Geodesic Order Types^{*}

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July 22, 2013

Abstract

The geodesic between two points a and b in the interior of a simple polygon P is the shortest polygonal path inside P that connects a to b. It is thus the natural generalization of straight line segments on unconstrained point sets to polygonal environments. In this paper we use this extension to generalize the concept of the order type of a set of points in the Euclidean plane to geodesic order types. In particular, we show that, for any set S of points and an ordered subset $\mathcal{B} \subseteq S$ of at least four points, one can always construct a polygon P such that the points of \mathcal{B} define the geodesic hull of S w.r.t. P, in the specified order. Moreover, we show that an abstract order type derived from the dual of the Pappus arrangement can be realized as a geodesic order type.

1 Introduction

Order types are one of the most fundamental combinatorial descriptions of sets of points in the plane. For each triple of points the order type encodes its orientation and thus reflects most of the combinatorial properties of the given set. We are interested in how much the order type of a point set changes when the points lie inside a simple polygon, and the orientation of point triples is given with respect to the geodesic paths connecting them. As depicted in Figure 1, this orientation can change depending on the polygon. In this paper we develop a generalization of point set order types to the concept of geodesic order types.

In set theory, order types impose an equivalence relation between ordered sets. Two sets have the same *order type* if there is a bijection between them

^{*}Research supported by the ESF EUROCORES programme EuroGIGA - ComPoSe, Austrian Science Fund (FWF): I 648-N18 and grant EUI-EURC-2011-4306. M. K. received support of the Secretary for Universities and Research of the Ministry of Economy and Knowledge of the Government of Catalonia and the European Union. A.P. is recipient of a DOC-fellowship of the Austrian Academy of Sciences at the Institute for Software Technology, Graz University of Technology, Austria.

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Figure 1: The radial order of shortest paths to points around a point p can be different in unconstrained and geodesic settings.

that is order preserving [11, pp. 50–51]. Goodman and Pollack [6] extend this concept to finite, multidimensional sets. They define that two d-dimensional point sets S_1 and S_2 have the same point set order type when there exists a bijection σ between the sets such that each (d+1)-tuple in S_1 has the same orientation (i.e., the side of the hyperplane defined by $p_1 \dots p_d$ on which the point p_{d+1} lies) as its corresponding tuple in S_2 . It is also common to consider two point sets to be of the same order type if all orientations are inverted in the second set. In the plane, this means that for two sets of the same order type, the ordered point triple u, v and w has the same orientation (clockwise or counterclockwise) as $\sigma(u)$, $\sigma(v)$, $\sigma(w)$. The infinitely many different point sets of a given cardinality can therefore be partitioned into a finite collection of order types. The orientations of all triples of the point set determine for any two given line segments whether they cross. Therefore, the order type defines most of the combinatorial properties of a point set.¹ For example, its convex hull, planarity of a given geometric graph (e.g., a triangulation), its rectilinear crossing number, etc. only depend on the order type. One might wonder whether every (consistent) assignment of orientations to triples of an abstract set allows a realization as a point set in the Euclidean plane. This is in general not true, not even if the assignment fulfills axiomatic requirements. See Knuth's monograph [12] for a detailed and self-contained discussion of this topic.

Generalizing classic geometric results to geodesic environments is a wellstudied topic. For example, Toussaint [16] generalized the concept of convex hulls of point sets to geodesic environments. Other topics like Voronoi Diagrams [2], Ham-sandwich Cuts [3], Linear Programming [4], etc. have also been covered. However, to the best of our knowledge, the concept of geodesic order types has not been studied in the literature. Hence, it constitutes a natural and general extension to the above results.

The classic order type is often used to identify extremal settings for combinatorial problems on point sets. For example, finding sets which minimize the number of crossings in a complete geometric graph, or maximize the number of elements of a certain class of graphs (spanning trees, matchings, etc.)

¹It is common to regard the properties defined by orientations of triples as the combinatorial ones. There are further settings on point sets that can be seen as being combinatorial as well, e.g., asking whether the fourth point of a quadruple lies inside the circle defined by the first three ones (see [12]). Also, the circular sequence of a point set is a richer way of describing the combinatorics of point sets, totally implying the order type [9].

are typical applications. In a similar spirit, the geodesic order type might be used to investigate extremal properties in geodesic environments. Examples might be problems on pseudo-triangulations (the side chains of a pseudo-triangle are geodesics), guarding problems inside polygonal boundaries (there, shortest paths are geodesics), and related problems; see, e.g., [15] for a recent survey on pseudo-triangulations.

1.1 Preliminaries

A closed polygonal path P is called a *simple* polygon if no point of the plane belongs to more than two edges of P, and the only points that belong to exactly two edges are the vertices of P. A closed polygonal path Q is a *weakly simple* polygon if every pair of points on its boundary separates Q into two polygonal chains that have no proper crossings, and if the angles of a complete traversal of the boundary of Q sum up to 2π [16]. Observe that a simple polygon is a weakly simple polygon, but the reverse is not true. Unless stated otherwise, all polygons are considered to be simple herein. We will follow the convention of including both, the interior and the boundary of a polygon, when referring to it. The boundary of polygon P will be denoted by ∂P .

