# On the structure of sets attaining the rectilinear crossing number *

Oswin Aichholzer[†] David Orden[‡] Pedro A. Ramos[§]

**Abstract**

We study the structural properties of the point configurations attaining the rectilinear crossing number $\overline{cr}(K_n)$, that is, those $n$-point sets that minimize the number of crossings over all possible straight-edge embeddings of $K_n$ in the plane. As a main result we prove the conjecture that such sets always have a triangular convex hull.

The techniques developed allow us to show a similar result for the halving-edge problem: For any $n$ there exists a set of $n$ points with triangular convex hull that maximizes the number of halving edges. Moreover, we provide a simpler proof of the following result from [13]: any set of points in the plane in general position has at least $3\binom{j+2}{2}$ $(\leq j)$-edges. This bound is known to be tight for $0 \leq j \leq \lfloor \frac{n}{3} \rfloor - 1$. In addition, we show that for point sets achieving this bound the $\lfloor \frac{n+3}{6} \rfloor$ outermost convex layers are triangles.

## 1 Introduction

Given a graph $G$, its *crossing number* is the minimum number of edge crossings over all possible drawings of $G$ in the plane. Crossing number problems have both, a long history, and several applications to discrete geometry and computer science. We refrain from discussing crossing number problems in its generality, but instead refer the interested reader to the early works of Tutte [15] or Erdös and Guy [8], the recent survey by Pach and Tóth [14], or the extensive online bibliography by Vrt'o [16].

In 1960 Guy [10] started the search for the *rectilinear crossing number* of the complete graph, $\overline{cr}(K_n)$, which considers only straight-edge drawings. The study of $\overline{cr}(K_n)$ is commonly agreed to be a difficult task and has attracted a lot of interest in recent years, see e.g. [1, 2, 3, 6, 7, 13]. In particular exact values of $\overline{cr}(K_n)$ are only known up to $n = 17$, see [4], and also the exact asymptotic behavior is still unknown. Several relations to other structures, like for example $k$-sets, are conjectured [11], but surprisingly little is known about the combinatorial properties of optimal sets.

Therefore in this paper we consider structural properties of point sets minimizing the number of crossings, that is, attaining the rectilinear crossing number $\overline{cr}(K_n)$. Relations are obtained by using basic techniques, like e.g. motion flips and rotational sweeps.

Moreover we draw connections to $j$-edges and $k$-sets which provide additional insight for two other prominent problems (see [5, 13]): Maximizing the number of halving edges and counting $(\leq k)$-edges. Let us recall that a *$j$-edge*, $0 \leq j \leq \lfloor \frac{n-2}{2} \rfloor$, is a segment spanned by the points $p, q \in S$ such that precisely $j$ points of $S$ lie in one open half space defined by the line through $p$ and $q$. In other words a $j$-edge splits $S \setminus \{p, q\}$ into two subsets of cardinality $j$ and $n-2-j$, respectively. Note that we consider non-oriented $j$-edges, i.e., the edge $pq$ equals the edge $qp$. A *$(\leq j)$-edge* has at most $j$ points in this half space, that is, it is a $k$-edge for some $0 \leq k \leq j$.

Due to the lack of space, we will omit proofs in this abstract.

## 2 Minimizing the number of rectilinear crossings

### 2.1 Order type flip events

Let $S = \{p_1, ..., p_n\}$ be a set of $n$ points in the plane in general position, that is, no three points lie on a common line. It is well known that crossing properties of edges spanned by points from $S$ are exactly reflected by the order type of $S$, introduced by Goodman and Pollack in 1983 [9]. The *order type* of $S$ is a mapping that assigns to each ordered triple $i, j, k$ in $\{1, ..., n\}$ the orientation (either clockwise or counter-clockwise) of the point triple $p_i, p_j, p_k$.

Consider a point $p_1 \in S$ and move it along a straight line in the plane in a continuous way. A change in the order type of $S$ occurs if, and only if, the orientation of a triple of points of $S$ is reversed during this process. This is the case precisely if $p_1$ is moved accross a line spanned by two other points, say $p_2$

*Research on this paper was carried out while the first author was a visiting professor at the Departamento de Matemáticas, Universidad de Alcalá, Spain.

