On 5-Gons and 5-Holes*

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Abstract. We consider an extension of a question of Erdős on the number of k-gons in a set of n points in the plane. Relaxing the convexity restriction we obtain results on 5-gons and 5-holes (empty 5-gons). In particular, we show a direct relation between the number of non-convex 5-gons and the rectilinear crossing number, provide an improved lower bound for the number of convex 5-holes any point set must contain, and prove that the number of general 5-holes is asymptotically maximized for point sets in convex position.

Introduction

Let S be a set of n points in general position in the plane. A k-gon is a simple polygon spanned by k points of S. A k-hole is an empty k-gon, that is, a k-gon which does not contain any points of S in its interior.

Erdős [12] raised the following questions for convex k-holes and k-gons. "What is the smallest integer $h(k)$ ($g(k)$) such that any set of $h(k)$ ($g(k)$) points in the plane contains at least one convex k-hole (k-gon)?"; and more general: "What is the least number $h_k(n)$ ($g_k(n)$) of convex k-holes (k-gons) determined by any set of n points in the plane?".

As already observed by Esther Klein, every set of 5 points determines a convex 4-hole (and thus 4-gon). Moreover, 9 points always contain a convex 5-gon and 10 points always contain a convex 5-hole, a fact proved by Harborth [16]. Only in 2007/08 Nicolás [19] and independently Gerken [15] proved that every sufficiently large point set contains a convex 6-hole, and it is well known that there exist arbitrarily large sets of points not containing any convex 7-hole [17]; see [3] for a brief survey.

In this paper we concentrate on 5-gons and 5-holes and generalize the above questions by allowing a 5-gon or 5-hole to be non-convex. Thus, when referring to a 5-gon or 5-hole, it might be convex or non-convex and we will explicitly state it when we restrict considerations to one of these two classes.

A preliminary version of this paper appeared as [6]. Similar results for 4-holes can be found in [5]. For 4-gons there is a one-to-one relation to the rectilinear crossing number of the complete graph, and thus results can be found in the respective literature (e.g. [10, 13]).

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1 Small sets

A set of five points in convex position obviously spans precisely one convex 5-gon. In contrast, already a set of only five points (with three extremal points) can span eight different 5-gons; see Fig. 1.

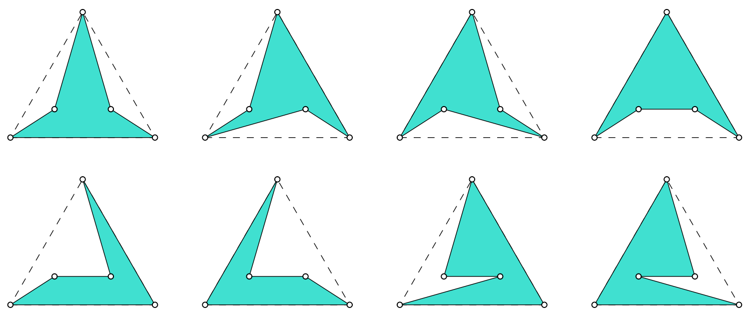

Fig. 1. The eight different (non-convex) 5-gons spanned by a set of five points with three extremal points (fixed order type).

For small point sets, Table 1 shows the numbers of 5-gons and 5-holes, respectively. We obtained these numbers by checking all point sets (with the corresponding number of points) from the order type database [7]. Given are: the minimum number of convex 5-gons and 5-holes; the maximum number of non-convex 5-gons and 5-holes; the minimum and maximum number of (general) 5-gons and 5-holes; and, for easy comparison, the number of 5-tuples.

	numbers of 5-gons				numbers of 5-holes				
n	convex min	non-convex max	general min	general max	convex min	non-convex max	general min	general max	$\binom{n}{5}$
5	0	8	1	8	0	8	1	8	1
6	0	48	6	48	0	31	6	31	6
7	0	156	21	157	0	76	21	77	21
8	0	408	56	410	0	157	56	160	56
9	1	900	126	909	0	288	126	292	126
10	2	1776	252	1790	1	492	252	501	252
11	7	3192	462	3228	2	779	462	802	462

Table 1. The number of 5-gons and 5-holes for $n = 5...11$ points.

