On 5-gons and 5-holes

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Abstract. We consider an extention of a question of Erdős on the number of k-gons in a set of n points in the plane. Relaxing the convexity restriction we obtain results on 5-gons and 5-holes (empty 5-gons).

Introduction

Let S be a set of n points in general position in the plane. A k-gon is a simple polygon spanned by k points of S. A k-hole is an empty k-gon, that is, a k-gon which does not contain any points of S in its interior.

Erdős [**9**] raised the following questions for convex k-holes and k-gons. "What is the smallest integer $h(k)$ ($g(k)$) such that any set of $h(k)$ ($g(k)$) points in the plane contains at least one convex k-hole (k-gon)?"; and more general "What is the least number $h_k(n)$ ($g_k(n)$) of convex k-holes (k-gons) determined by any set of n points in the plane?".

As already observed by Esther Klein, every set of 5 points determines a convex 4-hole (and thus 4-gon). Moreover, 9 points always contain a convex 5-gon and 10 points always contain a convex 5-hole, a fact proved by Harborth [**12**]. Only in 2007/08 Nicolás [**14**] and independently Gerken [**11**] proved that every sufficiently large point set contains a convex 6-hole, and it is well known that there exist arbitrarily large sets of points not containing any convex 7-hole [**13**]; see [**2**] for a brief survey.

In this paper we concentrate on 5-gons and 5-holes and generalize the above questions by allowing a 5-gon/5-hole to be non-convex. Thus, when referring to a 5-gon/5-hole, it might be convex or non-convex and we will explicitly state it when we restrict considerations to one of these two classes. Similar results for 4-holes can be found in [**3**]. For 4-gons there is a one-to-one relation to the rectilinear crossing number of the complete graph, and thus results can be found in the respective literature.

A set of five points in convex position obviously spans precisely one convex 5-gon. In contrast, already a set of only five points (with three extremal points) can span eight different 5-gons; see Figure 1(left). This makes the considered questions more challenging (and interesting) than they might appear on a first glance.

Due to space limitations all proofs are omitted in this extended abstract.

1 Small sets

For small point sets, Table 1 shows the numbers of 5-gons and 5-holes, respectively. Given are the minimum number of convex 5-gons/5-holes, the maximum number of non-convex 5-gons/5-holes, the minimum and maximum number of (general) 5-gons/5-holes, and, for easy comparison, the number of 5-tuples.

For counting convex 5-gons/5-holes it is easy to see that their number is maximized by sets in convex position and gives $\binom{n}{5}$. Of course these sets do not contain any non-convex 5-gons/5-holes. From Table 1 we also see that the minimum number of general

	numbers of 5-gons				numbers of 5-holes				
n	convex min	non-convex max	general min	general max	convex min	non-convex max	general min	general max	$\binom{n}{5}$
5	0	8	1	8	0	8	1	8	1
6	0	48	6	48	0	31	6	31	6
7	0	156	21	157	0	76	21	77	21
8	0	408	56	410	0	157	56	160	56
9	1	900	126	909	0	288	126	292	126
10	2	1776	252	1790	1	492	252	501	252
11	7	3192	462	3228	2	779	462	802	462

TABLE 1. The number of 5-gons and 5-holes for $n = 5...11$ points.

5-gons and 5-holes is $\binom{n}{5}$ for $5 \leq n \leq 11$. While for 5-gons this is obviously true in gerneral (a convex 5-tuple has exactly one polygonization, while a non-convex 5-tuple has at least four), this is not the case for 5-holes. In fact, we will show that for sufficiently large n, the convex set maximizes the number of 5-holes; see Theorem 3.6.