The geodesic $\pi(s, t, P)$ between two points $s, t \in P$ in a simple polygon P is defined as the shortest path that connects s to t, among all the paths that stay within P. If P is clear from the context, we simply write $\pi(s, t)$. It is well known from earlier work that there always exists a unique geodesic between any two points [13], even if P is weakly simple. Moreover, this geodesic is either a straight line segment or a polygonal chain whose vertices (other than its endpoints) are reflex vertices of P. Thus, we sometimes denote the geodesic as the sequence of these reflex vertices traversed in the geodesic (i.e., $\pi(s,t) = \langle s = v_0, v_1, \ldots, v_k = t \rangle$). When the geodesic $\pi(s,t)$ is a segment, we say that s sees t (and vice versa).

For any fixed polygon P, a region $C \subseteq P$ is geodesically convex (also called relative convex) if for any two points $p, q \in C$, we have $\pi(p, q, P) \subseteq C$. The geodesic hull (relative convex hull) $\operatorname{CH}_P(U)$ of a set U is defined as the smallest (in terms of inclusion) geodesically convex region C that contains U. We will denote by $\operatorname{CH}(U)$ the standard Euclidean convex hull. Whenever a point $p \in U$ is in the boundary of $\operatorname{CH}_P(U)$, we say that p is an extreme point of U (with respect to P). The set of all such extreme points is called the extreme set of U, and is denoted by $E_P(U)$.

Although these definitions are valid for any subset U of P, in this paper we will only use them for a finite set of points $S = \{p_1, \ldots, p_n\}$. Further note that the geodesic hull is a weakly simple polygon; see Figure 2. From now on we assume that the points in the union of S with the set V of vertices of P are in *strong general* position. That is, there are no three collinear points, and, for any four distinct points $p_1, p_2, p_3, p_4 \in S \cup V$, the line passing through p_1 and p_2 is not parallel to the line passing through p_3 and p_4 .



Figure 2: Seven points inside a polygon P and their geodesic hull (marked in gray). Observe that the boundary of the geodesic hull consists of the concatenation of the shortest paths connecting the extreme vertices of S, in circular order. Further note that a vertex of the geodesic hull that stems from P can be a convex and a reflex vertex of the geodesic hull at the same time (like vertex u) or only a reflex vertex (like v).

1.2 Orientations and Geodesics

The concept of clockwise order of a triple of points (p, q, r) naturally extends to geodesic environments. Let $\pi(p,q) = \langle p = v_0, \ldots, v_k = q \rangle$ and $\pi(p,r) = \langle p = u_0, \ldots, u_{k'} = r \rangle$ be the geodesics connecting p with q and r, respectively. Also, let i > 0 be the smallest index such that $v_i \neq u_i$. We say that (p,q,r) are in geodesic clockwise order if (v_{i-1}, v_i, u_i) are in (Euclidean) clockwise order. It is easy to see that, due to the strong general-position assumption, any triple is oriented either clockwise or counterclockwise in the geodesic environment. We adopt the common phrasing, and say that r is to the right of q (with respect to p) whenever (p,q,r) are in geodesic clockwise order (or that r is to the left, otherwise). By definition, if (p,q,r) are in geodesic clockwise order, then for any $a < i \leq b, c$, the triple (v_a, v_b, u_c) must also be in geodesic clockwise order. Hence, this definition also accounts for the intuitive perception of "left" and "right" when traversing the geodesics.

Note that "left" and "right" differ between the geodesic and the unconstrained setting, since we can use reflex vertices of the surrounding polygon to "reorder" unconstrained point triples. An illustration is shown in Figure 3; in this example, the polygonal chain crosses two edges of the triangle and the supporting line of the third one. In general, this operation is not local, and might alter the order type of other triples (more details of this operation will be given in Section 3).

The orientation predicate can also be defined in terms of the geodesic hull $CH_P(\{p,q,r\})$. When traversing this hull counterclockwise, the points appear in that order if and only if their geodesic orientation is counterclockwise.

1.3 Contribution

The triple orientation in geodesic environments extends the one in Euclidean environments. Since the latter defines the order type of a point set, we obtain a generalization of point set order types to geodesic order types. It is easy to see that the order type of a fixed point set S can change with different enclosing



Figure 3: Reordering a triangle using a polygonal chain. The triple (a, b, c) is in (Euclidean) counterclockwise order. However, upon introducing the polygon (right figure) the same triple is now in (geodesic) clockwise order.

polygons. In particular, some points that appear in the (Euclidean) convex hull may not be present in the geodesic hull and, vice versa, some non-extreme points of S may appear on the geodesic hull.

