Compatible Matchings for Bichromatic Plane Straight-line Graphs

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Abstract

Two plane graphs with the same vertex set are compatible if their union is again a plane graph. We consider bichromatic plane straight-line graphs with vertex set S consisting of the same number of red and blue points, and (perfect) matchings which are compatible to them. For several different classes $\mathcal{C}$ of graphs, we present lower and upper bounds such that any given graph $G(S) \in \mathcal{C}$ admits a compatible (perfect) matching with this many disjoint edges.

1 Introduction

We consider bichromatic point sets $S = R \cup B$ where the red set R and the blue set B have the same cardinality n. An edge spanned by two points of S is called *bichromatic*, if it has one red and one blue endpoint. A graph $G(S)$ is called bichromatic, if all its edges are bichromatic. Accordingly, an edge where both endpoints have the same color is called *monochromatic*, and a graph is called monochromatic if all its vertices have the same color. Two plane graphs with the same vertex set S are called *compatible* if their union is again a plane graph and *disjoint* if their intersection does not contain any edge. In the following, we solely consider plane straight-line graphs, and refer to them just as graphs for the sake of brevity.

There exist several results on compatible graphs, for bichromatic as well as for uncolored point sets. For example, Ishaque et al. [5] recently showed that any (uncolored) geometric matching with an even number of edges admits a disjoint compatible matching. In a similar direction, Abellanas et al. [2] showed upper and lower bounds for how many edges a compatible matching for a graph of a certain class can admit.

In a different work, Abellanas et al. [1] showed how many edges are needed at least to augment an (uncolored) connected graph to a 2-vertex or 2-edge connected graph. According results on bichromatic graphs have been obtained by Hurtado et al. [4], who also considered the question of augmenting a (disconnected) bichromatic graph to be connected. Among others, they showed how to connect a bichromatic perfect matching to a (bichromatic) spanning tree. Hoffmann and Tóth [3] extended this work to spanning trees with maximum vertex degree three.

In this work we investigate the following question. Given a bichromatic graph $G(S)$, we want to find a matching $M(S)$ (of some type) that is compatible with $G(S)$. Following the lines of [2], we call such a matching $G(S)$-compatible. Similarly, if $M(S)$ is disjoint from $G(S)$, we also say that it is $G(S)$-disjoint. We consider two classes of $G(S)$-compatible matchings: (1) bichromatic $G(S)$-disjoint / perfect matchings (Section 2) and (2) monochromatic matchings (Section 3).

For a $G(S)$-compatible matching $M(S)$, we denote the number of edges in $M(S)$ that are disjoint from $G(S)$ by $d(G(S), M(S))$. Similar to the work in [2], we focus on bounds for this number for the considered classes of matchings and the graph classes of spanning trees (*tree*), spanning paths (*path*), spanning cycles (*cycle*), and perfect matchings (*match*).

A preliminary version of this work can be found in [7], Section 3.3.

2 Bichromatic matchings

We start with bichromatic (perfect) matchings which are compatible to a given bichromatic graph. To simplify reading, we mostly omit the attribute bichromatic in this section.

It is well known that every set S with $|R| = |B|$ admits a bichromatic perfect matching [6]. On the other hand, there exists a large class of point sets with $|R| = |B|$, for which there exists only one such matching $M(S)$. Note that for any plane graph $G(S)$ that is obtained by adding edges to $M(S)$, we get $d(G(S), M(S)) = 0$. Due to these observations, and to avoid trivial bounds, we restrict considerations in this section to point sets admitting strictly more than one bichromatic perfect matching.

Let $\mathcal{S}$ be the class of point sets admitting at least two different bichromatic perfect matchings, and let $\mathcal{S}_n \subset \mathcal{S}$ be the sets with $|R| = |B| = n$. For a given class $\mathcal{C}$ of graphs, we denote by

$$b_{\mathcal{C}}(n) = \min_{S \in \mathcal{S}_n} \min_{G(S) \in \mathcal{C}} \max_{M(S)} d(G(S), M(S))$$

the maximum number such that for every point set $S \in \mathcal{S}_n$ and every graph $G(S) \in \mathcal{C}$, we can find a *bichromatic disjoint compatible matching* $M(S)$ of cardinality at least $b_{\mathcal{C}}(n)$. Accordingly, we denote by

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$$b^p_{\mathcal{C}}(n) = \min_{S \in \mathcal{S}_n} \quad \min_{G(S) \in \mathcal{C}} \quad \max_{M(S)} \quad d(G(S), M(S))$$

the maximum number such that for every point set $S \in \mathcal{S}_n$ and every graph $G(S) \in \mathcal{C}$, there exists a *bichromatic compatible perfect matching* $M(S)$ with (at least) $b^p_{\mathcal{C}}(n)$ edges disjoint from $G(S)$.