For counting convex 5-gons and 5-holes it is easy to see that their number is maximized by sets in convex position and gives $\binom{n}{5}$. Of course these sets do not contain any non-convex 5-gons or 5-holes. From Table 1 we observe that the minimum number of general 5-gons and 5-holes is $\binom{n}{5}$ for $5 \leq n \leq 11$. While for 5-gons this is obviously true in general (a convex 5-tuple has exactly one polygonization, while a non-convex 5-tuple has at least four), this is not the case for 5-holes. In fact, we will show that for sufficiently large n, the convex set maximizes the number of 5-holes; see Theorem 4.

2 5-gons

The rectilinear crossing number $\bar{cr}(S)$ of a set S of n points in the plane is the number of proper intersections in the drawing of the complete straight line graph on S. It is easy to see that the number of convex 4-gons is equal to $\bar{cr}(S)$ and is thus minimized by sets minimizing the rectilinear crossing number. This is a well known, difficult problem in discrete geometry; see [10] and [13] for details. Tight values for the minimum number of convex 4-gons are known for $n \leq 27$ points; see e.g. [2]. Asymptotically we have at least $c_4\binom{n}{4} = \Theta(n^4)$ convex 4-gons, where c_4 is a constant in the range $0.379972 \leq c_4 \leq 0.380488$ [1]. As any 4 points in non-convex position span three non-convex 4-gons, we get $3\binom{n}{4} - 3\bar{cr}(S)$ non-convex and $3\binom{n}{4} - 2\bar{cr}(S)$ general 4-gons for a set S. Thus, sets which minimize the rectilinear crossing number also minimize the number of convex 4-gons, and maximize both the number of non-convex 4-gons and the number of general 4-gons.

Surprisingly, a similar relation can be obtained for the number of non-convex 5-gons. To see this, consider the three combinatorial different possibilities (order types) of arranging 5 points in the plane, as depicted in Fig. 2.

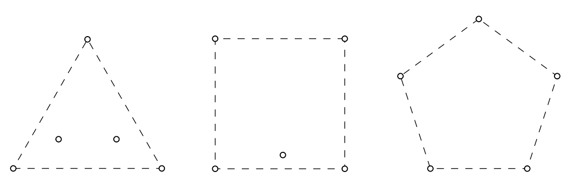

Fig. 2. The three order types for $n = 5$. For each set its number of different 5-gons and the number of crossings for the complete graph is shown.

Theorem 1. *Let S be a set of $n \geq 5$ points in the plane in general position. Then S contains $10\binom{n}{5} - 2(n-4)\bar{cr}(S)$ non-convex 5-gons.*

Proof. We denote with $o_3(S)$, $o_4(S)$, and $o_5(S)$ the number of 5-tuples of points with 3, 4, and 5, respectively, points on their convex hull. Summing over all such sets we get $o_3(S) + o_4(S) + o_5(S) = \binom{n}{5}$.

Note that every four points spanning a crossing pair of edges in S show up in $(n-4)$ 5-tuples of points in S. Using the number of crossings for each order type from Fig. 2 we get $\bar{cr}(S) = \frac{o_3(S)+3o_4(S)+5o_5(S)}{n-4}$.

Considering the numbers of different 5-gons given in Fig. 2, we see that the total number of non-convex 5-gons in S is $8o_3(S) + 4o_4(S)$. Using these three equations, it is straight forward to obtain the following relation for the number of non-convex 5-gons in S: $8o_3(S) + 4o_4(S) = 10\binom{n}{5} - 2(n-4)\bar{cr}(S)$. □

Taking the constant c_4 for the rectilinear crossing number into account, we see that asymptotically we can have up to $10\binom{n}{5} - 2(n-4)c_4\binom{n}{4} = 10(1-c_4)\binom{n}{5}$ non-convex 5-gons. This number is obtained for point sets minimizing the rectilinear crossing number and by a factor ≈ 6.2 larger than the maximum number of convex 5-gons.

For the number of convex 5-gons, no simple relation to the rectilinear crossing number is possible: There exist two different sets (order types) S_1 and S_2, both of cardinality 6 with 4 extremal points, with $\bar{cr}(S_1) = \bar{cr}(S_2) = 8$, where S_1 contains one convex 5-gon, while S_2 does not contain any convex 5-gon; see Fig. 3.