†Institute for Software Technology, Graz University of Technology, oaich@ist.tugraz.at. Partially supported by the FWF (Austrian Fonds zur Förderung der Wissenschaftlichen Forschung) under grant S09205-N12, FSP Industrial Geometry

‡Departamento de Matemáticas, Universidad de Alcalá, david.orden@uah.es. Partially supported by grants MTM2005-08618-C02-02 and S-0505/DPI/000235.

§Departamento de Matemáticas, Universidad de Alcalá, pedro.ramos@uah.es. Partially supported by grants TIC2003-08933-C02-01, BMF2002-04402-C02-01 and S-0505/DPI/000235.

and $p_3$, of $S$. We call this event a *flip*. The three-dimensional analogous of these order type changes have been used in the study of the halving-edge problem in [5], where they were called mutations.

Assume that at time $t_0$ the three points $p_1, p_2, p_3$ are collinear, then the orientation of that triple at time $t_0 + \epsilon$ is inverse to its orientation at time $t_0 - \epsilon$ for an arbitrarily small constant $\epsilon > 0$. Let us assume that $p_1$ moves over the line segment $p_2p_3$ as indicated in Figure 1; otherwise we can interchange the role of $p_1$ and $p_2$ (or $p_3$, respectively) for the time interval $[t_0 - \epsilon, t_0 + \epsilon]$. We say that $p_1$ plays the *center role* of the flip. Note that for the last assumption we make use of the fact that no three points of $S$ are collinear, except for the moment when a flip is performed. Therefore we further assume that we do not stop the movement of $p_1$ during the flip, that is, in a collinear position.

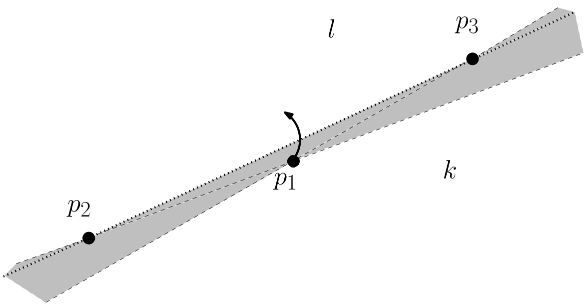


Figure 1: The point $p_1$ is flipped over the segment $p_2p_3$, changing the orientation of the triple $p_1, p_2, p_3$.

We call the above defined flip a $(k,l)$-*flip* if there are $k$ points on the same side of the line through $p_2$ and $p_3$ as $p_1$, excluding $p_1$, and $l$ points on the opposite side. Note that $k+l = n-3$. Our first goal is to study how flips affect the number of crossings of $S$, that is, the number of crossings of a straight-line embedding of $K_n$ on $S$, which will be denoted by $\overline{cr}(S)$. Note that we are only considering rectilinear crossings.

**Lemma 1** *A $(k,l)$-flip increases the number of crossings of $S$ by $k - l$.*

## 2.2 Halving rays

Since we know how flips affect the number of crossings, we are now interested in good moving directions. A point $p \in S$ is called *extreme* if it is a vertex of the convex hull of $S$. Two extreme points $p, q \in S$ are called non-consecutive if they do not share a common edge of the convex hull of $S$. We define a *halving ray* $\ell$ to be an oriented line passing through one extreme point $p \in S$, avoiding $S \setminus \{p\}$ and splitting $S \setminus \{p\}$ into two subsets of cardinality $\frac{n}{2}$ and $\frac{n-2}{2}$ for $n$ even and $\frac{n-1}{2}$ each for $n$ odd, respectively. Furthermore, we orient $\ell$ away from $S$: For $H$ a half plane through $p$ containing $S$, the 'head' of $\ell$ lies in the complement of $H$ and the 'tail' of $\ell$ splits $S$.

**Lemma 2** *Let $p$ be an extreme point of $S$ and $\ell$ be a halving ray for $p$. Moving $p$ along $\ell$ in the given orientation, every occurring flip event decreases the number of crossings of $S$.*

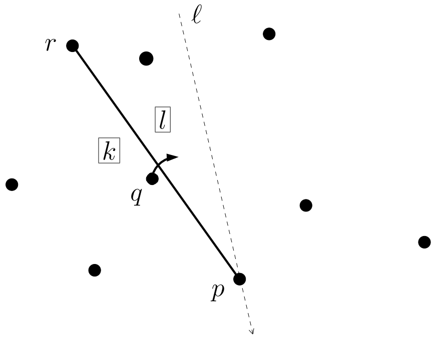


Figure 2: Idea of the proof of Lemma 2.