2 5-gons

The rectilinear crossing number $\bar{cr}(S)$ of a set S of n points in the plane is the number of proper intersections in the drawing of the complete straight line graph on S. It is easy to see that the number of convex 4-gons is equal to $\bar{cr}(S)$ and is thus minimized by sets minimizing the rectilinear crossing number, a well known, difficult problem in discrete geometry; see [**7**] and [**10**] for details. Tight values for the minimum number of convex 4-gons are known for $n \leq 27$ points; see e.g. [**1**]. Asymptotically we have at least $c_4\binom{n}{4} = \Theta(n^4)$ convex 4-gons, where c_4 is a constant in the range $0.379972 \leq c_4 \leq 0.380488$. As any 4 points in non-convex position span three non-convex 4-gons, we get $3\binom{n}{4} - 3\bar{cr}(S)$ non-convex and $3\binom{n}{4} - 2\bar{cr}(S)$ general 4-gons for a set S. Thus, sets which minimize the rectilinear crossing number also minimize the number of convex 4-gons, and maximize both the number of non-convex 4-gons and the number of general 4-gons.

Surprisingly, a similar relation can be obtained for the number of non-convex 5-gons. To see this, consider the three combinatorial different possibilities (order types) of arranging 5 points in the plane, as depicted in Figure 1(right). The proof of the following theorem is based on relations between the number of 5-gons and the numbers of crossings of these configurations.

Theorem 2.1 *Let S be a set of $n \geq 5$ points in the plane in general position. Then S contains $10\binom{n}{5} - 2(n-4)\bar{cr}(S)$ non-convex 5-gons.*

Taking the constant c_4 for the rectilinear crossing number into account, we see that asymptotically we can have up to $10\binom{n}{5} - 2(n-4)c_4\binom{n}{4} = 10(1-c_4)\binom{n}{5}$ non-convex 5-gons. This number is obtained for point sets minimizing the rectilinear crossing number and by a factor $\approx 6,2$ larger than the maximum number of convex 5-gons.

For the number of convex 5-gons, no simple relation to the rectilinear crossing number is possible: There exist two different sets (order types) S_1 and S_2, both of cardinality 6 with 4 extremal points, with $\bar{cr}(S_1) = \bar{cr}(S_2) = 8$, where S_1 contains one convex 5-gon, while S_2 does not contain any convex 5-gon.

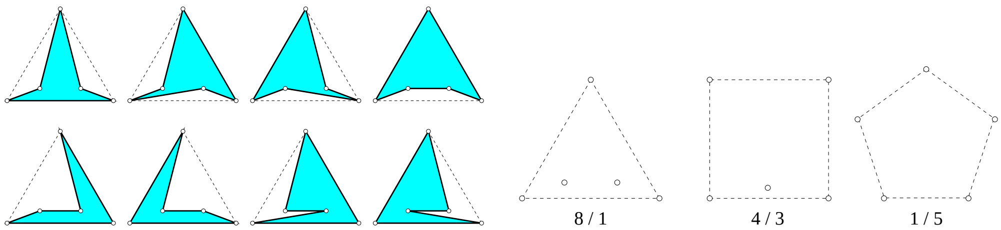

FIGURE 1. Left: The eight different (non-convex) 5-gons spanned by a set of five points with three extremal points. Right: The three order types for $n = 5$. For each set its number of different 5-gons and the number of crossings for the complete graph is shown.

3 5-holes

3.1 A new lower bound for the number of convex 5-holes

Let $h_5(S)$ denote the number of convex 5-holes of a point set S, and let $h_5(n) = \min_{|S|=n} h_5(S)$ be the number of convex 5-holes any point set of cardinality n has to have. The best upper bound $h_5(n) \leq 1.0207n^2 + o(n^2)$ can be found in [**6**]. The previous best lower bound $h_5(n) \geq \lfloor \frac{n-4}{6} \rfloor$ has been obtained by Bárány and Károlyi [**5**].

Here we give a slight improvement on this bound, which still remains linear in n. It is based on an observation by Dehnhardt [**8**] that every set of 12 points contains at least three convex 5-holes.

