Abstract Order Type Extension and New Results on the Rectilinear Crossing Number

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Abstract

We extend the order type data base of all realizable order types in the plane to point sets of cardinality 11. More precisely, we provide a complete data base of all combinatorial different sets of up to 11 points in general position in the plane. In addition, we develop a novel and efficient method for a complete extension to order types of size 12 and more in an abstract sense, that is, without the need to store or realize the sets. The presented method is well suited for independent computations. Thus, time intensive investigations benefit from the possibility of distributed computing.

Our approach has various applications to combinatorial problems which are based on sets of points in the plane. This includes classic problems like searching for (empty) convex k-gons (happy end problem), decomposing sets into convex regions, counting structures like triangulations or pseudo-triangulations, minimal crossing numbers, and more. We present some improved results to several of these problems. As an outstanding result we have been able to determine the exact rectilinear crossing number of the complete graph K_n for up to $n = 17$, the largest previous range being $n = 12$, and slightly improved the asymptotic upper bound.

1 Introduction

A finite point set in the plane belongs to the most common ingredients for computational and combinatorial geometry problems. For quite many, especially combinatorial problems, the exact metric properties are not relevant, but the combinatorial properties of the underlying point set play the main role. More precisely, the crossing properties of the line segments spanned by the point set already determine the problem. Triangulations, crossing numbers, convexity problems are among other famous examples. Order types provide a means to encode the combinatorial properties of finite point sets. The order type of a point set $S = \{p_1, .., p_n\}$ is a mapping that assigns to each ordered triple (p_i, p_j, p_k) an orientation. Throughout this work we assume that S is in general position, that is, the orientation of each point triple is either clockwise or counter-clockwise. Two point sets S_1, S_2 are of the same order type if and only if there is a bijection between S_1 and S_2 such that either all (or none) corresponding triples are of equal orientation.

To achieve results for point sets of fixed size for the problems mentioned above, it is sufficient to check one instance of each order type instead of looking at all (infinitely many) point sets. A data base containing all order types of size up to 10 already exists [3] and has been applied fruitfully to many problems in computational and combinatorial geometry [8]. One discouraging fact when thinking about an extension of this data base is the huge number of different order types of cardinality n. A lower bound for the asymptotics of the number of order types is $n^{4n+O(\frac{n}{log n})}$, cf. [21]. Moreover, point set representation of order types is highly desirable for applications as many implementations of algorithms are based on point coordinates. Again discouraging, we know that asymptotically this kind of representation may require exponentially many bits for one order type, see Goodman, Pollack and Sturmfels [22]. Felsner [18] provides an order type encoding using only $O(n^2)$ bits. In Section 2, we present new techniques and an extension of the data base to order types of size 11. Moreover, in Section 3, we develop a framework to investigate problems for order types of larger cardinality.

Order types have played a crucial role in gathering knowledge about rectilinear crossing numbers. The crossing number of a graph G is the least number of edge crossings attained by a drawing of G in the plane. Crossing number problems have a quite long history, see, e.g., Tutte [36] and Erdös and Guy [16] or the recent overview paper by Pach and Tóth [30]. We consider the problem of finding the rectilinear (edges are required to be straight line segments) crossing number $\overline{cr}(K_n)$ of the complete graph K_n on n vertices [25]. Determining $\overline{cr}(K_n)$ is commonly agreed to be a difficult task, see [4] for details. So far the exact values of $\overline{cr}(K_n)$ have been known for $n \leq 12$ [3, 4, 15, 16]. There exist combinatorial proofs for $\overline{cr}(K_n)$ for $n \leq 10$, although the proof of $\overline{cr}(K_{10}) = 62$ is quite voluminous. The calculation of $\overline{cr}(K_{11})$ and $\overline{cr}(K_{12})$ is related to the order type data base. In Section 4, we extend this range to $n \leq 17$. Moreover, we also present an improvement on the asymptotic upper bound of $\overline{cr}(K_n)$. Our results are available online [2]. A discussion of the latest results and an improved asymptotic lower bound can be found in [11]. Finally, Section 5 provides further applications and results of our extension technique.

A preliminary version of this work is presented in [7].

2 Order type data base for n=11

In [3] a complete data base of order types for sets with up to 10 points had been established. We present an extension to this data base for point sets of size 11. As many of our techniques are strongly related to [3], we briefly repeat the backbone of necessary concepts and definitions.

Let $S = \{p_1, .., p_n\}$ be a set of $n > 3$ points in the plane in general position. In the data base we identify the order type of S by its λ-matrix. The λ-matrix of S is an $n \times n$ matrix that stores for each ordered pair (p_i, p_j) the number of points of S that lie to the left of the oriented line through points p_i and p_j. Goodman and Pollack [20] proved that the order type of a point set can be completely encoded by its λ-matrix. That is, the numbers of points on the left of oriented lines through all pairs (p_i, p_j) determine which points these are.

The λ-matrix of S is not a unique encoding of an order type, since it depends on the labeling of the points. For identification purposes, we use the unique lexicographically minimal λ-matrix of S like a fingerprint. More precisely, we sort the λ-matrices of a point set by lexicographically comparing their elements row by row, and we take the minimal one in this order. Observe that the first row of a lexicographically minimal λ-matrix of S is given by the sequence $0, .., n-2$. Thus, it corresponds to a labeling of S such that one point on the convex hull of S is the first point p_1 and the other points $p_2, .., p_n$ are sorted clockwise around p_1. We call a λ-matrix satisfying this property natural. Of course, not all natural λ-matrices of S are lexicographically minimal in general.

For certain duality functions between point sets and line arrangements, the order type of S can be read of the local intersection sequences of the dual line arrangement. Generalizing line arrangements to pseudoline arrangements, such a duality transforms pseudoline arrangements to abstract order types. An abstract order type that allows a representation by a point set is called realizable. Not all abstract order types corresponding to pseudoline arrangements are realizable. In fact, for sufficiently large cardinality, most abstract order types are not realizable, cf. [21]. The smallest non-realizable abstract order type in the plane consists of 9 elements and is derived from the non-stretchable arrangement related to Pappus' theorem. In this paper, the term "order types" always refers to realizable abstract order types, whereas the class of "abstract order types" contains all realizable and all non-realizable order types.