In this paper, we study the ways in which the set of extreme points of a given set S can change with the shape of the polygon. We show that any subset \mathcal{B} of four or more points of S can become the extreme set of S (i.e., there exists a polygon P such that $E_P(S) = \mathcal{B}$). Moreover, we can make them appear in any predefined order along the boundary of the geodesic hull. We also characterize when this property is fulfilled for sets of size 3. Finally, we show in Section 3 that the abstract order types that can be realized as geodesic order types are a proper superset of the abstract order types realizable as Euclidean order types. Specifically, we show that the non-realizable abstract order type derived from Pappus' Theorem via duality can be realized as a point set inside a polygon.

Our approach can also be seen as the class of inverse problems to the classic questions for geodesic environments, where the polygon is usually part of the input.

2 Geodesic Hull versus Convex Hull

In this section, we study how much the geodesic hull of a given point set can alter from the Euclidean convex hull. We partition S into two sets of blue and red points (\mathcal{B} and \mathcal{R} , respectively). A set \mathcal{B} is said to be *separable* from \mathcal{R} if there exists a polygon with at most $|\mathcal{B}|$ convex vertices (i.e., a pseudo- $|\mathcal{B}|$ -gon) that contains all points of \mathcal{R} and no point of \mathcal{B} in its interior. From now on, we assume that the set S is fixed. Thus we omit writing "from \mathcal{R} " and simply refer to \mathcal{B} as a separable point set. The following theorem draws a nice connection between the separability of point sets and their geodesic hull.

Theorem 1. For any separable point set \mathcal{B} and any permutation σ of \mathcal{B} , there exists a polygon P such that $E_P(S) = \mathcal{B}$ and the clockwise ordering of \mathcal{B} on the boundary of $CH_P(S)$ is exactly σ .

Proof. Let $k = |\mathcal{B}|$ and P be a separating polygon of \mathcal{B} . If P has strictly less than k convex vertices, we introduce more by replacing any edge e by two edges,



Figure 4: Illustration of the proof of Theorem 1: with a one-to-one correspondence between the convex vertices c_1, \ldots, c_7 of P and the points s_1, \ldots, s_7 of \mathcal{B} , we can obtain a weakly simple polygon P' such that $E_{P'}(S) = \{s_1, \ldots, s_7\}$ (left), which then can be transformed to a polygon (right).

adding a convex vertex arbitrary close to the center point of e. Thus, we assume that P has k convex vertices c_1, \ldots, c_k .

Let s_1, \ldots, s_k be an arbitrary ordering of the vertices of \mathcal{B} . For all $i \leq k$, we connect point $s_i \in \mathcal{B}$ to c_i by a polygonal chain. Observe that we can always do this in a way that no two chains cross. Now let P' be the union of P and the polygonal chains; see Figure 4 (left). The union of geodesics connecting s_i with s_{i+1} (and s_k with s_1) exactly corresponds to the boundary of P'. Moreover, all points of \mathcal{R} are in the interior of P'. Notice that P' is not a polygon, but a weakly simple polygon. As illustrated in Figure 4 (right), we obtain a polygon from P' by transforming polygonal paths into narrow passages of width at most ε (for a sufficiently small ε , and such that no blue point sees any other blue point).

We now study the separability of a point set as a function of its size. Surprisingly, the separability of the set \mathcal{B} does not strongly depend on the set \mathcal{R} .

Theorem 2. Any set \mathcal{B} with cardinality $|\mathcal{B}| \geq 5$ is separable with a polygon with at most $2|\mathcal{B}| - 2$ vertices.

In order to prove the above theorem, we first consider some simpler cases and then show how to deal with larger point sets.

Lemma 1. Any set \mathcal{B} of five points is separable.

Proof. It is well-known that any set of five points contains a convex quadrilateral abcd that does not contain the fifth point e. The supporting lines of the edges ab and cd cross due to the strong general position assumption (analogously for the supporting lines of bc and da). These pairs of supporting lines define two wedges that contain abcd, and at least one of them does not contain the fifth point e (since their only region of intersection is the quadrilateral). W.l.o.g., let this be the wedge defined by the supporting lines of ab and cd and let m be the crossing point of its two supporting lines. Further, assume that m lies on the

ray from a through b and that the supporting line of ab separates e from c and d; see Figure 5 (left). We build two narrow polygonal spikes that contain the blue points and end sufficiently far away from the point set. Each spike has a positive aperture angle at its unbounded end and a sufficiently small aperture such that it does not contain any red point. The first spike starts on the line through cand d in a way that it contains c, d, and m. At m, the spike bends towards band a (with a slightly positive aperture angle). The second spike contains e, has its bisector parallel to the supporting line of ab, and is directed in the opposite direction of the first spike; see again Figure 5 (left). Let l and l' be two lines which are parallel to the supporting line of *ab* and slightly outside the convex hull of S (one line on each side). Since, by construction, the bisectors of the (last parts of the) two spikes are parallel to l and l', the pair of rays emanating from the end of each spike intersect l and l'. These intersection points become the end points of the spikes. Thus, they form a convex quadrilateral containing all the points of S, implying that the resulting polygon is a separating pseudo-5-gon; see Figure 5 (right).