**Sketch of the Proof.** When point $p$ (see Figure 2) is moved along a halving ray $\ell$, every flip involves $p$ and the center role is played by a different point $q$. Line $\ell$ being a halving ray implies that the flip is a $(k,l)$-flip with $k < l$. $\square$

**Lemma 3** *For every pair of non-consecutive extreme points $p$ and $q$ of $S$, we can choose halving rays that cross in the interior of the convex hull of $S$.*

**Remark 4** Using order type preserving projective transformations it can also be seen that a triangular convex hull can be obtained by projection along the halving ray. This is a rather common tool when working with order types, see e.g. [12]. However, we have decided to use a self-contained, planar approach.

We now have the ingredients to go for our first main results:

**Theorem 5** *Let $S$ be a set of $n$ points in the plane in general position with $h > 3$ extreme points. Then there exists a set $S'$ of $n$ points in general position with fewer crossings than those of $S$ and fewer than $h$ extreme points.*

As a consequence of Theorem 5 we prove the following common belief (see e.g. [7]) for which evidence was provided by all configurations attaining $\overline{cr}(K_n)$ for $n \leq 17$, [2, 4]:

**Theorem 6** *Any set $S$ of $n \geq 3$ points in the plane in general position attaining the rectilinear crossing number has precisely 3 extreme points, that is, a triangular convex hull.*

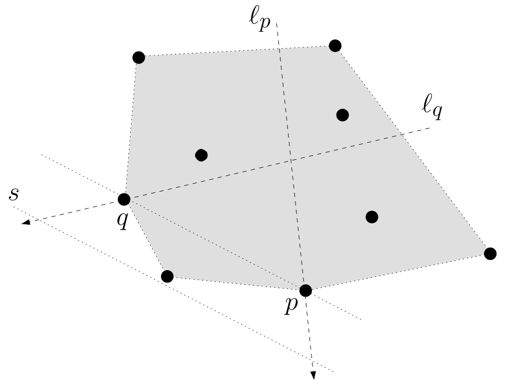


Figure 3: Idea of the proof of Theorem 5.

**Observation 1** If $S$ has 3 extreme points, from our proof of Theorem 5 it follows that for an optimal set $S$ the three extreme points have to be 'far away' in the following sense: For every extreme point $p$ of $S$, the cyclic sorted order of $S \setminus \{p\}$ around $p$ has to be the same as its sorted order in the direction orthogonal to the halving ray of $p$. (Otherwise another flip event would occur when we keep on moving $p$).

## 3 Halving edges, $j$-edges and $k$-sets

A *$k$-set* of $S$ is a set $S' \subset S$ of $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ points that can be separated from $S \setminus S'$ by a line (hyperplane in general dimension). In dimension 2 there is a one-to-one relation between the numbers of $k$-sets and $(k-1)$-edges, since each of these objects can be derived from precisely two of its corresponding counterparts. Thus, in this paper we will solely use the notion of $j$-edges, although all the results can also be stated in terms of $k$-sets.

A *halving edge* is a $j$-edge with $j = \lfloor \frac{n-2}{2} \rfloor$. Similarly to Lemma 1, we now consider how the numbers of $j$-edges and halving edges change during a flip.

**Lemma 7** *A $(k,l)$-flip changes the number of $j$-edges in the following way: For $k < l$ it decreases the number of $k$-edges by one and increases the number of $(k+1)$-edges by one. For $k > l$ it decreases the number of $(l+1)$-edges by one and increases the number of $l$-edges by one. It thus changes the number of halving edges by 0 or 1 for $k < l$, 0 or $-1$ for $k > l$. For $k = l$ everything remains unchanged.*

**Lemma 8** *A $(k,l)$-flip either leaves the number of halving edges unchanged or increases it by 1 for $k < l$ (decreases it by 1 for $k > l$, respectively).*

Results similar to Lemmas 7 and 8 have been obtained for dimension 3 in [5]. We are now ready to show our main result for halving edges.