For the realization problem of order types we travel from the real plane to the real projective plane. Planar order types can be grouped into classes of equivalent order types in the real projective plane, that is, two different planar order types are considered of the same projective order type if they are order type isomorphic under projective transformations. A planar order type is realizable if and only if its corresponding projective order type is realizable. Thus if one planar order type of a group is realizable, all order types of this group are realizable. Note that abstract order types in the projective plane are isomorphic to oriented matroids of rank 3.

Our approach to build the complete date base for $n = 11$ uses improved techniques to cover the following three steps.

1. Generating a complete candidate list of abstract order types
2. Grouping abstract order types into projective classes and deciding realizability
3. Realizing all realizable order types by point sets with "nice" coordinate representation

For the first step, we calculated the list of abstract order types with the same tools as in [3]. Although we acquired 2 343 203 071 inequivalent abstract order types, we only stored one representative of each projective class explicitly at this time. This evaluates to 41 848 591 abstract projective order types of size $n = 11$, see Table 1. Considering only one member of each projective class is crucial for the next step, that is, for deciding the realizability for all 11-element abstract order types in reasonable time. As a representation of a projective class, we choose the lexicographically minimal λ-matrix from all λ-matrices encoding abstract order types within a class. This can be achieved without explicitly calculating all natural λ-matrices of a projective class, that is, by some kind of "parallel" approach. We start with a list of candidate labelings corresponding to all possible natural λ-matrices of the abstract order types from the same projective class. We get these labelings from the dual pseudoline arrangement in the projective space in the following way: First, we choose the pseudolines corresponding to p_1 and p_2. For each choice of pseudolines 1 and 2, we get exactly two labelings that correspond to a natural labeling in a projection from the projective plane to the Euclidean plane. That is, we label the other pseudolines with labels $3, .., n$ according to the local intersection sequence of the pseudoline 1 and starting from the intersection of the pseudolines 1 and 2. As there are two choices of direction, we get two such labelings (for each choice of 1 and 2) which we take as candidate labelings. Note that there are only $2n(n-1)$ candidate labelings in total for a projective abstract order type, which is only a small fraction of the $n!$ possible labelings.

To determine the lexicographically minimal λ-matrix, we keep a list of all candidate labelings and determine only one relevant element $\lambda(i, j)$ for each corresponding candidate λ-matrix at a time. For this, we can use the facts that the first line of a natural λ-matrix is completely determined and every λ-matrix has the symmetry property $\lambda(i, j)+\lambda(j, i) = n-2$. Thus, $\lambda(2, 3)$ is the first element which has to be calculated explicitly. Obsolete labelings are removed until the labeling corresponding to the unique minimal λ-matrix remains. Note, that for deciding whether a given λ-matrix is lexicographically minimal or not, the first smaller element from some other candidate λ-matrix is sufficient for a negative decision. Moreover, we also applied a similar method to determine the lexicographically minimal λ-matrix of an (abstract) order type in the Euclidean plane. That is, we start with candidate labelings corresponding to all natural λ-matrices of an abstract order type in the plane.

The second step - deciding realizability - is the hardest part of the construction. This has not even been carried out for projective order types before. The trouble is, that this decision problem is known to be NP-hard [29] and no practical algorithms are known, not even for small sets, say of size 10 or 11. Some abstract projective order types can easily be identified as non-realizable by checking the data base for $n = 10$, as each 10-point subset of a realizable 11-point order type has to be realizable too. For a realizability proof the realizing point set, i.e., the x- and y-coordinates of the points, is an instance of the proof itself. We tried to find realizations and started by applying refined versions of the heuristic methods from [3] for each projective order type class. For a projective class of cardinality $n = 11$, we removed one of its points and read a realizing 10-point set in the data base. Then we tried to extend the given set by an 11th point to obtain the desired order type, either directly or including random, but order type preserving perturbation. In general (excluding symmetric cases), there are $11 \times (\binom{10}{2} + 1) = 506$ such 10-point sets, one for each choice of removed point and one for each choice of projection. This simple heuristic worked for most of the abstract order types in question, but left us with less than 0.1% projective classes which we could not settle. For classifying non-realizable order types, we make use of a well-known practical algorithm for a non-realizability proof developed by Bokowski and Richter [13], which is known as finding biquadratic final polynomials in oriented matroid theory, cf. [12]. This method proves unrealizability by showing infeasibility of a system of linear equations. This linear system is based on the sign pattern of Grassmann-Plücker equations (which defines an oriented matroid or a projective abstract order type, respectively). Although the method itself is deterministic, it is still an heuristic in the sense that for a non-realizability proof it does not work with all non-realizable order types. Either it proves the non-realizability of an abstract order type, or we are not able to decide the realizability of an abstract order type with this method. The smallest known non-realizable abstract order type, such that its non-realizability cannot be decided by applying this heuristic, consists of 14-elements and is due to Richter [32]. To our benefit, the heuristics for finding realizations and proving non-realizability were sufficient to completely settle the case for $n = 11$, see Tables 1 and 2.