Figure 5: A set of five points is always separable. A narrow, bent spike can be built around the empty convex quadrilateral of the set. A second spike is chosen parallel to the first one and in opposite direction. Sufficiently far away, the spike end points span a quadrilateral around the whole set.

Note that we can use the same construction when \mathcal{B} consists of four points in convex position. In such a case, we do not need the second spike (containing e), and only place one convex vertex of the pseudo-4-gon on the supporting line of ab in the opposite direction of the spike. If the three convex points on the convex hull of the pseudo-4-gon are chosen sufficiently far away from the points of \mathcal{B} , the pseudo-4-gon will always cover the red point set. This implies Corollary 1.

Corollary 1. Any set \mathcal{B} of four points in convex position is separable.

Remark. The separating polygon used in the above construction is likely to have a "bad aspect ratio", in the sense that its horizontal dilation is far larger than the one of the convex hull of the point set. While examples can be constructed where this cannot be avoided, we note that for subsets \mathcal{B} of cardinality ≥ 6 , we might obtain more elegant separating polygons using a different construction. In essence, that approach removes pairs of points of \mathcal{B} with thin wedges and uses a large enclosing triangle; see Figure 6 for an example.

The complete construction requires some case analysis on the order type of the point set, and is thus omitted.



Figure 6: One case for a construction to obtain a more "nicely" shaped polygon for $|\mathcal{B}| = 6$.

Lemma 2. For any set \mathcal{B} separable by a polygon P and a point $q \notin S$, the set $\mathcal{B} \cup \{q\}$ is separable by a polygon P' having at most two more vertices than P.



Figure 7: Proof of Lemma 2. Regardless of whether q is in ∂P , q sees c or a reflex vertex v_1 , we can separate $\mathcal{B} \cup \{p\}$. In all of the above cases, at most one convex vertex is added to P (as well as two edges).

Proof. Let P be the polygon that separates \mathcal{B} . Clearly, if $q \notin P$, the same polygon separates $\mathcal{B} \cup \{q\}$. If $q \in \partial P$, it is easy to do a small perturbation to P such that q is not contained in P anymore; see Figure 7 (left).

Thus, we assume that q is in the interior of P. Let c be any convex vertex of P. If q sees c, we remove q from P by adding a small spike emanating from c towards q; see Figure 7 (middle). In this operation we replace a single convex vertex with two. Since we also increased the size of the set by one, the separability invariant still holds.

It remains to consider the case in which q does not see c. Then, the geodesic connecting q and c is of the form $\langle q = v_0, v_1, \ldots, v_k = c \rangle$ for some $k \geq 2$. By definition, q sees v_1 and the interior angle $\angle pv_1v_2$ is larger than π (otherwise we could connect q towards v_2 directly). In this case we replace a reflex vertex with two vertices, but only one of them will be convex; see Figure 7 (right). \Box

The class of polygons constructed in the proof of Lemma 1 will never have more than 8 vertices. Moreover, by Lemma 2, each additional point of \mathcal{B} will add at most 2 additional vertices to the separating polygon. In particular, we will always have a separating polygon P whose number of edges is at most $2|\mathcal{B}| - 2$, which completes the proof of Theorem 2.

By definition, any point set of size 1 or 2 cannot be separated (since we cannot construct a simple polygon with one or two convex vertices). Hence, it remains to consider the cases in which $|\mathcal{B}| \in \{3, 4\}$. Let d be the shortest distance between any pair of blue points. We say that a set $\mathcal{R} \varepsilon$ -densely covers \mathcal{B} (for any $\varepsilon > 0$) if any wedge emanating from $p \in \mathcal{B}$ and not containing any point of \mathcal{R} inside a circle with center p and radius d/2 has an opening angle of at most ε . Observe that, if $\mathcal{R} \varepsilon$ -densely covers \mathcal{B} , no point of \mathcal{B} can appear on the boundary of CH(S). Moreover, if $\varepsilon \leq \pi/3$, any convex region that contains three or more blue points must contain a red point. Showing that for any set \mathcal{R} that ε -densely covers \mathcal{B} (for some sufficiently small ε), \mathcal{B} cannot be separated from \mathcal{R} , we obtain the following result.

Theorem 3. For any set \mathcal{B} of three points or four points in non-convex position, there exists a set \mathcal{R} such that \mathcal{B} is not separable from \mathcal{R} .

Proof. We claim that for any set \mathcal{R} that ε -densely covers \mathcal{B} (for some sufficiently small ε), \mathcal{B} cannot be separated from \mathcal{R} . Assume that this is not true, and let P be a separating polygon. Since the red set \mathcal{R} is ε -dense, every blue point must be inside a pocket of P (where a pocket is a simple polygon defined by an edge of CH(P) and the sub-sequence of edges of P between the two vertices of that edge).