**Theorem 9** *For any fixed $n \geq 3$, there exists a point set with triangular convex hull that maximizes the number of halving edges.*

One might wonder whether we can obtain a stronger result similar to Theorem 6 stating that any point set maximizing the number of halving edges has to have a triangular convex hull. But there exist sets of 8 points with 4 extreme points bearing the maximum of 9 halving edges, see [5], and similar examples exist for larger $n$. Hence, the stated relation is tight in this sense. We leave as an open problem the existence of a constant $h$ such that any point set maximizing the number of halving edges has at most $h$ extreme points. We conjecture that such a constant exists, and the results for $n \leq 11$ suggest that $h = 4$ could be the tight bound.

From Lemma 1 and Lemma 7 we get a relation between the number $\overline{cr}(S)$ of rectilinear crossings of $S$ and the number of $j$-edges of $S$, denoted by $f_j$. An equivalent relation can be found in [13].

**Lemma 10**

$$\overline{cr}(S) + \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} (j-1)(n-j-3) f_j = \frac{n^4 - 10n^3 + 27n^2 - 18n}{8}$$

**Lemma 11** *The number $\overline{cr}(S)$ of rectilinear crossings of $S$ can be computed in $O(n^2)$ time.*

Define the *$j$-edge vector* of $S$ as $(f_0, \ldots, f_{\lfloor \frac{n-2}{2} \rfloor})$. Another consequence of Lemma 10 is that any two sets of $n$ points with the same $j$-edge vector necessarily have the same number of crossings. The reverse is in general not true, as we have examples of sets with 11 points and 106 crossings each, but $j$-edge vectors $(3, 6, 9, 15, 22)$ and $(3, 6, 10, 12, 24)$, respectively. But it is conjectured that for point sets attaining the rectilinear crossing number this relation is in fact a bijection [11]. This conjecture is known to be true for $n \leq 16$, and it turned out that the distribution of the number of $j$-edges follows some interesting patterns, see [4] for details.

Let us consider the flipping operation as some kind of local improvement operation, in order to obtain a global optimum that minimizes the number of crossings. The idea would be to start with an arbitrary point set and to repeat an improving flip until no more improving flips exist. However, as one would expect, there exists an example of 9 points with 40 crossings and a $j$-edge vector $(3, 6, 11, 16)$ which is locally optimal w.r.t. flips. But the global optimum has only 36 crossings and a $j$-edge vector $(3, 6, 9, 18)$.

## 4 On $j$-edge vectors and $(\leq j)$-edges

Recall that a $(\leq j)$-edge is a segment spanned by two points $a, b \in S$ that has at most $j$ points of $S$ in one open half space defined by the line through $ab$.

Using similar notations as above, $f_{(\leq j)}$ counts the number of $(\leq j)$-edges of $S$ and $(f_{(\leq 0)}, \ldots, f_{(\leq \lfloor \frac{n-2}{2} \rfloor)})$ is the $(\leq j)$-*edge vector* of $S$.

**Lemma 12** *Let $S$ be a set of $n$ points with $h > 3$ extreme points, having $(\leq j)$-edge vector $(f_{(\leq 0)}, \ldots, f_{(\leq \lfloor \frac{n-2}{2} \rfloor)})$. Then there exists a set $S'$ of $n$ points with triangular convex hull and $(\leq j)$-edge vector $(f'_{(\leq 0)}, \ldots, f'_{(\leq \lfloor \frac{n-2}{2} \rfloor)})$ with $f'_{(\leq i)} \leq f_{(\leq i)}$ for all $i = 0, \ldots, \lfloor \frac{n-2}{2} \rfloor$, where at least one inequality is strict.*

This allows us to give a geometric proof of the following result, which was proved in [13] using circular sequences:

**Theorem 13** *Let $S$ be a set of $n$ points in the plane. The number of $(\leq j)$-edges of $S$ is at least $3\binom{j+2}{2}$ for $0 \leq j < \frac{n-2}{2}$. This bound is tight for $j \leq \lfloor \frac{n}{3} \rfloor - 1$.*

That Theorem 13 is not tight for $j \geq \lfloor \frac{n}{3} \rfloor$ can be seen for $n = 5$: For any order type, all $\binom{5}{2} = 10$ segments are $(\leq 1)$-edges. Still we can prove some property of the lexicographic minimal $j$-edge vector:

**Corollary 14** *Let $v = (f_0, \ldots, f_{\lfloor \frac{n-2}{2} \rfloor})$ be the lexicographic minimal $j$-edge vector for a set $S$ of $n$ points in the plane. Then, $f_i = 3$ for $i = 0, \ldots, \lfloor \frac{n}{3} \rfloor - 1$ and $f_{\lfloor \frac{n}{3} \rfloor} \geq 3$.*