The main goal of the third step is to store the data base in an application friendly way. To this end, we provide two representations of the data base. An explicit version of the data base contains one point set for each planar order type, all in 16-bit integer representation. The required storage for this kind of representation amounts to nearly 100 Gigabyte. A more efficient way to store order types is to explicitly store only one member of each projective order type class. This contains realizing point sets for all corresponding planar order types

	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	
projective abstract o.t.	4	11	135	4 382	312 356	41 848 591	A006248
− thereof non-realizable				1	242	155 214	
= projective order types	4	11	135	4 381	312 114	41 693 377	A018242
abstract order types	16	135	3 315	158 830	14 320 182	2 343 203 071	A006247
− thereof non-realizable				13	10 635	8 690 164	
= order types	16	135	3 315	158 817	14 309 547	2 334 512 907	A063666

Table 1: Number of order types of cardinality n. The last column gives the sequence identification codes in Sloane's On-line Encyclopedia of Integer Sequences [34].

	n=6	n=7	n=8	n=9	n=10	n=11
$h=3$	6	49	1 178	55 235	4 876 476	783 505 062
$h=4$	6	59	1 468	70 475	6 319 019	1 026 727 086
$h=5$	3	22	570	28 232	2 628 738	438 721 262
$h=6$	1	4	90	4 552	450 176	78 870 441
$h=7$		1	8	311	33 969	6 443 494
$h=8$			1	11	1 146	241 522
$h=9$				1	22	4 006
$h=10$					1	33
$h=11$						1
$\sum$	16	135	3 315	158 817	14 309 547	2 334 512 907

Table 2: Number of order types with n points, thereof h extreme points.

implicitly. For unique identification purposes, we choose the order type representation that obtains the lexicographically minimal λ-matrix among all order types of its projective class. Storing the projective representative of each class in 16-bit integer coordinates, the size of this data base is about 1.7 Gigabyte. To determine x- and y-coordinates realizing these planar order types, we use simple, but efficient deterministic projection techniques. For nearly all point sets 16-bit integer coordinates suffice as well, but there are a few requiring 24-bit. Note that obtaining integer coordinates by the projection technique critically depends on the quality of the stored realization.

Supporting the reliability in the construction of our data base, we applied improved implementations of the methods that have already been used to establish the complete and reliable data base for $n \leq 10$. We had to develop new techniques for satisfying the enormous requirements of time and space for the $n = 11$ data base. All algorithms to generate the complete data base of abstract order types are of purely combinatorial nature. The applied methods for deciding realizability are heuristics, but the acquired results can be checked in a deterministic way. To assure that no duplicate order types appear in our data base, we sorted all order types according to their lexicographically minimal λ-matrices.

The vast storage and the lack of applicability are the two main reasons - apart from calculation time - that we do not have a complete data base of order types with 12 or more points. We estimate the number of order types of cardinality 12 to be approximately 750 billion, thereof about 11 billion projective classes. This would require over 30 Terabyte of storage.

Integer realizations are always possible for order types corresponding to point sets in general position. If collinear points are allowed, realizing point sets might force irrational coordinate representation [23]. In [19] one can find an extension of our work by releasing the general position assumption and dealing with arbitrary dimensions. As the difficulty of the realizability question becomes far worse with the restriction on general position being dropped, mainly results on abstract (projective) order types are presented. We restrict ourselves to point sets in general position as all applications we have in mind are dealing only with such sets.

3 Complete abstract point extension

For several problems and conjectures the complete order type data base of sets of up to 11 points has been sufficient to give a final answer, cf. [4]. Many problems tend to be harder, however, and cannot be settled just by checking all cases for size up to 11. It still looks highly plausible to gain significantly more insight with a few additional points, say sets of 12 or 13 points.

For many problems on point sets there exist inductive restrictions, so-called subset properties. For example, there exists an inequality for the rectilinear crossing number of the complete graph K_{n+1} that provides a measurement for the quality of its subgraphs K_n. This means that not every subgraph K_n of an optimally drawn K_{n+1} can have a large number of crossings, see Section 4. Exploiting relations of this kind between $(n+1)$-point sets and their subsets of size n cuts down a lot on the number of "interesting" point sets. For the rectilinear crossing number example, only a small fraction of the 2 334 512 907 order types of size 11 appears as a (possible) subset in an optimal drawing of K_{12}, K_{13}, and so on.

Definition 1. *(**Subset property**) Let S_n be an order type consisting of n elements, $n \geq 4$, and consider some property that is valid for S_n. Then this property is called a subset property if and only if there exists some $S_{n-1} \subset S_n$ of $n-1$ elements such that a related property holds for S_{n-1}.*

Our general idea is to exploit subset properties for order type based problems to obtain results beyond point sets of size 11. First, we apply the order type data base to completely determine the problem for point sets of small size, that is, up to $n = 11$. This gives a set of result order types of cardinality 11, all realized by point sets. Next, we enumerate all order types of size 12 that contain one of the 11-point result order types as a subset. Applying the subset property, we are able to filter these 12-point order types. Only order types that fulfill the subset property are kept. Then we repeat this procedure, theoretically extending the set of result order types to arbitrary n.

For this technique, we require an algorithm that calculates for a given order type of cardinality n all $(n+1)$-point order types that contain the input order type as a sub-order type. We call this step complete point extension. Unfortunately, similar to the case of generating a complete data base, an extension technique relying only on the geometric realizations of the data base cannot guarantee completeness of the extension, see Figure 1. The set of order types of size $n+1$ that can be generated by adding a single point to a point set of size n does not only depend on the order type of the n-point set but also on the point coordinates. In other words, for a specific n-point realization of an order type we cannot derive all required $n+1$-point order types just by adding a new point to this realization. To achieve completeness of the extension, we use an abstract extension method, that is, applying a combinatorial extension technique. We provide a one-element extension to an abstract order type by adding a pseudoline to the dual pseudoline arrangement in all combinatorially possible ways.

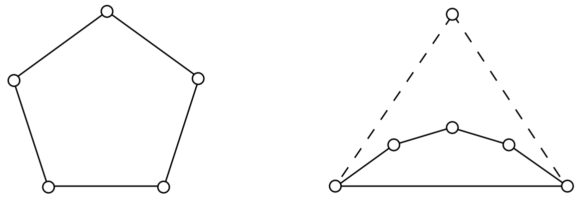

Figure 1: Two realizations of the order type of five points in convex position. Only the right point set can be extended in a way such that the resulting point set has three points on its convex hull.