Note that $b_{\mathcal{C}}(n) \geq b^p_{\mathcal{C}}(n)$, as any compatible perfect matching $M(S)$ for a given graph $G(S)$ contains a $G(S)$-disjoint matching $M'(S)$ of size $d(G(S), M'(S)) = d(G(S), M(S))$.

We start with the class of perfect matchings, and with disjoint compatible matchings for them.

Theorem 1 $\lceil \frac{n-1}{2} \rceil \leq b_{match}(n) \leq \frac{3n}{4}$.

Proof. Consider a perfect matching $PM(S)$. Using the result of Hoffman and Tóth [3], we augment $PM(S)$ to a bichromatic spanning tree $T(S)$ with maximum vertex degree three; see Figure 1.

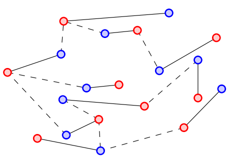

Figure 1: A perfect matching (solid edges) augmented to a spanning tree (augmenting edges are dashed).

Now consider the graph $A(S) = T(S) \backslash PM(S)$ of the augmenting edges. $A(S)$ contains exactly $n - 1$ edges. Further, as every vertex is incident to exactly one edge of $PM(S)$, the maximum vertex degree in $A(S)$ is at most two. In other words, $A(S)$ is a collection of paths $\mathcal{P}$ and isolated vertices. Every path P with $k_P + 1$ vertices has k_P edges, of which $\lceil \frac{k_P}{2} \rceil$ form a matching. Thus, $A(S)$ contains a matching with $\sum_{P \in \mathcal{P}} \lceil \frac{k_P}{2} \rceil \geq \lceil \frac{n-1}{2} \rceil$ edges, yielding a lower bound of $b_{match}(n) \geq \lceil \frac{n-1}{2} \rceil$ for the number of edges in a maximum $PM(S)$-disjoint compatible matching.

For an upper bound on $b_{match}(n)$ consider the perfect matching $PM(S)$ shown in Figure 2. Every red vertex that is incident to one of the small edges inside a triangle does not "see" any blue vertex except for the one it is matched to. Thus, in any $PM(S)$-disjoint compatible matching $M(S)$ all these red vertices must stay unmatched, implying an upper bound of $b_{match}(n) \leq d(PM(S), M(S)) \leq \frac{3n}{4}$. □

The upper bound for disjoint matchings (induced by Figure 2) directly implies an according upper bound of $b^p_{match}(n) \leq \frac{3n}{4}$ for perfect matchings that are compatible to a given perfect matching. But for this case we can say even more.

Theorem 2 $b^p_{match}(n) \leq \frac{n}{2}$.

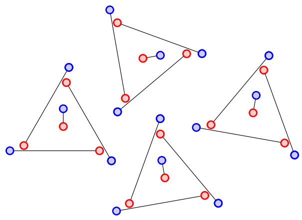

Figure 2: An upper bound example for $b_{match}(n)$.

Proof. Consider the perfect matching $PM(S)$ illustrated in Figure 3. Every vertex that is incident to one of the short edges can only be compatibly matched in two ways; either to the other vertex of the edge it is incident to, or to the accordingly colored vertex of the long edge next to it. As matching one vertex of a short edge to the according vertex of the long edge next to it would force the other vertex of the short edge to stay unmatched, any perfect matching $M(S)$ that is compatible with $PM(S)$ must contain all short edges of $PM(S)$. □

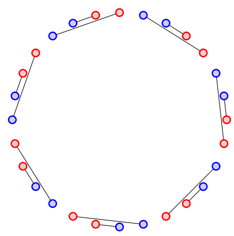

Figure 3: An upper bound example for $b^p_{match}(n)$.

For the classes of spanning trees, spanning cycles, and spanning paths, the examples illustrated in Figure 4 imply bounds of $b_{tree}(n) = b_{cycle}(n) = 0$ and $b_{path}(n) \leq 1$.

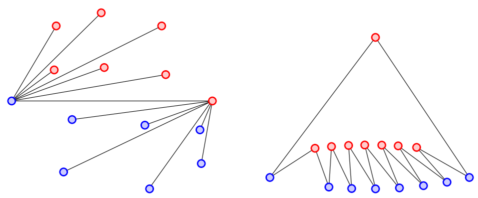

Figure 4: Upper bound examples for $b_{tree}(n)$ and $b_{cycle}(n)$.

3 Monochromatic matchings

We continue with monochromatic compatible matchings for bichromatic graphs. In the following, we denote by

$$m_{\mathcal{C}}(n) = \min_{|S|=2n} \quad \min_{G(S) \in \mathcal{C}} \quad \max_{M(S)} \quad d(G(S), M(S))$$

the maximum number such that for every graph $G(S) \in \mathcal{C}$ there exists a *monochromatic compatible matching* $M(S)$ with (at least) $m_{\mathcal{C}}(n)$ edges.

