  of a simple planar graph G there exists a pointed plane drawing  $\mathcal{F}_q(G)$  with quadratic Bézier curves as edges such that  $\mathcal{F}_q(v) = \mathcal{F}_s(v)$  for all  $v \in V$ . Moreover, for every  $v \in V$  the cyclic order of the edges incident to v in  $\mathcal{F}_s(G)$  is the same as in  $\mathcal{F}_q(G)$ .

*Proof.* Without loss of generality assume that in  $\mathcal{F}_s(G)$  no two vertices have identical *x*-coordinates or *y*-coordinates. Assume further, that the vertices are sorted by *y*-coordinates in increasing order.

We construct  $\mathcal{F}_q(G)$  by iteratively replacing the straight-line edges of  $\mathcal{F}_s(G)$  with quadratic Bézier curves. We first replace the edges incident to the bottom most vertex  $v_1$ , then the edges incident to  $v_2$ , and so on. During the construction we maintain the following two invariants:

- (1) For every vertex  $v_i$ , the tangent rays of all already redrawn edges lie in the open halfplane  $H_i^-$  below the horizontal line through  $\mathcal{F}_q(v_i)$ .
- (2) Every intermediate drawing is plane.

When replacing the edges incident to a vertex  $v_i$ , all edges incident to a vertex  $v_j$ , with j < i, have already been redrawn by our algorithm (as all vertices below  $v_i$  have already been processed). Let  $E_i = e_1, \ldots, e_k$  be the edges which have not been replaced yet, sorted by absolute slope, such that  $e_1$  has the smallest absolute slope. We redraw these edges in increasing order.

Let  $e = v_i v_j$ , j > i be the current edge we want to process (see Figure 2). Due to invariant (1) and the processing order of the edges incident to  $v_i$  we can choose a point  $p_m$  in  $H_i^-$  such that the triangle  $t = \mathcal{F}_q(v_i)p_m\mathcal{F}_q(v_j)$  does not contain any vertex or part of an edge of the current drawing in its interior. By convention we place  $p_m$  to the right of  $\mathcal{F}_q(v_i)$  if e has positive slope, otherwise to the left. We use the triangle tas a control polygon for a quadratic Bézier curve b with endpoints  $\mathcal{F}_q(v_i)$  and  $\mathcal{F}_q(v_j)$ , which we take as replacement for the straight-line edge  $\mathcal{F}_s(e)$ . Note that b is tangent to  $\mathcal{F}_q(v_i)p_m$  at  $\mathcal{F}_q(v_i)$ , and thus invariant (1) still holds for  $v_i$ . As b lies completely inside t, and  $t \setminus \{\mathcal{F}_q(v_j)\}$  lies completely inside  $H_j^-$ , invariant (1) for  $v_j$  and invariant (2) remain fulfilled as well.



Figure 2: Constructing a plane pointed drawing where the edges are quadratic Bézier curves (intermediate step).

Having redrawn all edges in this way, we obtain a drawing whose pointedness follows directly from invariant (1), and that is plane due to invariant (2).

It is possible to augment the drawing  $\mathcal{F}_s(G)$  to a triangulation by adding edges and deleting the corresponding arcs in the final drawing. For a triangulation with fixed boundary face the order of the edges around a vertex is unique up to a global reflection [25]. Hence, this order has to be preserved in  $\mathcal{F}_q(G)$ .

The technique used in the proof of Theorem 2.1 can be modified to show a similar statement for (tangent continuous) biarcs due to the following observation.

**Lemma 2.1.** Consider a triangle spanned by three points  $p_1$ ,  $p_m$  and  $p_2$ . There exists a tangent continuous biarc connecting  $p_1$  with  $p_2$  that lies inside the triangle. Furthermore, the biarc is tangent to  $p_1p_m$  at one end and tangent to  $p_2p_m$  at the other end.

*Proof.* Assume that the segment  $p_1p_m$  is shorter than  $p_2p_m$ . We place a point  $\tilde{p}$  on the segment  $p_2p_m$  such that the length of  $p_m\tilde{p}$  is equal to the length of  $p_1p_m$  (see Figure 3).



Figure 3: Drawing an edge as a tangent continuous biarc in a triangle.