The algorithm of complete abstract point extension of a single point set works as follows. Given a lexicographically minimal λ-matrix describing the order type of a point set, we calculate the local intersection sequence of the dual pseudoline arrangement. Any natural λ-matrix is transformed to a pseudoline arrangement of a special kind. From the properties of the duality transformation, we know that an abstract point is extreme (corresponding to a point on the convex hull of an order type realization) if and only if its dual pseudoline shares some segment with the vertically unbounded cells. Moreover, the pseudoline corresponding to the first point of the order type intersects the pseudolines corresponding to the remaining points in an increasing sequence, see Figure 2. To extend this arrangement with a new pseudoline, we fix the first intersection of the new pseudoline (from left to right) to be a crossing with the first pseudoline. There are n possibilities to fix this first intersection for the extension of an order type of cardinality n. Then, we apply a back-tracking algorithm for all possible ways of further intersections of the new pseudoline, intersecting each other pseudolines exactly once. Visiting cell by cell, this method reports all possible extending pseudolines conforming with the initial pseudoline arrangement. By abstract duality, we obtain from the local intersection sequences all abstract order types that contain the initial order type as an abstract sub-order type. The duality transformation can be applied directly to obtain all extended abstract order types if the first point keeps its extreme position. Otherwise, the new pseudoline has to correspond to an extreme point. Then we transform the pseudoline arrangement by order type preserving projective transformations, that is, we fix the vertically unbounded cells in the projection and rotate other pseudoline intersections over the line at infinity by altering the direction of projection. In this way, we are able to project the pseudoline arrangement such that the dual abstract order type is unchanged and the new pseudoline fulfills the required properties for the first pseudoline afterwards, cf. [27].

For applications, the problem of complete extension often comes with a slightly different flavor. Instead of extending only one abstract order type, we usually have a set of n-point order types with some property and want to completely extend it to a set of $(n+1)$-point order types that retain that property, e.g. drawings with few crossings to calculate the rectilinear crossing number $\overline{cr}(K_n)$. Checking a subset property like this can be integrated as a filter for the output, but one can hope for much more benefits if such a known property can be included in the construction process more directly. For specific applications with a subset property, we define an order type extension graph. In this graph each order type is represented by a

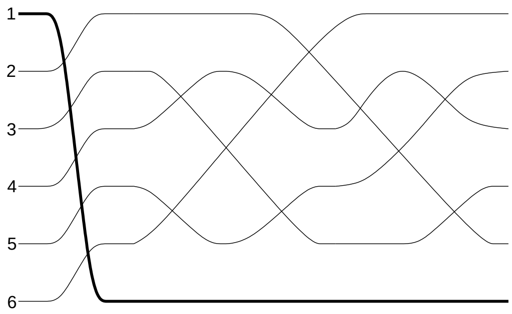

Figure 2: Pseudoline arrangement corresponding to an order type encoding by a natural λ-matrix. Pseudoline 1 intersects the other pseudolines in the sequence $2, 3, .., n$.

node. For each order type of size $n+1$ (son), there is exactly one connection by an edge to a predecessor sub-order type of size n (father). By this definition we have that each order type corresponds to a unique predecessor order type by removal of a single point. On the other hand, an extension process that only extends corresponding to the edges of an order type extension graph (from father to son) enumerates each extended order type exactly once.

Note that the choice and the unique predecessor order type may depend on the application of the extension process. For the example of the rectilinear crossing number, we define the unique predecessor relationship as follows. For an $n+1$-point order type we count the number of crossings determined by the removal of a single point separately. We take the points that determine the minimum number of crossings as a candidate list. From these points we remove the one with the largest index in the lexicographically minimal λ-matrix representation of the initial order type. Thus, this method supplies us with a unique n-point sub-order type as a predecessor for the order type extension graph.

In general, the algorithm of complete abstract point extension extends one input order type point by point, then continuing on the remaining set of order types. After extension with one abstract point, we check if the created order type of size $n+1$ (son) has the initial order type of size n as its predecessor order type (father) in the order type extension graph. Only if this is the case we keep it as a candidate for the output. Note that there is a difference between filtering out order types which do not fulfill some property and order type extensions violating the extension graph. The very general approach with the order type extension graph guarantees to avoid duplicates in the construction process, thus it can be used for recursive enumeration techniques, known as reverse search, cf. Avis and Fukuda [10]. An additional benefit of this technique is that it can be applied iteratively, i.e., extending from n points to $n+1$, $n+2$, and so on, without storing intermediate results. In fact, only the order types corresponding to a single path of the order type extension graph have to be kept in memory, that is, the edges describing the father-son relationship between order types of size n, $n+1$, $n+2$, and so on. This allows calculations which otherwise would not be possible because of enormous storage requirements for intermediate steps. In addition, applications based on the order type extension graph are easily executed in parallel. Thus, highly time intensive problems may be settled through distributed computing approaches. Currently a system implementing this idea is set up at [37].

4 New Rectilinear Crossing Numbers

In the present and the following sections we show how abstract extension can be successfully applied to notoriously hard combinatorial problems.

4.1 Subset Property for $\overline{cr}(K_n)$

To efficiently apply abstract extension we need a measurement on the number of crossings in the n sub-drawings K_{n-1} of K_n. Note that it is not enough to consider only optimal drawings of K_{n-1} for extension, as not all optimal drawings of K_n contain an optimal sub-drawing of K_{n-1}. In fact, it is an open problem whether there always exists at least one optimal drawing of K_n which contains an optimal sub-drawing of K_{n-1}.

The next two well-known lemmas (see e.g. Guy [24] for references) provide the necessary relations to obtain a subset property for $\overline{cr}(K_n)$. For an arbitrary, rectilinear drawing of K_n each of its n sub-drawings K_{n-1} has at least $\overline{cr}(K_{n-1})$ crossings. Summing up the numbers of crossings in the n sub-drawings K_{n-1}, we count each crossing exactly $n-4$ times, as each quadruple of points determining a crossing shows up in all but 4 of the sub-drawings K_{n-1}. This implies the following recurrence relation.

Lemma 2. $\overline{cr}(K_n) \geq \lceil \frac{n}{n-4}\, \overline{cr}(K_{n-1}) \rceil$

Corollary 3. *(***Crossing number subset property***) For any drawing of K_n with c crossings there exists at least one sub-drawing K_{n-1} with at most $\lfloor \frac{n-4}{n}\, c \rfloor$ crossings.*

For example, a drawing of K_{12} with 153 (or fewer) crossings has to contain at least one sub-drawing of size 11 with $\frac{8}{12}\; 153 = 102$ (or fewer) crossings. The following, interesting parity property leads to further improvements for sets of odd cardinality, see e.g. [4] for a simple proof.