If $|\mathcal{B}| = 3$, the separating polygon has to be a pseudo-triangle, and every pocket is a side chain of P. We define the *aperture* of a side chain as the inner angle between the supporting lines of the first and the last edge of the side chain. Since the red set is ε -dense, at most two blue points can be separated via the same side chain, and thus there must be at least two side chains enclosing blue points. Moreover, any such side chain has an aperture angle of at most ε . Consider the angular turn at a vertex of P, i.e., the signed angular change of direction when traversing the boundary of P. Recall that the sum of the angular turns a simple polygon is 2π and observe that due to the aperture of the pockets, the sum of the angular turns of the two pockets containing at least one blue point is $(-2\pi)+2\varepsilon$. The angular turns of the three convex vertices can only add an amount strictly smaller than 3π to that sum, which implies that we would need a fourth convex vertex to close the polygonal chain.

We now consider the case $|\mathcal{B}| = 4$. Recall each pocket of P is associated to a sub-sequence of edges that starts and ends at convex vertices of P. Moreover, convex vertices of P in such a sub-sequence (other than the endpoints) correspond to reflex vertices of the pocket (and *vice versa*). Since CH(P) must have at least three vertices, P can have at most one single pocket that is non-convex (and this situation can only happen when CH(P) is a triangle).

As \mathcal{B} is non-convex, any pocket containing all four blue points (and no red point) would need at least two convex vertices. This implies that there have to

be at least two pockets containing blue points. Let k be the number of pockets of P that contain blue points, and let β_1, \ldots, β_k be the number of blue points contained in each pocket (in decreasing order).



Figure 8: Scheme of the possible configurations in which we can place four blue points in up to four pockets. In all cases, we obtain a contradiction, hence a separating polygon cannot exist.

Since $\sum_i \beta_i = 4$ and $\beta_i \in \{1, \ldots, 3\}$, we distinguish between the following cases (depicted in Figure 8):

- **Case** k = 2, $\beta_1 = 3$, $\beta_2 = 1$. This case (depicted in Figure 8 (a)) is similar to the case $|\mathcal{B}| = 3$. As a pocket containing three points must have a convex vertex, the convex hull of P has three vertices and the two pockets containing the blue points must share a convex hull vertex (i.e., the side chains associated to each pocket share an endpoint). One of the tree blue points sharing a pocket must see both convex hull vertices of that pocket, and therefore the aperture of that pocket is at most ε . As before, the sum of angular turns is too small, and P cannot be closed without introducing additional convex vertices.
- **Case** k = 2, $\beta_1 = \beta_2 = 2$. First consider the case in which there is a pocket Q_1 that is not convex. If Q_1 does not contain a blue point or it contains a blue point that sees the convex hull vertices of Q_1 , then we can argue in the same way as in the case where $|\mathcal{B}| = 3$ (since the two pockets will be consecutive and each will have aperture at most ε , see Figure 8 (d)).
 - Otherwise, no blue point sees both convex hull vertices of Q_1 (Figure 8 (e)). In this case, we know that both blue points inside Q_1 see each other.



Figure 9: Angles used in the proof that no pocket with a convex vertex contains two blue points. Note that alternatively, the supporting line of b_1b_2 could intersect b_3b_4 .



Figure 10: No separating polygon can exist with two blue points in a pocket with a convex vertex.

Let Q_2 be the second pocket containing blue points. As in the previous cases, we know that the aperture of Q_2 is at most ε . Let u and v be the convex hull vertices defining Q_2 and let w be the third vertex of the triangular convex hull of P. W.l.o.g., v and w define the pocket Q_1 . Let b_1 and b_2 be the two vertices of \mathcal{B} in the pocket Q_2 and let b_3 and b_4 be the ones in Q_1 . Further, let $\alpha > 0$ be the smallest angle between b_1b_2 and b_3b_4 . We argue using bounds on the angles of the resulting polygon, see Figure 9.

Since the angular turns have to sum up to 2π , we observe that the sum of the inner angles of all convex hull vertices is at most ε . The smallest angle between the convex hull edge vw and b_1b_2 is at most ε , since the aperture of Q_2 is at most ε , and the inner angle at v is at most ε , but in the other direction. This implies that the angle between vw and b_3b_4 is at least $\alpha - \varepsilon$. In particular, the supporting line of b_3b_4 intersects the segment vw if we choose $2\varepsilon < \alpha$. W.l.o.g., let b_3 and b_4 be arranged in a way that the ray from b_3 through b_4 intersects vw. Barring symmetries, we have the situation shown in Figure 10. Let c be the convex vertex of Pin the pocket Q_2 . Observe that c has to be in the same closed half-plane defined by the supporting line of vb_4 as the edge vw, as otherwise b_4 sees both v and w or the boundary of P has another convex vertex between v and c. Since c is separated from b_3 by the supporting line of vb_4 , the interior of the triangle defined by the supporting lines of vb_4 , b_4b_3 , and b_3w is disjoint from P, and has an angle of at least $\alpha - 2\varepsilon$ at b_3 . However, this contradicts the assumed ε -density for a suitable choice of ε .