Let  $l_1$  be the line perpendicular to  $p_1p_m$  through  $p_1$  and let  $l_2$  be the line perpendicular to  $p_2p_m$  through  $\tilde{p}$ . The quadrilateral spanned by the intersection point of  $l_1$  and  $l_2$ ,  $p_1$ ,  $p_m$ , and  $\tilde{p}$  is a kite. Thus, there exists a circular arc passing through  $p_1$  and  $\tilde{p}$  with center  $l_1 \cap l_2$ . Because  $l_1$  is perpendicular to  $p_1p_m$ , and  $l_2$  is perpendicular to  $p_2p_m$ , the arc is tangent to  $p_1p_m$  at  $p_1$ , and to  $p_2p_m$  at  $\tilde{p}$ . Let this arc be the first part of our biarc. The second part is given by the straight-line segment  $\tilde{p}p_2$  (a degenerate circular arc). The biarc is tangent continuous because the circular arcs are tangent in the meeting point  $\tilde{p}$ .

**Theorem 2.2.** For every plane straight-line drawing  $\mathcal{F}_s(G)$  of a simple planar graph G there exists a pointed plane drawing  $\mathcal{F}_b(G)$  with tangent continuous biarcs as edges such that  $\mathcal{F}_b(v) = \mathcal{F}_s(v)$  for all  $v \in V$ . Moreover, for every  $v \in V$  the cyclic order of the edges incident to v in  $\mathcal{F}_s(G)$  is the same as in  $\mathcal{F}_b(G)$ .

*Proof.* We re-use the construction from the proof of Theorem 2.1. Whenever we have chosen an appropriate empty triangle for an edge replacement, we place a tangent continuous biarc in it (as described in Lemma 2.1).  $\Box$ 

We conclude this section with a negative result on pointed redrawings.

**Theorem 2.3.** There exist planar graphs G = (V, E) with plane straight-line drawings  $\mathcal{F}_s(G)$ , for which there are no pointed plane drawings  $\mathcal{F}_c(G)$  with circular arcs as edges such that  $\mathcal{F}_c(v) = \mathcal{F}_s(v)$  for all  $v \in V$ .

*Proof.* Consider the graph G shown in Figure 4(a). Vertex  $v_c$  is placed at the origin, vertex  $v_t$  at (0,2), vertex  $v_l$  at (-0.2,1), and vertex  $v_r$  at (0.2,1). The positions of the remaining vertices are obtained by rotating these vertices by ±120 degrees. Since G is 3-connected and planar, its combinatorial embedding is fixed for any non-crossing drawing [25]. This implies that in any such drawing the edge between  $v_c$  and  $v_t$  has to pass through the narrow passage between  $v_l$  and  $v_r$ . Since we are restricted to circular arcs, the arc connecting  $v_t$  and  $v_c$  has to lie in the shaded region depicted in Figure 4(b). This region is the intersection of the disk touching  $v_t, v_l, v_c$  with the disk touching  $v_t, v_r, v_c$ .



Figure 4: Example of a straight-line drawing that can not be redrawn pointed with circular arcs.

The region lies inside a wedge of angle  $\alpha = 45.3$  degrees. Thus, the tangents of two arcs from  $v_c$  to the convex hull are separated by an angle of at most  $\beta = 165.3$  degrees. But in order to make the vertex  $v_c$  pointed, one of these angles would have to be larger than  $\pi$ .

Larger examples can be constructed easily. As long as a straight-line drawing similar to Figure 4(a) is contained inside another drawing, a pointed redrawing with circular arcs is impossible. Moreover, with a construction similar to the one shown in Figure 4(a), but with many "spokes" (instead of just three), one can force the largest possible angle free of incident edges at the central vertex to be arbitrary small.

## 3. Pointed Drawings

#### 3.1. Pointed Drawings with Bézier curves and Biarcs

In the last section the placement of the points was determined by a given plane straight-line drawing. If the location of the vertices can be chosen arbitrarily, we get the following easy consequence of Theorem 2.1.

**Theorem 3.1.** For any  $\varepsilon > 0$  and any planar graph G, there exists a plane drawing  $\mathcal{F}_a(G)$  with quadratic Bézier curves where all vertices are  $\varepsilon$ -pointed.