Lemma 4. *Let $n \in \mathbb{N}$ be odd. Consider a straight-line drawing of K_n with c crossings. Then: $c \equiv \binom{n}{4} (mod\ 2)$.*

A drawing of K_{13} with 229 (or fewer) crossings contains at least one sub-drawing K_{12} with $\lfloor \frac{9}{13}\; 229 \rfloor = 158$ (or fewer) crossings. Recursive application shows that there exists a sub-drawing of size 11 with $\lfloor \frac{8}{12}\; 158 \rfloor = 105$ crossings. By the parity property we can further reduce the number of crossings for the 11-point subset to at most 104. Thus to achieve a data base of all order types of size 13 with 229 (or fewer) crossings, one can start with a complete data base of order types of size 11 defining drawings of K_{11} with at most 104 crossings, i.e., either 102 or 104 crossings. Table 3 shows the number of inequivalent drawings of K_{11} having a fixed number of crossings. We can conclude that for our example, instead of taking the complete data base of 2 334 512 907 order types into account, we only need to consider 374 order types of size 11 with 102 crossings and 3 984 order types with 104 crossings, less than 0.0002%.

number of crossings	102	104	106	108	110	112
number of inequivalent sets	374	3 984	17 896	47 471	102 925	228 497

Table 3: Number of inequivalent drawings of K_{11} obtaining a specific, small crossing number.

4.2 Extension of $\overline{cr}(K_n)$ for $n \geq 12$

For $n = 12, ..., 20$ Table 4 gives the bounds that result from Corollary 3 and Lemma 4 in the calculation of the rectilinear crossing number $\overline{cr}(K_n)$ by abstract order type extension. For each n the entry in column n and row n_b gives the number of crossings of the best (i.e., minimizing) drawings found so far. Thus the values 153, 229, 324, 447, 603, 798, 1029, 1318, and 1657 are conjectured to represent $\overline{cr}(K_n)$, $n = 12, ..., 20$. With our approach we want to prove or disprove these conjectures.

n	11	12	13	14	15	16	17	18	19	20
12_a	≤ 100	≤ 152								
12_b	≤ 102	≤ 153								
13_a	≤ 104	≤ 157	≤ 227							
13_b		≤ 158	≤ 229							
14_a				≤ 323						
14_b	≤ 106	≤ 159	≤ 231	≤ 324						
15_a				≤ 326	≤ 445					
15_b		≤ 161	≤ 233	≤ 327	≤ 447					
16_a	≤ 108	≤ 162	≤ 235	≤ 330	≤ 451	≤ 602				
16_b						≤ 603				
17_a		≤ 164	≤ 237	≤ 333	≤ 455	≤ 608	≤ 796			
17_b	≤ 110	≤ 165	≤ 239	≤ 335	≤ 457	≤ 610	≤ 798			
18_a								≤ 1028		
18_b						≤ 611	≤ 800	≤ 1029		
19_a		≤ 166	≤ 241	≤ 338	≤ 461	≤ 616	≤ 806	≤ 1038	≤ 1316	
19_b						≤ 617	≤ 808	≤ 1040	≤ 1318	
20_a	≤ 112	≤ 168	≤ 243	≤ 341	≤ 465	≤ 620	≤ 812	≤ 1045	≤ 1324	≤ 1656
20_b										≤ 1657

Table 4: Bounds on the number of crossings in subset drawings. For each $n = 12, ..., 20$ there are two rows: row n_a gives the bounds to determine $\overline{cr}(K_n)$ and row n_b contains the bounds to compute all inequivalent drawings obtaining this number. Empty entries indicate that the same bound as in the previous line holds.

The bounds on the number of crossings in subset drawings are displayed in each line from right to left, i.e., an entry of $\leq c_n$ in column n adjacent to an entry of $\leq c_{n-1}$ for column $n-1$ means that any drawing of K_n with c_n (or fewer) crossing has a subgraph K_{n-1} with c_{n-1} (or fewer) crossings. Note that empty entries in Table 4 indicate that the same bound as in the previous line holds. A calculation corresponding to row n_a yields the exact value of the corresponding rectilinear crossing number, i.e., it proves the best known drawing to be optimal in the case that there is no result. If our method verifies the best known drawing to be optimal, then using row n_b gives the number of inequivalent, optimal (abstract) drawings in addition. Note that row 13_b reflects the example from the end of the previous section.

Some interesting relations can be seen from Table 4. For example determining $\overline{cr}(K_{14})$ is significantly simpler than computing all inequivalent drawings obtaining $\overline{cr}(K_{14})$ as the later task also requires all sets of K_{11} with 106 drawings to be considered (row $14b$). On the other hand, if all inequivalent drawings obtaining $\overline{cr}(K_{17}) = 798$ have been computed, we can determine $\overline{cr}(K_{18})$ by just extending these sets, cf. Section 6.

4.3 Results on $\overline{cr}(K_n)$ for $n \geq 12$

Using the crossing number subset property we define an order type extension graph with a unique predecessor relationship (father-son) as suggested in Section 3. We were able to calculate the rectilinear crossing numbers for $n = 12, ..., 17$, see Table 5. For $n \geq 18$ the lower bounds are obtained from $\overline{cr}(K_{17}) = 798$ and by Lemma 2.

n	12	13	14	15	16	17	18	19	20
$\overline{cr}(K_n)$	153	229	324	447	603	798	≥ 1026 ≤ 1029	≥ 1300 ≤ 1318	≥ 1625 ≤ 1657
d_n	1	4 534	20	16 001	36	≥ 37269	≥ 1	≥ 13243	≥ 6

Table 5: Rectilinear crossing number $\overline{cr}(K_n)$ for $n = 12, ..., 17$ and the number d_n of inequivalent drawings of K_n obtaining $\overline{cr}(K_n)$. For $n = 18, ..., 20$ the best bounds known so far are given.