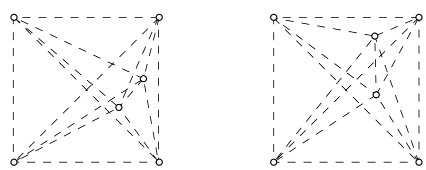

Fig. 3. Two point sets for $n = 6$, both with crossing number eight. One contains a convex 5-gon, the other one does not.

3 5-holes

Recall that a 5-hole is a 5-gon which does not contain any points from the underlying set in its interior.

3.1 An improved lower bound for the number of convex 5-holes

Let $h_5(S)$ denote the number of convex 5-holes of a point set S, and let $h_5(n) = \min_{|S|=n} h_5(S)$ be the number of convex 5-holes any point set of cardinality n has to have. The best upper bound $h_5(n) \leq 1.0207n^2 + o(n^2)$ can be found in [9]. Although $h_5(n)$ is conjectured to be quadratic in the size of S, to this date not even a super-linear lower bound exists. For quite some time, the best published lower bound was $h_5(n) \geq \lfloor \frac{n-4}{6} \rfloor$, obtained by Bárány and Károlyi [8]. García [14] improves this bound to $h_5(n) \geq \frac{2}{9}n - O(1)$. In the proceedings version of our paper [6], we presented a slightly better bound, showing $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$.

The following theorem further improves the lower bound for $h_5(n)$, but still remains linear in n. It is based on an idea of Clemens Huemer [18].

Theorem 2. *Any set of n points in the plane in general position contains at least $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$ convex 5-holes.*

Proof. Consider an arbitrary set S of n points. Assume that there is an extreme point $p \in S$ which is incident to (at least) one convex 5-hole spanned by S. We count these convex 5-holes (solely) for p, remove p from S, and continue with $S_1 = S \backslash \{p\}$. Assume further that we can repeat this $i \geq 0$ times. This way we count (at least) i different convex 5-holes, and obtain a point set S_i of cardinality $|S_i| = n - i$, for which all extreme points of S_i are not incident to any convex 5-hole.

Now take any extreme point $p \in S_i$. Sort all other points of S_i radially around p (such that its neighbours on the convex hull $\mathrm{CH}(S_i)$ are the first point p_1 and the last point p_{n-i-1} in the sorting, respectively). Split the sorted set $S_i \backslash \{p\}$ into consecutive groups G_l, for $1 \leq l \leq \lfloor \frac{n-i-5}{7} \rfloor$, of seven points each, such that the remainder R contains at least four points; see Fig. 4. Then every group G_l together with the sorting anchor p and the first four points of G_{l+1} (or R, respectively) gives a set $G'_l \subset S_i$ of 12 points.

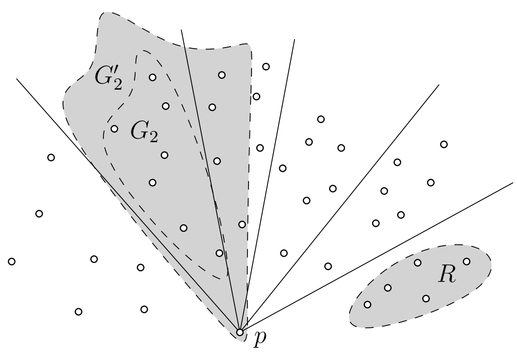

Fig. 4. Splitting S_i into groups G_l of seven points each, plus a remainder R of at least four points.

We know by Dehnhardt [11] that every set of 12 points, and thus also every set G'_l, contains at least 3 convex 5-holes. As p is not incident to any convex 5-hole, all convex 5-holes in any set G'_l must be incident to at least one point of its underlying set G_l and can thus be counted (solely) for G_l. As R must have at least four points, there are exactly $\lfloor \frac{n-i-1-4}{7} \rfloor$ groups G_l, and at least three times that many convex 5-holes in S_i. Adding the convex 5-holes we counted for points of $S \backslash S_i$, of which there are at least i, we obtain a lower bound of

$$\begin{aligned} i + 3\left\lfloor \frac{n-i-5}{7} \right\rfloor &\geq i + 3\frac{n-i-5-6}{7} \\ &= \frac{3n+4i-33}{7} \end{aligned}$$

for the total number of convex 5-holes in S. This term is minimized for $i = 0$, which leads to a lower bound of $\lceil \frac{3}{7}(n-11) \rceil$ for the minimum number $h_5(n)$ of convex 5-holes in any set of n points. □