Theorem 3.1 *Let S be a set of $n \geq 12$ points in the plane in general position. Then $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$*

3.2 A lower bound for the number of (general) 5-holes

We obtained the following observation for general 5-holes by checking all 14 309 547 according point sets from the order type data base [**4**].

Observation 3.2 *Let S be a set of $n = 10$ points in the plane in general position, and $p_1, p_2 \in S$ two arbitrary points of S. Then S contains at least 34 5-holes having p_1 and p_2 among their vertices.*

This observation implies the following result, using a similar approach as in [**3**].

Theorem 3.3 *Let S be a set of $n \geq 10$ points in the plane in general position. Then S contains at least $17n^2 - O(n)$ 5-holes.*

3.3 Maximizing the number of (general) 5-holes

The results for small sets shown in Table 1 suggest that the number of (general) 5-holes is minimized by sets in convex position. We not only show that this is in fact not the case, but rather prove the contrary: For sufficiently large n, sets in convex position maximize the number of 5-holes.

Lemma 3.4 *A point set S with triangular convex hull and i interior points contains at most $(4i+5)$ 5-holes which have the three extreme points among their vertices.*

Lemma 3.5 *Let Γ be a non-empty convex quadrilateral in S. There are at most four 5-holes spanned by the four vertices of Γ plus a point of S in the interior of Γ.*

Considering the size of the convex hull of each 5-tuple, these two lemmas lead to the following theorem.

Theorem 3.6 *For $n \geq 86$ the number of 5-holes is maximized by a set of n points in convex position.*

4 Conclusion

In this abstract we presented several results for a variant of a classic Erdős-Szekeres type problem for the case of 5-gons and 5-holes. The following questions remain open: What is the maximum number of general 5-gons and of non-convex 5-holes? Is there a super-linear lower bound for the number of convex 5-holes (cf. Theorem 3.1) or a super-quadratic lower bound for the number of general 5-holes (cf. Theorem 3.3)?

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References

[1] O. Aichholzer. On the rectilinear crossing number. *http://www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/*.

[2] O. Aichholzer. [Empty] [colored] k-gons - Recent results on some Erdős-Szekeres type problems. In *Proc. XIII Encuentros de Geometría Computacional*, pages 43–52, Zaragoza, Spain, 2009.

[3] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. In *Proc. 27th European Workshop on Computational Geometry EuroCG'11*, Morschach, Switzerland, 2011. To appear.

[4] O. Aichholzer and H. Krasser. The point set order type data base: A collection of applications and results. In *Proc. 13th Canadian Conference on Computational Geometry CCCG'01*, volume 13, pages 17–20, 2001.

[5] I. Bárány and G. Károlyi. Problems and results around the erdős–szekeres convex polygon theorem. *In* Discrete and Combinatorial Geometry, LNCS 2098:91–105, 2001.

[6] I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons. *Studia Scientiarum Mathematicarum Hungarica*, 41(2):243–269, 2004.

[7] P. Brass, W. Moser, and J. Pach. Research problems in discrete geometry. *Springer*, 2005.

[8] K. Dehnhardt. *Leere konvexe Vielecke in ebenen Punktmengen.* PhD thesis, TU Braunschweig, Germany, 1987.

[9] P. Erdős. Some more problems on elementary geometry. *Austral. Math. Soc. Gaz.*, 5:52–54, 1978.

[10] P. Erdős and R. Guy. Crossing number problems. *Amer. Math. Monthly*, 88:52–58, 1973.

[11] T. Gerken. Empty convex hexagons in planar point sets. *Discrete and Computational Geometry*, 39(1-3):239–272, 2008.

[12] H. Harborth. Konvexe fünfecke in ebenen punktmengen. *Elemente Math.*, 33:116–118, 1978.

[13] J. Horton. Sets with no empty convex 7-gons. *Canad. Math. Bull.*, 26(4):482–484, 1983.

[14] C. Nicolás. The empty hexagon theorem. *Discrete and Computational Geometry*, 38(2):389–397, 2007.