The numbers d_n of inequivalent drawings of K_n minimizing the number of crossings are given in the last row of Table 5. To obtain these numbers we had to perform the more challenging task of deciding the realizability of the calculated abstract order types. Our heuristics - see Section 2 - found realizing point coordinates for all optimal abstract drawings for $n \leq 16$. Thus, the calculated values are exact. Note that the numbers of inequivalent optimal drawings of K_n follow a parity pattern. There are relatively few drawings of K_n with $\overline{cr}(K_n)$ crossings for even n compared to the case of odd n. This property is the main reason that allows complete abstract extension to work so well, as the problem itself cuts down on the number of interesting sets periodically.

Let us explain some details of our enumerative investigations. The computations to determine $\overline{cr}(K_n)$ and d_n for $n \leq 16$ using row 16_b of Table 4 have taken 58 days on a 2 GHz PC under Linux. The most difficult sets among the data base with 11 points have been the sets with 102 crossings. The intuitive reason is that starting with these sets iteratively produces the highest number of extended order types which fit into the given bounds for their crossing number. The average computation time for such sets was approximately 3.7 hours, though there was one set requiring over 56 hours, producing 11 optimal drawings for K_{16} (and 3166 optimal drawings for K_{15}). On the other hand the 'fastest' 11-sets with 102 crossings took only 16 seconds, not producing any optimal drawing.

An interesting observation is that for $n = 12, ..., 16$ all optimal drawings of K_n contain an optimal sub-drawing of K_{11}. This has to be seen in spite of the fact that there exist optimal drawings of K_n which do not contain an optimal drawing of K_{n-1}. In other words, it would have been sufficient to only start with 11-sets with 102 crossings to obtain all optimal drawings - but we would not have known for sure that we did not miss some sets.

To determine $\overline{cr}(K_{17})$ in reasonable time we had to distribute our computations to several different machines. In fact we already did this for the computations for $\overline{cr}(K_{16})$ and could thus complete this task within a few days. The ability of distributing the computations is another important advantage of our approach: the enumeration of sets from the data base can be done entirely independently. There is no specific order in which the sets have to be investigated, nor is there any need to know about the status of other sets. That means in principle we can load every set to its own machine without any communication overhead. Using row 17_a of Table 4 we found within a few months that there is no drawing of K_{17} with

796 or fewer crossings, and thus we have $\overline{cr}(K_{17}) = 798$. Although we found more than 37 000 inequivalent drawings of K_{17} obtaining 798 crossings, we expect d_{17} to be over 100 000.

To determine $\overline{cr}(K_n)$ for $n \leq 20$ we have to refer to row 20_a of Table 4. Starting with sets of 11 points and 108, 110, and 112 crossings, respectively, the computations showed that no sets with more than 14 vertices are generated within the given bounds. Starting with all 11-sets with 106 crossings the largest generated sets have 18 points and 1043, 1044 or 1045 crossings, but no 19-point sets are generated. No optimal drawings have been produced by these computations. This means that for further investigations for $n \leq 20$ only sets of 11 points with either 102 or 104 crossings need to be considered. We already partially performed the computations for the sets with 104 crossings (about 3760 of 3984 sets, that is, over 94%), and the results so far coincide with the results for sets with larger crossing numbers. So we conjecture that all optimal drawings for $12 \leq n \leq 20$ have optimal sub-drawings of K_{11}. This would imply that for further investigations the 374 sets with 11 points and 102 crossings are a sufficient starting base.

In addition to new results on $\overline{cr}(K_n)$ for constant n, we also achieved an improvement on the asymptotic upper bound. We constructed a set of 54 points with 115 999 crossings such that with the strategy of lens replacement [4] we were able to prove the next theorem. The previously best known upper bound was $\overline{\nu}^* < 0.38074$.

Theorem 5. $\overline{\nu}^* = \lim_{n\to\infty} \overline{cr}(K_n)/\binom{n}{4} < 0.38058$

For possible further improvement, we constructed an abstract order type of 96 points. If this abstract order type is realizable, we will obtain a drawing of K_{96} with 1 238 505 crossings. This would prove an upper bound of $\overline{\nu}^* < 0.38047$.

In [28] the first lower bound breaking the barrier of $\overline{\nu}^* > 3/8$ has been established. This has to be seen in contrast to the general crossing number $cr(K_n)$, that is, the minimal number of crossings attained by a drawing of K_n in the plane with general Jordan arcs as edges. It is known that K_n can be drawn with at most $3/8\binom{n}{4} + O(n^3)$ crossings, cf. [24]. Consequently, there is a significant asymptotic difference between the rectilinear crossing number $\overline{cr}(K_n)$ and its counterpart $cr(K_n)$. The currently best known lower bound for the rectilinear case is $\overline{\nu}^* > 0.37533$ [11].

5 Further Applications

Among many other applications of our approach which come to mind, let us briefly mention three further examples which we have already partially investigated.

Counting Triangulations

Counting the number of triangulations of a set of points in the plane is another interesting geometric problem. Exact numbers, using our data base, are known for all sets with $n \leq 11$ points. The best asymptotic lower bound of $\Omega(2.33^n)$ for this problem is based on these results for small sets [6]. In fact, any improvement on the number of triangulations for sets of small, constant cardinality will affect the asymptotic bound.