We again start with the class of bichromatic perfect matchings, and with monochromatic compatible matchings for them.

Theorem 3 $\frac{n}{4} \leq m_{match}(n) \leq \frac{5n}{12}$.

Proof. Consider a perfect matching $PM(S)$. Assume w.l.o.g. that $PM(S)$ does not contain any vertical edge, and that for at least half of the edges, the left vertex is red. We augment the matching to a weakly simple polygon in the following way. First, we add a bounding box around the $PM(S)$. Next, we extend all edges with a left red vertex to the right until it hits the bounding box or (an extension of) an edge. Then we extend all other edges to the right as well, but with a slight turn. The result is a so-called weakly simple polygon, which can be transformed to a simple polygon by slightly “inflating” the edges; see Figure 5 (left). In the resulting polygon, all left endpoints of edges and all red right endpoints of edges appear as reflex vertices. All blue right endpoints of matching edges are “hidden” in the exterior of the polygon.

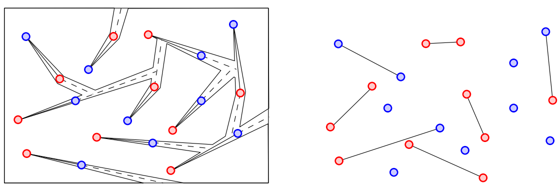

Figure 5: (left) Transforming a perfect matching to a simple polygon. (right) A resulting compatible matching.

Abellanas et al. [1] showed that for every simple polygon $P(V)$ with vertex set V and every subset $V' \subseteq V$ containing all reflex vertices of $P(V)$, there exists a perfect matching of the vertices V' where no edge is outside the boundary of $P(V)$. Applying this result to the set of reflex vertices of the constructed polygon, we obtain a matching $M(S)$ with at most $\frac{n}{2}$ bichromatic edges and thus $M(S)$ has at least $\frac{n}{4}$ red edges, implying $m_{match}(n) \geq \frac{n}{4}$. Figure 5 (right) shows a possible resulting matching.

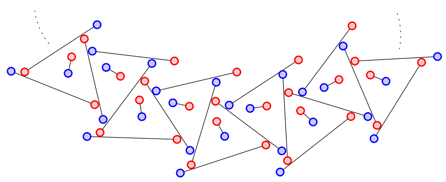

Figure 6: Scheme for a bichromatic perfect matching where any monochromatic compatible matching has at most $\frac{5n}{12}$ edges.

For an upper bound on the number of edges in a monochromatic matching, we can recycle the idea from Figure 2. Inverting the colors of every second triangle construction and combining them to a closed cycle, we obtain a perfect matching $PM(S)$ where every sixth point of each color must remain unmatched in any monochromatic $PM(S)$-compatible matching; see Figure 6 for a schematic illustration. $\square$

The principle of the lower bound part of the above proof can be reused to provide a lower bound for the size of maximum monochromatic compatible matchings for trees, in dependence of the number of interior vertices (of one color) of the tree. The basic idea for this bound was developed during a research week which was also the starting point for the work [2].

Theorem 4 *Let $T(S)$ be a bichromatic tree $T(S)$ with i_r interior (non-leaf) red vertices. There exists a red matching $M(S)$ with $d(T(S), M(S)) \geq \lceil \frac{i_r - 1}{6} \rceil$.*

Proof. We generate a simple polygon for $T(S)$ by adding a bounding box, connecting $T(S)$ to the box and then inflating the whole construction. Every vertex v of $T(S)$ with vertex degree $d(v)$ corresponds to $d(v)$ vertices in the polygon P, at most one of them being reflex. We choose one of these $d(v)$ vertices for each vertex v of $T(S)$, (if v corresponds to a reflex vertex, we choose that one). Additionally, we choose a second vertex for each of the i_r red interior vertices; see Figure 7 (left) for a resulting polygon.

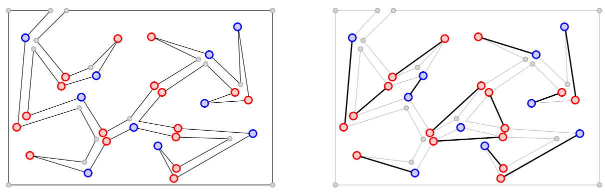

Figure 7: (left) A simple polygon generated from a bichromatic tree (non-selected vertices and vertices on the bounding box are drawn gray). (right) A (nearly) perfect matching of the selected vertex set.