In the above proof we used a result by Dehnhardt [11], stating that every set of 12 points contains at least three convex 5-holes. In fact, Dehnhardt showed $3 \leq h_5(12) \leq 4$, and conjectured that $h_5(12) = 4$. Using the order type database [7], we found point sets of 12 points that contain only three convex 5-holes, disproving Dehnhardt's conjecture and thus settling $h_5(12)$. A point set attaining this lower bound $h_5(12) = 3$ is shown in Fig. 5. Note, that this point set has 4 extreme points. This answers the question of Dehnhardt (in [11]), whether there exist sets of n points whose convex hull is not a triangle, but which minimize $h_5(n)$.

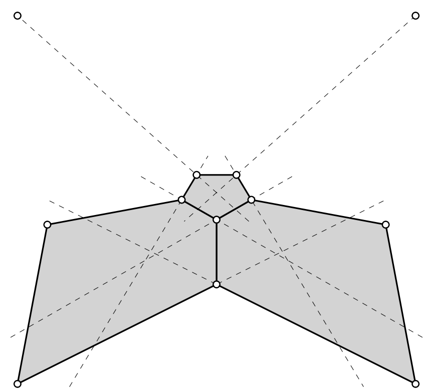

Fig. 5. A set of 12 points containing only three convex 5-holes, implying $h_5(12) = 3$.

Note that on the one hand, for $n \leq 17$ the best known lower bound is still only $h_5(n) \geq 3$. On the other hand, from the examples we found so far it follows that $h_5(13) \leq 4$, $h_5(14) \leq 6$, and $h_5(15) \leq 9$; see [21] for point sets attaining these bounds.

Remark (added during revision): Valtr [20] recently presented a bound of $h_5(n) \geq \frac{n}{2} - O(1)$. In a forthcoming paper (by Aichholzer, Fabila-Monroy, Hackl, Huemer, Pilz, and Vogtenhuber) this bound is further improved to $h_5(n) \geq \frac{3}{4}n + o(n)$.

3.2 A lower bound for the number of (general) 5-holes

We obtained the following observation for general 5-holes by checking all 14 309 547 according point sets from the order type database [7].

Observation 1. *Let S be a set of $n = 10$ points in the plane in general position, and $p_1, p_2 \in S$ two arbitrary points of S. Then S contains at least 34 5-holes which have p_1 and p_2 among their vertices.*

Based on this simple observation we derive the following lower bound for the number of 5-holes, following the lines of a similar proof for the number of 4-holes in [5, Theorem 5].

Theorem 3. *Let S be a set of $n \geq 10$ points in the plane in general position. Then S contains at least $17n^2 - O(n)$ 5-holes.*

Proof. We consider the point set S in x-sorted order, $S = \{p_1, \dots, p_n\}$, and sets $S_{i,j} = \{p_i, \dots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ having at least 10 points is

$$\sum_{i=1}^{n-9} \sum_{j=i+9}^{n} 1 = \frac{n^2}{2} - O(n)$$

For each $S_{i,j}$ consider the eight points of $S_{i,j} \setminus \{p_i, p_j\}$ which are closest to the segment $p_i p_j$ to obtain a set of 10 points, including p_i and p_j. By Observation 1, each such set contains at least 34 5-holes which have p_i and p_j among their vertices. Moreover, as p_i and p_j are the left- and rightmost point of $S_{i,j}$, they are also the left- and rightmost point for each of these 5-holes. This implies that any 5-hole of S counts for at most one set $S_{i,j}$, which gives a lower bound of $17n^2 - O(n)$ for the number of 5-holes in S. □

3.3 Maximizing the number of (general) 5-holes

The results for small sets shown in Table 1 suggest that the number of (general) 5-holes is minimized by sets in convex position. In this section we will not only show that this is in fact not the case, but rather prove the contrary: For sufficiently large n, sets in convex position maximize the number of 5-holes.