Thus, to improve the general asymptotic lower bound, it will be useful to obtain a tight lower bound for sets of size $n = 12, 13, \ldots$. Currently the best minimizing example is the so-called double circle [6], which has $\sqrt{12}^{\,n-\Theta(\log(n))}$ triangulations. For $n = 12$ the double circle gives 2 236 triangulations. To prove that this value is minimal we have to consider all sets of 12 points which might contain less triangulations. As a subset property for this task we can use the fact that adding an interior point to a given set increases the number of triangulations by at least some constant factor c. Making use of specific properties of our extension method, the currently best known factor is $c = 2$ [33]. For a point set S this factor can be obtained if the additional point p lies on the second convex hull layer of S. That is, if S' is the set of points we obtain when removing the extreme points of S, then p is an extreme point of S'. Thus, to extend a given point set we require p to be the last interior point in the clockwise sorting around the point p_1 of S. Note that this fits perfectly in the framework of our order type representation and extension, cf. Sections 2 and 3. An additional requirement to obtain the factor 2 is that the cardinality of S' is at least 2. Therefore, we also have to consider all sets where p is the only interior point. As the number of such sets is rather small, we ignore them in the further discussion.

	min. number of	upper bound	Number of start sets	
n	triangulations	for $n = 11$	for $n = 11$	%
12	2 236	1 118	2 351	0.0001
13	7 147	1 786	845 829	0.0362
14	20 979	2 622	25 877 327	1.1085
15	68 448	4 278	436 539 440	18.6994

Table 6: Computing a lower bound for the number of triangulations: Number for currently minimizing examples (double circle), upper bounds for starting base for $n = 11$ and number (percentage) of elements in starting base to be considered.

The number of triangulations has already been computed for all sets of the data base for $n = 11$, determining a minimum of 776 triangulations for the double circle. As shown in Table 6 we thus know how many sets we will have to extend for a given maximum number of triangulations.

The factor $c = 2$ implies that to determine the tight lower bound for $n = 12$ we need to start with the 2 351 sets with 11 points achieving at most 1 118 triangulations each. To be precise it would be sufficient to start with all sets for $n = 11$ with at most 1 117 triangulations. But including the ones with 1 118 triangulations allows us to check whether the double circle is the unique minimal example for $n = 12$. Even for $n = 13$ this task seems to be reachable, as it is sufficient to consider 845 829 sets with at most 1 786 triangulations, less than 0.04% of the entire data base of 11 points. We expect to get these results in the near future and maybe to attack the case $n = 14$, too.

Let us finally mention that the discussed approach requires an efficient algorithm to count triangulations within an abstract framework, that is, without knowing the coordinates of the points. None of the currently available methods presented in [1] and [31] can directly be used for this task. So at the moment we are developing a modified version of [31], combining it with our extension algorithms.

Happy End Problem

Erdös and Szekeres asked in 1935 for the smallest number $g(k)$, such that each point set in the plane with at least $g(k)$ points contains a convex k-gon [17]. For $k > 5$ this problem is still unsolved, where it is known that $g(6) \leq 36$. The conjecture is that the true value for $g(6)$

is 17. To answer this conjecture our plan is to apply our abstract extension technique in order to obtain all sets without convex hexagons for $n \leq 17$. If we cannot find a set for $n = 17$ this will prove the conjecture to be true. The subset property for this problem is obvious: any $n - 1$ point subset of a set of n points to be considered must not contain a convex hexagon.

We already started the enumerations and achieved some first results: Starting with the 235 987 328 sets with 11 points without convex hexagons, we got 14 048 972 314 (abstract) drawings of 12 points without a convex hexagon. For $n = 13$ we expect this number to be approx. 800 10^9, see [14] for details. Though to our knowledge this approach is currently the only one with reasonable hope to finally settle the conjecture, the obtained numbers show that we will need further improvements in order to decide whether $g(6) = 17$ or not.

For arbitrary $k \geq 5$, Tóth and Valtr give the best known upper bound $g(k) \leq \binom{2k-5}{k-2} + 1$ [35], whereas the lower bound $g(k) \geq 2^{k-2} + 1$ from Erdös and Szekeres is conjectured to be tight [17].

Decomposition

Similar to the convex-decomposition problem of decomposing a point set into convex polygons one might allow the resulting faces to be either convex polygons or pseudo-triangles [5]. When investigating this problem it turned out to be important to know optimal decompositions of small sets. In this context we asked for independent (disjoint) empty convex polygons spanned by the set. Let us briefly mention two results we got from the data base, see [5] for details. First: Any set of 8 points contains either an empty convex pentagon or two disjoint empty convex quadrilaterals. And with a similar flavor: Any set of 11 points contains either an empty convex hexagon or an independent empty convex pentagon and an empty convex quadrilateral. We hope to extend these results for sets of size, say, $n = 13$ with our extension approach. The mentioned results directly lead to an upper bound of $7n/10$ for the number of convex or pseudotriangular faces used to decompose a set of n points [5].

6 Further Discussion and Open Problems

The next steps of our investigation will be to compute all inequivalent, optimal sets for $n = 17$, that is to determine d_{17}. From a computational point of view and by Table 4 this problem is more or less equivalent to compute $\overline{cr}(K_{18})$. The possible range for $\overline{cr}(K_{18})$ is $\{1026, 1027, 1028, 1029\}$, where our conjecture is $\overline{cr}(K_{18}) = 1029$. Using heavy distributed computing we consider this task to be realistic in the near future, see [37].

It is known that there exist optimal drawings of K_n that contain no optimal sub-drawing K_{n-1}. The smallest example is a drawing of K_9, cf. [27]. An interesting open problem is whether there always exists at least one optimal drawing of K_n which contains an optimal sub-drawing of K_{n-1}. A potential counter-example is $n = 18$, as all 17-point subsets of the only known drawing of K_{18} with 1029 crossings (depicted in Figure 3) determine more than $\overline{cr}(K_{17}) = 798$ crossings.