Proof. Consider an arbitrary straight-line drawing  $\mathcal{F}_s(G)$ . In the proof of Theorem 2.1 we showed a construction for a pointed drawing  $\mathcal{F}'_q(G)$ , in which for every vertex v and for every edge e incident to v, the tangent ray of  $\mathcal{F}'_q(e)$  at  $\mathcal{F}'_q(v)$  lies below the horizontal line through  $\mathcal{F}'_q(v)$ . By compressing the *x*-axis (i.e., scaling by a factor less than 1), the large angle at every vertex in the resulting drawing increases towards  $2\pi$ . This modification produces no crossings. Moreover, every quadratic Bézier curve is transformed to a quadratic Bézier curve (with respect to the compressed control polygon). Thus, sufficiently compressing  $\mathcal{F}'_q(G)$  results in the desired  $\varepsilon$ -pointed drawing  $\mathcal{F}_q(G)$ .  $\Box$ 

By similar arguments, it is possible to obtain an  $\varepsilon$ -pointed drawing  $\mathcal{F}_b(G)$  with biarcs. In this case the argumentation is more involved, because compressing a biarc in one direction does not result in another biarc. However, we can modify the proof of Theorem 2.1 in the following way: Recall that we used as invariant, that for every vertex  $v_i$ , the tangent rays of all already redrawn edges lie in the open halfplane  $H_i^-$  below the horizontal line through  $\mathcal{F}_s(v_i)$ . To obtain a stronger result, we consider vertical double-wedges centered at the embedded vertices with wedge angle  $\varepsilon$ , and redefine the region  $H_i^-$  to be the wedge below the horizontal line through the embedded vertex. We compress the x-axis until all edges of the compressed straight-line drawing lie strictly within the double-wedges of their endpoints, and apply the previous approach with the changed invariant to this compressed drawing.

A disadvantage of this construction is that the biarcs tend to consist of a circular arc with small radius and a circular arc with infinite radius. Thus, these drawings are unlikely to be aesthetically pleasant. For this reason, we present a completely different approach, which also fulfills an even stronger criterion of pointedness. This criterion, namely that all arcs incident to a vertex share a common tangent at this vertex, implies  $\varepsilon$ -pointedness for any  $\varepsilon > 0$ .

**Theorem 3.2.** Every planar graph G = (V, E) has a plane pointed drawing  $\mathcal{F}_b(G)$  with tangent-continuous biarcs as edges such that  $\mathcal{F}_b(G)$  is pointed. Moreover, for every vertex v all edges incident to v share a common tangent at  $\mathcal{F}_b(v)$  in  $\mathcal{F}_b(G)$ . The directions of these tangents can be independently specified for each vertex, with the exception of finitely many directions.

*Proof.* According to the Koebe-Andreev-Thurston circle packing theorem [1, 10, 22], every planar graph admits a disk packing, where each disk belongs to a vertex (which is the center of the disk), and two disks touch if and only if the corresponding vertices share an edge.

We start with such a disk packing of the graph G (see [4, 12, 5] for algorithmic aspects of such packings). To get our drawing  $\mathcal{F}_b(V)$  of the vertices, we place every vertex  $v_i$ arbitrarily on the boundary of its disk  $D_i$ , avoiding touching points of the disks.



Figure 5: Construction of a tangent-continuous biarc from two touching disks  $D_i$ ,  $D_j$ .

Now consider an edge  $v_i v_j \in E$ . For the embedded vertex  $\mathcal{F}_b(v_i)$  let  $t_i$  be the tangent through  $\mathcal{F}_b(v_i)$  to its disk  $D_i$ . Furthermore, let  $p_{ij}$  be the touching point of the two adjacent disks  $D_i$  and  $D_j$  and let  $t_{ij}$  be the tangent to  $D_i$  and  $D_j$  through  $p_{ij}$  (see Figure 5). We draw a circular arc  $C_i$  from  $\mathcal{F}_b(v_i)$  to  $p_{ij}$  inside  $D_i$ , the center of  $C_i$  being the crossing of  $t_i$  and  $t_{ij}$ . Similarly, we draw an arc  $C_j$  from  $\mathcal{F}_b(v_j)$  to  $p_{ij}$  inside  $D_j$ , with center  $t_j \cap t_{ij}$ . Both arcs meet in  $p_{ij}$  with the same tangent (orthogonal to  $t_{ij}$ ). Therefore, the concatenation of  $C_i$  and  $C_j$  gives a tangent-continuous biarc. We use  $C_iC_j$ as drawing for  $v_iv_j$  and apply this construction for all edges in E.



Figure 6: The situation at a vertex  $v_i$  that shows that the biarcs do not intersect.