It remains to consider the case in which all pockets are convex. By the nonconvexity of \mathcal{B} , pockets containing blue points cannot share an endpoint (Figure 8 (f)). However, in this case, the convex hull of the four pocket endpoints cannot contain all points of \mathcal{R} , implying that P cannot be a separating polygon.

Case $k \geq 3$. Regardless of how many points are on the convex hull of P, notice that the pockets must share at least two endpoints (Figure 8 (b) and (c)), and that all extreme vertices of P must be pocket endpoints. As the total aperture angle of the three pockets cannot be larger than $k\varepsilon < \pi$, the polygon cannot be closed.



Figure 11: Two pseudo-triangles containing many red points such that the four blue points are not separable. However, they are the extreme vertices w.r.t. some polygon.

The example in Figure 11 shows a point set where the four blue points lie on the geodesic hull but are not separable. This implies, in contrast to sets of larger cardinality, that for $|\mathcal{B}| = 4$, the concepts of separability and geodesic hull are not equivalent. Thus, we switch back to the geodesic setting and consider the remaining cases $|\mathcal{B}| \in \{3, 4\}$.

Lemma 3. For any set S, any set $\mathcal{B} \subset S$ of four points, and any permutation σ of \mathcal{B} , there exists a polygon P such that $E_P(S) = \mathcal{B}$ and the clockwise ordering of \mathcal{B} on the boundary of $CH_P(S)$ is exactly σ .

Proof. If the points of \mathcal{B} are in convex position, then the statement follows directly from Corollary 1 and Theorem 1. Thus, assume that \mathcal{B} is not in convex position. Consider a line l_1 spanned by two of the extreme points of \mathcal{B} , and a line l_2 that is parallel to l_1 and passes through the third extreme point of \mathcal{B} (see Figure 12). We construct two pseudo-triangles P_1 and P_2 , each with four edges, with the following properties: (1) P_1 has a convex and a reflex vertex on l_1 , such that the reflex vertex is between the convex vertex and both blue points on l_1 . (2) Accordingly, P_2 has a convex and a reflex vertex on l_2 , such that the reflex vertex is between the convex vertex and the blue vertex on l_2 . (3) Both, P_1 and P_2 , have a vertex between l_1 and l_2 , and the edges connecting the convex point on l_1 (l_2) to these vertices are parallel. (4) The non-extreme point of \mathcal{B} lies between P_1 and P_2 . (5) All red points lie inside P_1 or P_2 . Note that these properties can always be fulfilled, as the convex points of the pseudo-triangles



Figure 12: Construction for a polygon P with $E_P(S) = \mathcal{B}$ based on two pseudotriangles that contain all red points (depicted with dots) and none of the blue points (drawn as crosses).

can be far away, and thus the reflex angles can be made arbitrarily small and the area covered by the pseudo-triangles can be arbitrarily "thick".

As indicated in Figure 12, we can merge the two pseudo-triangles to form a polygon by adding a narrow passage from a convex vertex of P_1 to a convex vertex of P_2 . To obtain our final polygon P with $E_P(S) = \mathcal{B}$ in the desired order, we proceed like in the proof of Theorem 1, connecting the blue points to the four convex vertices of P_1 and P_2 that were not used for the passage between P_1 and P_2 .

If we combine this result with Theorems 1 and 2 we obtain the following statement.

Theorem 4. For any set S, any set $\mathcal{B} \subset S$ of at least four points, and any permutation σ of \mathcal{B} , there exists a polygon P such that $E_P(S) = \mathcal{B}$ and the clockwise ordering of \mathcal{B} on $E_P(S)$ is exactly σ .

We conclude this section by studying what happens when the set \mathcal{B} has cardinality three.

Theorem 5. Let $\mathcal{B} \subset S$ be a set with $|\mathcal{B}| = 3$ such that \mathcal{B} spans the geodesic hull of S for some polygon P. Then \mathcal{B} is separable.

Proof. Recall that the geodesic hull of S is a weakly simple polygon which has all points of \mathcal{B} on its boundary, and contains all points of $S \setminus \mathcal{B}$ in its interior. Moreover, a vertex v of the geodesic hull can only be convex if (1) $v \in \mathcal{B}$, or (2) v is part of some weakly simple polygonal chain and thus coincides with a reflex vertex of the geodesic hull. Thus, as $|\mathcal{B}| = 3$, the geodesic hull must consist of a pseudo-triangle Δ , possibly with polygonal chains attached to the convex vertices of Δ , where each blue vertex corresponds to one convex vertex of Δ ; see Figure 13. By slightly shrinking Δ , we obtain a pseudo-triangle Δ' still having all points of $S \setminus \mathcal{B}$ in its interior that leaves all points of \mathcal{B} outside. Thus Δ' is a separating polygon for \mathcal{B} .