Lemma 1. *A point set S with triangular convex hull and i interior points contains at most $(4i+5)$ 5-holes which have the three extreme points among their vertices.*

Proof. Let Δ be the convex hull of S, a, b, and c the three extreme points of S (in counterclockwise order), and $I = S \setminus \{a, b, c\}$ the set of inner points of S, $|I| = i$. As all 5-holes we consider have a, b, and c among their vertices, they contain either one or two edges of Δ.

First, we derive an upper bound for the number of 5-holes that contain only one edge of Δ. If two points $p, q \in I$ form a 5-hole that contains only the edge ab of Δ, they have to be neighboured in a circular order of I around c; see Fig. 6(a).

Let p be before q in the counterclockwise order around c. We say that p *starts a 5-hole (with ab)*. Note that q is uniquely defined by p and ab, and that the triangular area bounded by the supporting lines of cp, ap, and ab must not contain any points of I.

Assume that p starts a 5-hole with each edge of Δ, implying that the according areas for all three edges of Δ have to be empty; see Fig. 6(b). Then any other point $q \in I$ can start 5-holes with at most two edges of Δ, as p lies in one of the three areas that would have to be empty for q; see again Fig. 6(b). Using

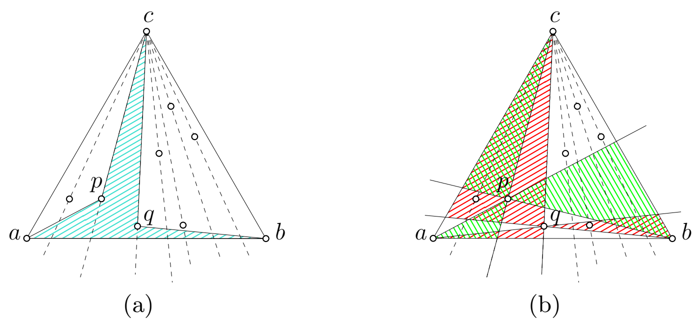

Fig. 6. (a) A 5-hole containing only the edge ab of Δ. (b) Shaded areas have to be empty if p (or q, respectively) starts a 5-hole with each edge of Δ.

this fact, we conclude that at most one point of I might start three such 5-holes and all other inner points start at most two such 5-holes. This gives a total of at most $(2i+1)$ 5-holes that contain only one edge of Δ.

Second, we consider 5-holes that contain two edges of Δ where one of the two vertices of I is reflex and the other is convex. Assume that there exists such a 5-hole without the edge ab, and with p_{ab} as reflex vertex; see Fig. 7(a).

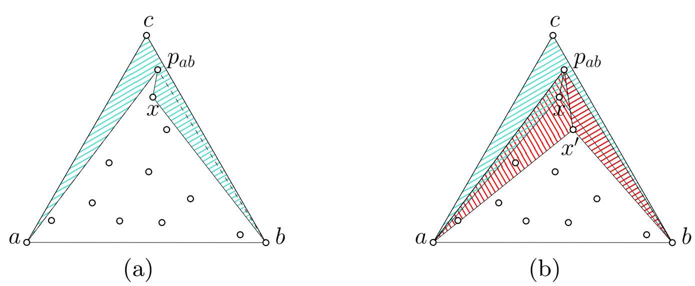

Fig. 7. (a) A 5-hole $ap_{ab}xbc$ containing two edges of Δ. (b) Only one point of I can span two 5-holes for ab.

Then the non-convex quadrilateral $ap_{ab}bc$ must not contain any points of I, which implies that all other such 5-holes without the edge ab have p_{ab} as reflex vertex as well. Let x be the convex vertex in a 5-hole without ab. We say that x *spans the 5-hole (for ab)*.

Note that a point x might span two 5-holes for ab, namely $axp_{ab}bc$ and $ap_{ab}xbc$. But this situation can happen for at most one point x, as all other points have to lie inside the triangle axb and thus for each of them, x lies inside exactly one of the two according possible 5-holes; see Fig. 7(b).

Now assume that for every edge e of Δ, there exist 5-holes skipping (solely) e. Then for every edge e there is one unique point $p_e \in I$ that is the single reflex vertex in all 5-holes for e, and each non-convex quadrilateral spanned by $\Delta\setminus\{e\}$ and p_e is empty; see Fig. 8.