The investigation of lexicographic minimal $j$-edge vectors is driven by the following conjecture, for which evidence is also provided by Lemma 10:

**Conjecture 1** *Point sets attaining the rectilinear crossing number have a lexicographic minimal $j$-edge vector.*

The next result provides insight about the structure of point sets minimizing the $j$-edge vector. For a set $S$ of $n$ points in the plane we call the convex hull of $S$ its *1-st convex layer*. The *$j$-th convex layer* of $S$ is the convex hull of $S_j$, where $S_j$ is the set of points we get after removing all points from $S$ which lie on the $k$-th convex layer for $1 \leq k < j$.

**Theorem 15** *If $S$ has $3\binom{j+2}{2}$ $(\leq j)$-edges for every $j$, $0 \leq j \leq 2J < \frac{n-2}{2}$, then for $1 \leq k \leq J+1$ the $k$-th convex layer of $S$ is a triangle, consisting of three $(2(k-1))$-edges.*

Finally, from Theorem 13 we know that sets minimizing the $j$-edge vector (or equivalently the $(\leq j)$-edge vector) have precisely $3\binom{j+2}{2}$ $(\leq j)$-edges for $j \leq \lfloor \frac{n}{3} \rfloor - 1$. We thus get:

**Corollary 16** *Let $S$ be a set lexicographically minimizing the $j$-edge vector. Then the outermost $\lfloor \frac{n+3}{6} \rfloor$ convex layers of $S$ are triangles. Among them, the $k$-th convex layer consists of three $(2(k-1))$-edges.*

## References


[1] B.M. Ábrego and S. Fernández-Merchant, *A lower bound for the rectilinear crossing number.* Graphs and Combinatorics, 21:3 (2005), 293–300.

[2] O. Aichholzer, *Rectilinear Crossing Number Page.* `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`

[3] O. Aichholzer, F. Aurenhammer, H. Krasser, *On the crossing number of complete graphs.* Computing 76 (2006) 165–176.

[4] O. Aichholzer, H. Krasser, Abstract Order Type Extension and New Results on the Rectilinear Crossing Number. In *Proc. Proc. 21th Ann. Sympos. Comput. Geom.*, Pisa, Italy (2005), 91–98.

[5] A. Andrzejak, B. Aronov, S. Har-Peled, R. Seidel, E. Welzl, Results on $k$-Sets and $j$-Facets via Continuous Motion. In *Proc. Proc. 14th Ann. Sympos. Comput. Geom.*, Minneapolis, USA, (1998), 192–199.

[6] J. Balogh, G. Salazar, Improved bounds for the number of $(\leq k)$-sets, convex quadrilaterals, and the rectilinear crossing number of $K_n$. In *Proc.12th Int. Symp. on Graph Drawing.* LNCS 3383 (2005), 25–35.

[7] A. Brodsky, S. Durocher, E. Gethner, *Toward the rectilinear crossing number of $K_n$: new drawings, upper bounds, and asymptotics.* Discrete Mathematics 262 (2003), 59–77.

[8] P. Erdös, R.K. Guy, *Crossing number problems.* American Mathematical Monthly 80 (1973), 52–58.

[9] J.E.Goodman, R.Pollack, *Multidimensional sorting.* SIAM J. Computing 12, 484-507, 1983.

[10] R.K. Guy, *A combinatorial problem.* Nabla (Bulletin of the Malayan Mathematical Society) 7 (1960), 68–72.

[11] H.F.Jensen, *Personal communication.* (2004/05)

[12] H.Krasser, *Order Types of Point Sets in the Plane.* PhD-Thesis, TU-Graz (2003)

[13] L.Lovász, K.Vesztergombi, U.Wagner, E.Welzl, *Convex Quadrilaterals and k-Sets.* Contemporary Mathematics 342 (2004), 139–148.

[14] J. Pach, G. Tóth, *Thirteen problems on crossing numbers* Geombinatorics 9 (2000), 195–207.

[15] W.T. Tutte, *Toward a theory of crossing numbers.* Journal of Combinatorial Theory 8, (1970), 45–53.

[16] I. Vrt'o, Crossing numbers of graphs: A bibliography. `http://www.ifi.savba.sk/~imrich`