Figure 13: A set $\mathcal{B} \subset S$ with $|\mathcal{B}| = 3$, and a polygon P (dark shaded) with $E_P(S) = \mathcal{B}$ (depicted \times). The geodesic hull is drawn light shaded.

$ \mathcal{B} $	Pushable		Separable
≤ 2	never (Def.)	\Leftrightarrow	never (Def.)
3	not always	\Leftrightarrow (Thm. 1 and 4)	not always (Thm. 3)
4	always (Thm. 3)	\Leftarrow (Thm. 1)	convex position: always (Cor. 1)
			non-convex: not always (Thm. 3)
≥ 5	always (Thm. 4)	\Leftrightarrow	always (Thm. 2)

Table 1: Overview of results and relationship between pushable and separable.

Together with Theorem 3 the above result implies that there exist point sets S with $|\mathcal{B}| = 3$ such that \mathcal{B} can not be used to define the geodesic hull of S. This is in contrast to the fact that for any set with $|\mathcal{B}| \ge 4$ this is always possible. Table 1 gives an overview of the obtained results and also shows the relation between a set being 'pushable' (meaning that there is a polygon such that \mathcal{B} is on the geodesic hull) and 'separable' for different cardinalities of \mathcal{B} .

3 Realizing the Non-Pappus Arrangement

By duality, every set of points in the *d*-dimensional Euclidean space corresponds to an arrangement of hyperplanes in the same space (see e.g. [5] for details on this mapping). This dual is incidence and order preserving. When traversing a line u^* in the plane, the order in which the lines v^* and w^* are crossed gives the orientation of the corresponding point triple u, v, w in the primal setting [8]. Hence, the crossings in the line arrangement determine the order type of the corresponding point set. An arrangement of pseudo-lines is a set of simple curves such that each pair has exactly one point in common, and at this point the pair crosses. The crossings in the pseudo-line arrangement define an *abstract order type*. Obviously, if we can stretch the curves to straight lines without changing the order of all crossings, we obtain a realization of the order type defined by the crossings. This has been used in the exhaustive enumeration of point set order types [1]. However, for sets of size 9 or more, it is known that there exist non-realizable abstract order types (i.e., pseudo-line arrangements that are non-stretchable). The example for 9 pseudo-lines is based on the well-known Pappus' Theorem [10, 14].

Using the axiomatic system of [12, p. 4], one can show that geodesic order types are in fact a subset of abstract order types, i.e., of those that are defined by pseudo-line arrangements. Let the predicate cc(u, v, w) be true whenever the point triple (u, v, w) is oriented counterclockwise. We already observed that $cc(u, v, w) \Rightarrow cc(v, w, u), cc(u, v, w) \Rightarrow \neg cc(u, w, v),$ and $cc(u, v, w) \lor cc(u, w, v)$. Note that the latter holds since we require all points to be strictly inside the surrounding polygon. What remains to show is that

 $cc(x, u, v) \wedge cc(x, v, w) \wedge cc(x, w, u) \Rightarrow cc(u, v, w)$ and

 $cc(a, b, u) \wedge cc(a, b, v) \wedge cc(a, b, w) \wedge cc(a, u, v) \wedge cc(a, v, w) \Rightarrow cc(a, u, w).$

In other words, if x is left of $\pi(u, v), \pi(v, w)$, and $\pi(w, u)$ then $\operatorname{CH}_P(\{u, v, w\})$ is given by the sequence $\langle u, v, w \rangle$, and the points to the left of $\pi(a, b)$ are in transitive radial order around a. For the first of these statements, observe that since x cannot be on the geodesic hull of the four points, it is inside the pseudotriangular region of the hull. Hence, it is easy to see that the implication is analogous to the Euclidean setting. For the second implication, consider the geodesics from a to u, v, and w. If they split at a, transitivity follows from the analogy to the Euclidean setting. The same is the case if they split at the same vertex r, as r is reflex. If, say, u splits first (the other case is symmetric), it is clear that the orientation of (a, u, v) is the same as of (a, u, w). It follows that all parts of the axiomatic system are fulfilled, and therefore all geodesic order types are realizations of abstract order types (cf. [12, pp. 23–35]).

3.1 The Arrangement



Figure 14: A non-stretchable pseudo-line arrangement derived from Pappus' Theorem, adapted from [7, Fig. 5.3.2] (left). The transformed arrangement, having all lines crossing line 1 first (right).

278	345	467	567	678
279	348	468	568	679
	368	469	569	
	378	478	578	
	379	479	579	
			589	

Table 2: All ascending counterclockwise point triples derived from the arrangement.