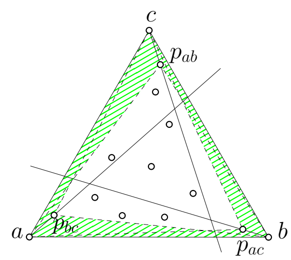

Fig. 8. If for each combination of two sides of Δ there is a 5-hole where one vertex of I is convex, then the shaded area must be empty.

Assume further that there is a point y, that spans a 5-hole for each edge e of Δ. Note that if a point x lies below the supporting line of ap_{bc}, then the 5-gon $axp_{ab}bc$ contains p_{bc}. Accordingly, if x lies below the supporting line of bp_{ac}, then p_{ac} lies inside $ap_{ab}xbc$. Thus, no point inside the triangle formed by the supporting lines of ap_{bc}, bp_{ac}, and ab can span a 5-hole for ab because any such 5-gon contains either p_{bc} or p_{ac}. As similar statements hold for the other edges of Δ as well, y has to lie outside all these triangles, and thus inside the triangle formed by the supporting lines of ap_{bc}, bp_{ac}, and cp_{ab}.

Note that y can span only one 5-hole for each side, as for each reflex point there is a line l supporting one of the segments cp_{ab}, ap_{bc}, or bp_{ac} such that y and the reflex point lie on opposite sides of l. (Recall that the shaded area in Fig. 8 must be empty of points from S and that y lies inside the triangle formed by the supporting lines of ap_{bc}, bp_{ac}, and cp_{ab}.)

Fig. 9 shows the three possible 5-holes spanned by y both separately and altogether. As by assumption, the whole shaded area in Fig. 9(d) does not contain any points of I, all other points must be located in the non-shaded wedges.

Now, if a point lies in the wedge from y towards p_{ab}, then it cannot span a 5-hole for ac, as y lies inside one candidate and p_{ab} lies inside the other. Accordingly, a point in the wedge from y to p_{bc} cannot span a 5-hole for ab, and a point in the wedge from y to p_{ac} cannot span a 5-hole for bc. Thus, at most one point might span a 5-hole for e, for each edge e of Δ. We obtain an upper bound of $(2i+4)$ for the number of 5-holes that contain two edges of Δ where one of the two vertices of I is reflex and the other is convex: at most two such 5-holes per point of I, plus one for the special point spanning a 5-hole for each edge of Δ, plus one additional 5-hole per edge of Δ.

Finally, consider 5-holes that contain two edges of Δ, where the two additional vertices are both reflex, like the one shown in Fig. 10.

There is at most one such 5-hole per non-used side of Δ. Moreover, the existence of such a 5-hole for an edge e of Δ implies that there is no 5-hole for e

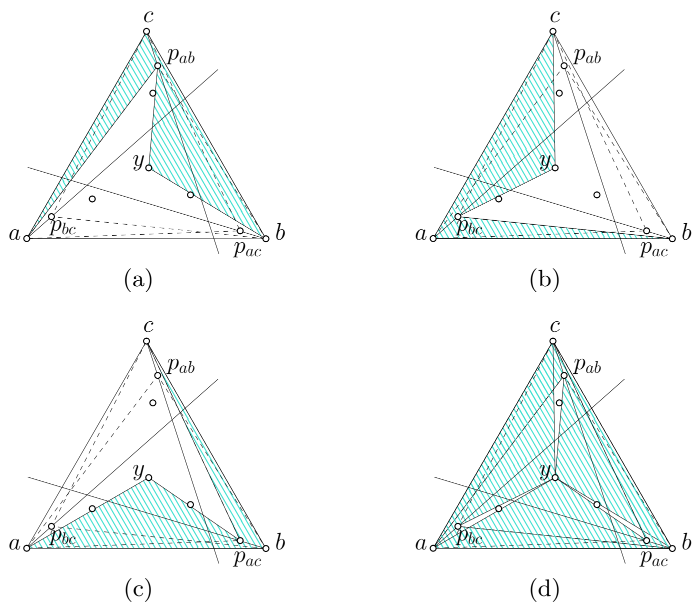

Fig. 9. Three 5-holes spanned by y, each one leaving out a different side of Δ.