It is left to show that the constructed drawing is non-crossing. Two biarcs could cross only within a disk of the disk packing. Consider all circular arcs incident to the embedded vertex  $\mathcal{F}_b(v_i)$  as depicted in Figure 6. All corresponding circles have their centers on  $t_i$  and are passing through  $\mathcal{F}_b(v_i)$ , which lies on  $t_i$  as well. Thus, any two of these circles intersect only in  $\mathcal{F}_b(v_i)$ , and the constructed drawing is plane.

All biarcs incident to an embedded vertex  $\mathcal{F}_b(v_i)$  have a common tangent orthogonal to  $t_i$ . We can determine this tangent by placing the vertex  $v_i$  on  $C_i$  appropriately, avoiding the finitely many touching points of  $D_i$ .

The above proof leaves some freedom to place the vertices on the boundaries of the corresponding disks. If in the drawing  $\mathcal{F}_s(G)$  no two disk centers have the same *x*-coordinate, we can place each vertex on the bottommost point of the boundary of its disk. By this, all biarcs have positive curvature and we have no "S-shaped" biarcs (see Figure 7).

Another possibility is to place each vertex  $v_i \in V$  farthest away from any touching point of its disk  $D_i$ . In this way we can guarantee the radius of any circular arc inside  $D_i$  to be at least  $R_i \cdot \tan \frac{\pi}{2k_i}$ , where  $R_i$  is the radius of  $D_i$ , and  $k_i \geq 2$  is the degree of  $v_i$ . Unfortunately, in general, we have no control over the radii  $R_i$  in the disk packing.

#### 3.2. Pointed Drawings with Circular Arcs

We assume in this section that no two vertices will get the same *y*-coordinate in the drawing. We aim at constructing drawings that contain only the following special types of circular arcs.



Figure 7: A pointed drawing with biarcs as edges, constructed from a disk packing.

**Definition 3 (upper horizontally tangent arc, uht-arc).** Let  $p_1$  and  $p_2$  be two points, where  $p_1$  has the larger y-coordinate. We call a circular arc between  $p_1$  and  $p_2$  upper horizontally tangent if it passes through  $p_1$  and  $p_2$ , and has a horizontal tangent at  $p_1$ .

**Definition 4 (upper horizontally tangent triangle, uht-triangle).** We call a drawing of a triangle *upper horizontally tangent* if all of its edges are drawn as uht-arcs.

Note that for any two points the uht-arc is uniquely defined. Hence, for every point triple the uht-triangle is unique. The following lemmata show that under certain assumptions the uht-triangles behave nicely.

**Lemma 3.1.** Consider the uht-arc  $\mu$  between  $p_1$  and  $p_2$ . Let  $h_1$  be the horizontal line through  $p_1$ . Then the angle at  $p_1$  between  $h_1$  and  $\mu$  is twice as large as the angle at  $p_1$  between  $p_1p_2$  and  $h_1$ .

*Proof.* The situation stated in the lemma is depicted in Figure 8. Let  $\alpha$  be the angle at  $p_1$  between  $h_1$  and  $p_1p_2$ . This angle is the alternate angle to the angle at  $p_2$  between  $h_2$  and  $p_2p_1$ . Let  $p_t$  be the intersection of the tangents of  $\mu$  at  $p_1$  and  $p_2$ . The triangle  $p_1p_2p_t$  is isosceles and hence the angle between  $p_1p_2$  and  $p_1p_t$  is  $\alpha$  as well. Thus, the angle between  $\mu$  and  $p_1p_2$  equals  $2\alpha$ .

In the following lemma, we restrict the straight-line edges to have an absolute slope less or equal 1. This implies that the angle between the tangent of an uht-arc at the lower point and the horizontal line through this lower point is at most  $\pi/2$ . As a consequence, the uht-arc is x-monotone and is contained in the (axis parallel) bounding rectangle spanned by its endpoints.

**Lemma 3.2.** Consider three points  $p_1, p_2, p_3$ , sorted by their x-coordinates. If

(i) the absolute slope of the line segments  $p_1p_2$ ,  $p_2p_3$  and  $p_1p_3$  is smaller than 1, and



Figure 8: Construction used in the proof of Lemma 3.1.

(ii)  $p_2$  lies below the line through  $p_1$  and  $p_3$ , or  $p_2$  has the highest y-coordinate,

then  $p_1$ ,  $p_2$ , and  $p_3$  span a non-crossing uht-triangle that is oriented in the same way as the straight-line triangle  $p_1p_2p_3$ , that is, the clockwise order of the points around the faces is the same.