A k-set of a point set S is a subset of k points that is the intersection of S with an open half plane, see [9, 28] for interesting relations between k-sets and crossing numbers. Jensen [26] conjectured that any optimal drawing of $\overline{cr}(K_n)$ yields the same distribution of k-sets. For $n \leq 16$ we have been able to verify this conjecture, cf. Table 7. Moreover it turned out that this distributions follow some interesting patterns. So the hope is that our

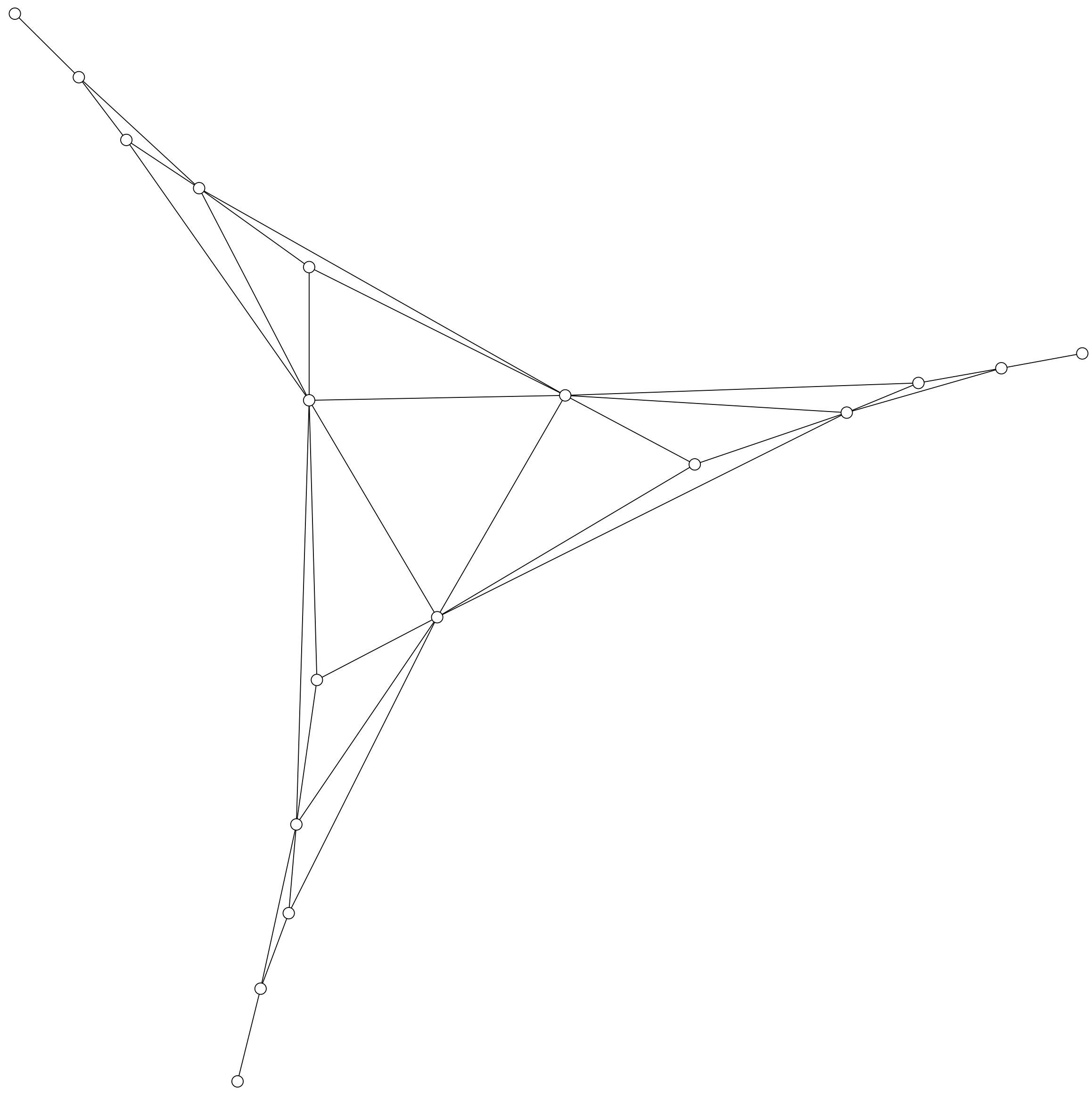

Figure 3: Set with $n = 18$ points, realizing a rectilinear drawing of K_{18} with 1029 crossings. This set is conjectured to be the unique minimal example. The drawn line segments are the halving segments of the point set. Coordinates are available from [2].

enumerative investigations will also lead to deeper theoretical insight into the relations of different combinatorial structures.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	3																
4	3	3															
5	3	7															
6	3	6	6														
7	3	6	12														
8	3	6	10	9													
9	3	6	9	18													
10	3	6	9	14	13												
11	3	6	9	13	24												
12	3	6	9	12	18	18											
13	3	6	9	12	17	31											
14	3	6	9	12	16	23	22										
15	3	6	9	12	15	21	39										
16	3	6	9	12	15	20	28	27									
17	3	6	9	12	15	19	25	47									
18	3	6	9	12	15	18	24	33	33								
19	3	6	9	12	15	18	23	29	56								
20	3	6	9	12	15	18	22	28	39	38							
21	3	6	9	12	15	18	21	27	33	66							
22	3	6	9	12	15	18	21	26	32	45	44						
23	3	6	9	12	15	18	21	25	31	38	75						
24	3	6	9	12	15	18	21	24	30	36	51	51					
25	3	6	9	12	15	18	21	24	29	35	43	85					
26	3	6	9	12	15	18	21	24	28	34	40	58	57				
27	3	6	9	12	15	18	21	24	27	33	39	48	96				
28	3	6	9	12	15	18	21	24	27	32	38	44	66	63			
29	3	6	9	12	15	18	21	24	27	31	37	43	55	105			
30	3	6	9	12	15	18	21	24	27	30	36	42	48	75	69		
31	3	6	9	12	15	18	21	24	27	30	35	41	47	62	115		
32	3	6	9	12	15	18	21	24	27	30	34	40	46	53	85	73	
33	3	6	9	12	15	18	21	24	27	30	33	39	45	51	69	126	
34	3	6	9	12	15	18	21	24	27	30	33	38	44	50	59	92	80
35	3	6	9	12	15	18	21	24	27	30	33	37	43	49	55	77	136

Table 7: The number of k-sets for drawings minimizing the rectilinear crossing number for K_n, $n \leq 17$. The values for $18 \leq n \leq 25$ reflect the number of k-sets for the best known drawings for K_n.

Very recently, it was proven that any optimal drawing of K_n has exactly three extreme vertices [9]. Another conjecture we plan to investigate, is that that there always exists at least one drawing which has a three-fold rotational symmetry if n is a multiple of 3 [26], cf. also Figure 3.

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