The non-stretchable arrangement whose abstract order type we realize in the geodesic setting is an adaption from the one shown in [7, p. 107]; see Figure 14 (left). It is well-known that this pseudo-line arrangement cannot be stretched and thus the corresponding abstract order type cannot be realized by a point set. From the correspondence between a straight line in the Euclidean plane to a great circle in the sphere model of the projective plane, it is easy to see that an arrangement is stretchable in the real plane if and only if it is stretchable in the projective plane, provided that no pseudo-line in the projective plane coincides with the line at infinity. We can therefore apply projective transformations to the arrangement without affecting its realizability. In this way we transform the arrangement of [7] to the *standard labeling*; see Figure 14 (right) for the resulting drawing. Roughly speaking, the crossings of a pseudo-line that happen before the crossing with l_1 are "moved" to the other side. Namely, these are the crossing of l_9 with l_8 and the crossings of l_5 with l_9, l_8, l_7 , and l_6 , in the given order. We do so in order to make all pseudo-lines cross pseudo-line l_1 before any other. In the primal, this corresponds to p_1 being on the convex hull boundary and points p_2, \ldots, p_9 being sorted clockwise around it. Note that this kind of projective transformation actually preserves the order type. Table 2 shows all triples with ascending indices that have counterclockwise orientation (which easily allows obtaining the orientation of all triples). For example, the entry "278" indicates that pseudo-line l_2 crosses l_8 before l_7 , inducing counterclockwise orientation of the point triple $p_2p_7p_8$ in the primal.

3.2 The Realization

Consider the point set $S = \{p_1, \ldots, p_9\}$ shown in Figure 15. The only triples whose orientations do not match those indicated by Figure 14 are the permutations of p_2, p_7 and p_9 . Equivalently, one can say that the triangle defined by the three points is the only one that has the wrong orientation among all triangular subgraphs of the complete graph of S. This triangle is shown with thick (blue) edges.

We already discussed how reflex vertices of a surrounding polygon can change the orientation of a triple. The problem with this tool is that the polygonal chain is likely to reorder other triangles as well. In the point set shown in Figure 15, this tool can, however, be applied. We create a polygon P that contains S. The



Figure 15: A point set that "almost" realizes the unrealizable arrangement. The point triple spanning the thick blue triangle $\Delta p_2 p_7 p_9$ is the one for which the orientation is wrong.

result of the construction is shown in Figure 16. We cross four edges during this operation. Note that the geodesics $\pi(p_1, p_9, P)$ and $\pi(p_1, p_8, P)$ are now no longer line segments, still the order defined by their end vertices has not changed. The triple p_2, p_7, p_9 , however, is now oriented counterclockwise, as demanded by the abstract order type. By checking all the point triples, the reader can verify that this geodesic order type indeed realizes the abstract order type of the non-Pappus arrangement.

Theorem 6. There exists a point set S and a polygon whose geodesic order type realizes an abstract order type that is not realizable as a point set in the plane.

We note that our construction is minimal; that is, there cannot exist a point set of nine points and a polygon of fewer vertices (than the one given in Figure 16) that realize the non-Pappus arrangement.

There are 13 non-stretchable pseudo-line arrangements of 9 lines; all these arrangements correspond to the same arrangement in the projective plane, i.e., the non-Pappus arrangement [14]. As already mentioned, the sphere model of the projective plane shows that a pseudo-line arrangement in the Euclidean plane is stretchable if and only if the corresponding arrangement in the projective plane is stretchable. We found one realization for one abstract order type of the non-Pappus arrangement, however, we do not know whether the remaining 12 non-realizable abstract order types are realizable as a geodesic order type as well.

4 Conclusion

In this paper, we made a first step into generalizing the concept of point set order types to geodesic order types. For a selection of four or more points out of a set S, we showed how to construct a polygon such that exactly these vertices



Figure 16: A geodesic realization of the arrangement (right). The shortest paths between the points are geodesics in the interior of the polygon (gray). The region of interest is shown in detail in the middle. The polygon closes with a convex vertex far on the right side, as indicated.

are on the geodesic hull of S, in any order desired. To the contrary, this is not always possible for three points. We further showed an example of an abstract order type that is not realizable in the Euclidean plane, but is realizable in geodesic environments.

Several interesting questions rise from our investigations. Which bounds on the number of vertices in the polygon that forces the desired geodesic hull can we derive? What is the complexity of minimizing the number of vertices? Even though we showed the realizability of the abstract order type derived from Pappus' Theorem, we have no general tools to realize order types inside polygons. Can every abstract order type (which is non-realizable in the Euclidean plane) be realized as a geodesic order type? And which of them can be realized in a given polygon? If not all of them can be realized, does realizability of an order type imply realizability of all abstract order types that correspond to the same pseudo-line arrangement in the projective plane?

5 Acknowledgements

The authors would like to thank Prosenjit Bose, Stefan Langerman, and Pat Morin for the introduction of the topic and for several fruitful discussions on it.

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