*Proof.* For  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ , we denote with  $y_i$  be the y-coordinate of  $p_i$ , with  $h_i$  the horizontal line passing through  $p_i$ , and with  $a_{ij}$  the uht-arc between  $p_i$  and  $p_j$ .

We prove the lemma by case distinction. Without loss of generality we assume that  $y_1 < y_3$ . Depending on the relative location of  $y_2$  we obtain three cases (see Figure 9).



Figure 9: The three cases discussed in the proof of Lemma 3.2.

**Case 1**  $(y_2 < y_1)$ :  $a_{13}$  and  $a_{23}$  cannot intersect since they have a common tangent at  $p_3$  and do not lie on the same circle. The other pairs of arcs have bounding rectangles with disjoint interior, and hence cannot intersect.

**Case 2**  $(y_1 < y_2 < y_3)$ : Again,  $a_{13}$  and  $a_{23}$  do not intersect since they have a common tangent at  $p_3$  and do not lie on the same circle. The arcs  $a_{12}$  and  $a_{23}$  have bounding rectangles with disjoint interior, and therefore do not intersect either. Since  $p_2$  lies below the line segment  $p_1p_3$  (condition (ii)),  $p_2$  lies below the arc  $a_{13}$  and  $p_1p_3$  has larger slope than  $p_1p_2$ . Thus, and due to Lemma 3.1, the angle between the tangent of  $a_{13}$  and  $h_1$  is larger than the angle between the tangent of  $a_{12}$  and  $h_1$ , meaning that  $a_{12}$  is incident to

 $p_1$  "below"  $a_{13}$ . As the second endpoint of  $p_2$  of  $a_{12}$  lies below  $a_{13}$  as well, an intersection of  $a_{12}$  and  $a_{13}$  (to the right of  $p_1$ ) would imply a second such intersection. This is impossible, because the two circles induced by  $a_{12}$  and  $a_{13}$  would intersect three times. **Case 3**  $(y_3 < y_2)$ : The pairs  $a_{23}/a_{12}$ , and  $a_{23}/a_{13}$  have bounding rectangles with disjoint interior and therefore do not intersect. For the remaining pair of arcs  $a_{12}$  and  $a_{13}$  we apply again Lemma 3.1 and observe that  $a_{12}$  is incident to  $p_1$  "above"  $a_{13}$ . As the second endpoint of  $p_2$  of  $a_{12}$  lies above  $a_{13}$  as well, it follows that an intersection of  $a_{12}$  and  $a_{13}$  (to the right of  $p_1$ ) would again imply that the two circles induced by  $a_{12}$  and  $a_{13}$  intersect three times, which is impossible.

Since in all three cases the order of incident edges at each vertex is preserved, the orientation of the uht-triangle is identical to the orientation of the straight-line triangle.  $\Box$ 

We continue by constructing a straight-line drawing that allows us to substitute its triangles by uht-triangles. The basic idea goes back to de Fraysseix, Pach and Pollack [6].

**Theorem 3.3** ([6]). A plane triangulated graph has a plane straight-line drawing on a  $(2n-4) \times (n-2)$  grid.

Let us briefly review the incremental construction used in [6], see Figure 10. The vertices are inserted in a special (so-called canonical) order, such that the next vertex  $p_{k+1}$  that is inserted can be drawn on the outer face of the graph  $G_k$  induced by the first k vertices. Thereby as invariant it is maintained that the outer boundary of the graph  $G_k$  (drawn so far) forms a chain of pieces of slope  $\pm 1$ , resting on a horizontal basis (Figure 10(a)). The next vertex  $p_{k+1}$  to be drawn is adjacent to a continuous subsequence of vertices on the outer boundary. To make space for the new edges incident to  $p_{k+1}$ , the boundary of  $G_k$  is split into three pieces, which are separated from each other by shifting them one unit apart (Figure 10(b)). The middle piece contains all neighbors of  $p_{k+1}$  except the first and the last one. They show that shifting the whole graph  $G_k$  apart (inside the shaded area) does not create any crossings.

We slightly modify this inductive procedure to prove the following theorem.

**Theorem 3.4.** A plane triangulated graph has a plane straight-line drawing on a grid of size  $(4n - 9) \times (2n - 4)$ , with the following additional properties:

- (a) No edge is vertical.
- (b) No edge is horizontal.
- (c) In each triangular face, the vertex with the middle x-coordinate is either the vertex with the highest y-coordinate, or it lies below the opposite edge.