Applying the result of [1] to the selected vertices, we obtain a (nearly) perfect matching with $\lfloor \frac{2n+i_r}{2} \rfloor$ edges, at least $\lfloor \frac{i_r}{2} \rfloor$ of them red. But it might happen that both red vertices corresponding to one red vertex in S are incident to such a red edge. Also, there might occur cycles of such red edges; see Figure 7 (right).

In general, the translated red edges form a set of cycles and paths. For such a path of length k, $\lceil \frac{k}{2} \rceil$ edges can simultaneously occur in a monochromatic $T(S)$-compatible matching. For a red cycle of length k, $\lfloor \frac{k}{2} \rfloor$ of these edges can be used. Assuming the worst case where (nearly) all red edges form 3-cycles, we obtain $\lceil \frac{i_r - 1}{6} \rceil$ edges for a red $T(S)$-compatible matching. $\square$

Applying Theorem 4 to the class of spanning paths, we obtain

Corollary 5 $m_{path} \geq \lceil \frac{n-2}{6} \rceil$.

Note that in a path all vertices have degree at most two. Thus, a cycle of odd length among the red edges (translated back to S) can only occur if the path ends inside this cycle. As there might be a sequence of cycles $C_1, \ldots, C_l$ such that each C_{i+1} contains C_i in its interior, this observation does not improve the above bound; see Figure 8.

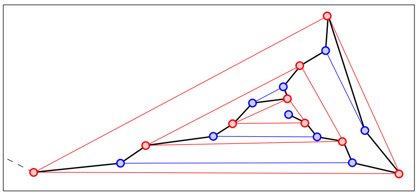

Figure 8: Multiple red 3-cycles for a path resulting from a matching of the according inflated polygon.

For spanning cycles, the proof of Theorem 4 can be slightly modified, yielding the following lower bound.

Corollary 6 $m_{cycle}(n) \geq \lceil \frac{n-1}{4} \rceil$.

Proof. For creating a simple polygon P for a given spanning cycle $C(S)$, we duplicate an extreme vertex v (w.l.o.g. a blue one), and cut the cycle between v and its duplicate v'. Then we extend one of the incident edge of v' until it hits the bounding box, and again inflate the resulting construction, by this hiding v' again. Applying the above construction, we obtain at least $\lfloor \frac{n}{2} \rfloor$ red matching edges. As the "end" vertex v of the inflated path is extreme, the red edges translated back to S cannot form any odd cycles. Thus we can use at least half of the obtained red edges for a $C(S)$-compatible monochromatic matching. $\square$

For upper bounds, consider the examples shown in Figure 9. In Figure 9 (left), any monochromatic compatible edge is incident to one of the two high degree vertices, implying $m_{tree}(n) \leq 1$. In Figure 9 (right), the only vertex to which an "end vertex" of a "spike" can be matched is the next vertex on the big circle, implying that $m_{cycle}(n), m_{path}(n) \leq \frac{5n+7}{12}$.

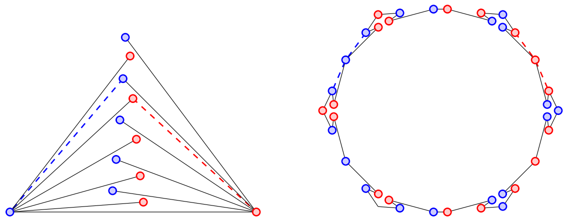

Figure 9: Upper bound examples for spanning trees (left) and spanning cycles (right).

4 Conclusion

We have shown the following bounds on the numbers of edges $m_{\mathcal{C}}(n)$ that can be reached by a monochromatic $G(S)$-compatible matching, with $G(S) \in \mathcal{C}$ and $\mathcal{C} \in \{tree, path, cycle, match\}$.

$$\begin{array}{rcccl} \frac{n}{4} & \leq & m_{match}(n) & \leq & \frac{5n}{12} \\ \lceil \frac{n-2}{6} \rceil & \leq & m_{path}(n) & \leq & \frac{5n+7}{12} \\ \lceil \frac{n-1}{4} \rceil & \leq & m_{cycle}(n) & \leq & \frac{5n+7}{12} \\ & & m_{tree}(n) & = & 1 \end{array}$$

Further, we showed that every bichromatic perfect matching admits a disjoint compatible bichromatic matching with at least $\lceil \frac{n-1}{2} \rceil$ edges, and that there exist point sets with non-unique bichromatic perfect matchings for which any compatible bichromatic (perfect) matching has at most $\frac{3n}{4}$ $(\frac{n}{2})$ disjoint edges.

We conclude with the following open problem. Given a bichromatic perfect matching $PM(S)$, can we always find a bichromatic compatible perfect matching $M(S)$ with $d(PM(S), M(S)) > 0$, supposed that $S = R \cup B$ is a point set admitting at least two different bichromatic perfect matchings?

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