*Proof.* The newly created triangles in the incremental construction described above always fulfill property (c), which can be checked directly, and no horizontal edges are created (property (b)). The only horizontal edge is the bottom base edge. This horizontal edge can easily be avoided by starting the construction with a non-horizontal base triangle in the first step.

To prevent vertical edges, one can split the middle part into two pieces and set them apart by two more units (Figure 10(c)). (*Two* units are necessary to ensure that the left



Figure 10: (a–b) The incremental step in the straight-line drawing algorithm of de Fraysseix, Pach and Pollack [6], and (c) the modification that prevents vertical edges.

and right part are separated in total by an even offset; this guarantees that the position of  $p_{k+1}$ , which is defined by the requirement that its leftmost and rightmost incident edges have slope +1 and -1 respectively, gets integer coordinates.)

Adding a vertex preserves the order of the old y-coordinates and the order of the edges incident to a vertex. As a consequence, properties (b) and (c) can be guaranteed to hold for previously added vertices after shifting. Property (a) is preserved because shifting decreases the absolute slope of an already inserted edge.

The dimensions of the grid increase by  $4 \times 2$  units for each new vertex. The initial drawing of the graph  $G_3$  with the first three vertices needs a  $3 \times 2$  grid.

We continue with the main result of this section.

# **Theorem 3.5.** Every planar graph G has a plane pointed drawing with circular arcs as edges.

Proof. We assume that the graph G is a triangulation (otherwise we add edges such that G becomes a triangulation and delete these edges in the end). Given an *n*-vertex plane triangulated graph, the algorithm of Theorem 3.4 constructs drawings in which for every edge its absolute slope is less than 2n. By scaling the *x*-axis by a factor of 2n, we obtain a drawing in which all edges have slopes in the range between -1 and +1. In this scaled drawing, all triangles fulfill the conditions of Lemma 3.2. We substitute every straight-line edge by its corresponding uht-arc. Note that by this substitution, the order of the edges around a vertex is preserved, and every straight-line triangle is replaced by its corresponding uht-triangle. Thus, and due to Lemma 3.2, the obtained circular drawing is crossing-free (Edges on the upper hull are non-crossing as they have bounding-rectangles with disjoint interior).

Around every vertex there is a number of edges that emanate in the horizontal direction, plus a number of additional edges that point upward. The latter type of edges have distinct tangent directions. Thus one can slightly bend every edge upward and achieve a pointed drawing with circular arcs.  $\hfill \Box$ 

Due to Theorem 3.4, pointed drawings constructed as above lie on an  $O(n) \times O(n^2)$  grid. An example of such a drawing is shown in Figure 11.



Figure 11: An example of a pointed drawing with circular arcs.

#### 3.3. Pointed Drawings with Combinatorial Pseudo-Triangulations

A different way to find a pointed drawing utilizes the framework established by Haas et al. [8] and Orden et al. [14]. Let us recall some terminology first.

A combinatorial pseudo-triangulation is a planar combinatorial embedding of an (abstract) connected planar graph G with an assignment of the tags big/small to the angles of G such that the following conditions are fulfilled.

- (1) Every interior face has exactly three *small* angles.
- (2) The outer face has only angles labeled *big*.
- (3) Every vertex is incident to at most one angle labeled *big*. If it is incident to a big angle, it is called pointed (in the face where is has its big angle).
- (4) A vertex of degree at most 2 is incident to one angle labeled *big*.

Due to [8, Section 5.2], every combinatorial pseudo-triangulation can be embedded as a pseudo-triangulation such that every angle with tag *big* is larger than  $\pi$  in the drawing, and every angle with tag *small* is smaller than  $\pi$  in the drawing. Furthermore, the shape of every face can be specified up to affine transformations.

**Lemma 3.3.** Every triangulation with n vertices can be turned into a combinatorial pointed pseudo-triangulation by subdividing at most n-3 edges, each with exactly one additional vertex. Furthermore, for every given vertex not on the outer face, the face in which it is pointed can be prescribed.

*Proof.* The combinatorial pointed pseudo-triangulation is constructed incrementally. At the beginning, every angle of an interior face gets the tag *small* (indicated by a  $\bullet$  in the figures), and every angle at the outer face gets the tag *big* (indicated by a  $\circ$  in the figures). This assignment fulfills the four desired properties of a combinatorial pseudo-triangulation, and none of the interior vertices is incident to a *big* angle tag. Now we

add the *big* tags for the interior vertices on the prescribed faces one by one. Changing a tag from *small* to *big* violates condition (1). This can be repaired by subdividing an incident edge of the enlarged angle, and by giving the new vertex the tags *small* (on the face where the deficit of small angles appears) and *big* (on the opposite face). The procedure is depicted in Figure 12. Now all three conditions hold again and we continue.



Figure 12: How to turn a small angle tag (indicated by a  $\bullet$ ) into a *big* angle tag (indicated by a  $\circ$ ) at the leftmost vertex, while maintaining a proper assignment of angle tags.

What remains to show is that we can choose the edges to be subdivided in a way that in total every edge is subdivided at most once. For every vertex we have two incident edges we could choose, and every edge could be chosen at most twice. This implies that for any subset V of vertices the number of candidate edges is at least |V|. Thus, the setting precisely fulfills Hall's condition of the marriage theorem (see e.g. [9, 3]). Thus, there exists a distinct representative (edge) for each vertex, i.e., each edge is subdivided at most once.

**Theorem 3.6.** Every planar graph G with n vertices has a plane pointed drawing with either quadratic Bézier curves, tangent continuous biarcs, or 2-chains (polygonal chains consisting of two line segments), which uses at most n-3 non straight-line edges. Moreover, for each inner vertex, one can arbitrarily choose a face in which it is pointed.

*Proof.* We assume that the graph G is a triangulation (otherwise we add edges such that G becomes a triangulation and delete these edges in the end), also determining the outer face. As a first step, we turn G into a combinatorial pointed pseudo-triangulation as done in Lemma 3.3, selecting the faces that get the *big* angle tags appropriately. We apply the algorithm of [8] to realize the combinatorial pseudo-triangulation. Recall that using this algorithm, we are allowed to specify the shape of the faces up to affine transformations. Thus we can ensure that in every interior face, every vertex incident to a reflex angle can see (or is incident to) all other vertices in the face. We depict the possible face shapes in Figure 13. Any affine transformation does not destroy the "visibility criteria" – otherwise the orientation of a point triple reverses, which is only possible if we reverse all orientations.

What we have obtained so far is a pointed drawing, where at most n-3 edges are drawn as 2-chains, which proves the theorem for the case of polygonal chains.

For the case of Bézier curves or biarcs, we consider for each 2-chain  $p_1, p_m, p_2$  (with  $p_m$  being the vertex of degree two) the triangle  $\Delta = p_1 p_m p_2$ .  $\Delta$  lies in a pseudo-triangle in which  $p_m$  has a small angle, and in which at least one of the points  $p_1$  and  $p_2$  is



Figure 13: Affine shapes of faces used for the drawing and possible control triangles for curve replacement inside these faces.

pointed (the one whose direction of pointedness caused the insertion of  $p_m$ ). Due to the affine shape of the faces, all triangles which are spanned by 2-chains have pairwise disjoint interior. Moreover, as we do not have 3-chains, every edge is part of at most one of these triangles, and every degree two vertex is part of exactly one such triangle. We use these triangles as control polygons as shown in Figure 13 to replace the 2-chains by Bézier curves or biarcs (similar as in Lemma 2.1).

In general it is not possible to draw a planar graph pointed using a larger number of (non-crossing) straight-line edges, since every maximal pointed straight-line graph has 2n - 3 edges [21], and due to Euler's formula a triangulation can have up to 3n - 6 edges. In this sense, Theorem 3.6 is optimal. Note that the underlying theory behind the algorithm of [8] is an (asymmetric) version of Tutte's famous barycentric embedding method [23, 24].

We demonstrate the construction used in the proof of Theorem 3.6 by an example. Let G be the graph depicted in Figure 14(a). We want to obtain a pointed drawing where every interior vertex is pointed in the central triangle. The appropriate combinatorial pseudo-triangulation (created with the methods of the proof of Lemma 3.3, the grey dashed edges are added to obtain a triangulation) is shown in Figure 14(b).

The realization of the combinatorial pseudo-triangulation is depicted in Figure 15(a). Replacing the 2-chains by Bézier curves, we obtain Figure 15(b).

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Figure 14: Construction of a combinatorial pseudo-triangulation as example.



Figure 15: Pointed drawing with 2-chains (a) and biarcs (b) of the graph of Figure 14.

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