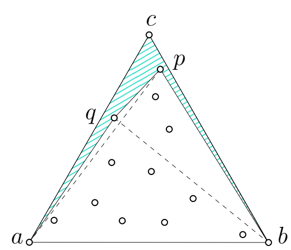

Fig. 10. Remaining points of I have to be located in the white areas.

where one of the vertices of I is convex. Thus, the upper bound for all 5-holes using two edges of Δ (with and without a point of I being convex) is still $(2i+4)$. Hence, together with the $(2i+1)$ 5-holes using one edge of Δ we obtain an upper bound for the total number of 5-holes of $(4i+5)$. $\square$

Note that the upper bound from Lemma 1 is most likely not tight. The best example we found so far is depicted in Fig. 11. It spans $3i+2$ (non-convex) 5-holes (of all eight types indicated in Fig. 1), where i is the number of points inside the triangle.

Lemma 2. *Let Γ be a non-empty convex quadrilateral in S. There are at most four (non-convex) 5-holes spanned by the four vertices of Γ plus a point of S in the interior of Γ.*

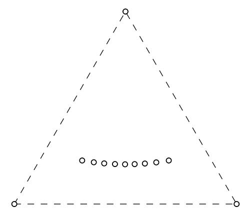

Fig. 11. A point set where the extreme triangle spans $3i+2$ (non-convex) 5-holes.

Proof. Let $p_1, \ldots, p_4$ be the vertices of Γ. Observe that any non-convex 5-hole has to use three edges of Γ. Thus there are four choices for the unused edge of Γ, and for each choice there is at most one way to complete the three used edges of Γ to a 5-hole. Assume to the contrary that two different 5-holes avoid the edge p_1p_2 and use points q_1 and q_2, respectively, in the interior. Then q_2 lies in the triangle formed by $p_1p_2q_1$. But then q_1 must lie inside the polygon $p_1q_2p_2p_3p_4$, a contradiction. □

Taking into account the size of the convex hull of each 5-tuple, these two lemmas lead to the following theorem.

Theorem 4. *For $n \geq 86$ the number of 5-holes is maximized by a set of n points in convex position.*

Proof. In the following we assign every non-convex 5-tuple to the (three or four) vertices of its convex hull, and call this convex hull the *representing* triangle (or quadrilateral) of the potential non-convex 5-holes.

From Lemma 1 we know that a non-empty triangle Δ with $i > 0$ interior points represents at most $4i+5$ non-convex 5-holes. In addition, each of the $o = n-3-i$ points outside Δ might form a convex quadrilateral Γ with Δ. According to Lemma 2, each such Γ represents at most 4 non-convex 5-holes. Thus, altogether we obtain (1) as an upper bound for the number of non-convex 5-holes which have the vertices of Δ on their convex hull.

$$4o + 4i + 5 = 4n - 7 \tag{1}$$

Note that if a (convex) quadrilateral is non-empty, then its vertices form at least one triangle which is non-empty as well. Thus, summing (1) for all non-empty triangles, we obtain an upper bound on the number of non-convex 5-holes.

Considering convex 5-holes, observe that every 5-tuple gives at most one convex 5-hole. Denote with N the number of 5-tuples that do *not* form a convex 5-hole, and with T the number of non-empty triangles. Then we get (2) as a first upper bound on the number of (general) 5-holes of a point set.

$$\binom{n}{5} - N + (4n - 7) \cdot T \tag{2}$$

To obtain an improved upper bound from (2), we need to derive a good lower bound for N. For this, consider again a non-empty triangle Δ. As Δ is not empty, each of the $\binom{n-3}{2}$ 5-tuples that contain all three vertices of Δ is either not convex or not empty. On the other hand, for such a 5-tuple, all of its $\binom{5}{3}$ contained triangles might be non-empty. Thus, we obtain $T\binom{n-3}{2}/\binom{5}{3}$ as a lower bound for N, and (3) as an upper bound for the number of 5-holes.

$$\binom{n}{5} + \left(4n - 7 - \frac{\binom{n-3}{2}}{\binom{5}{3}}\right) \cdot T \tag{3}$$

For $n \geq 86$ this is at most $\binom{n}{5}$, the number of 5-holes for a set of n points in convex position, which proves the theorem. □

Examples show that at least for $n \leq 16$ the number of general 5-holes is not maximized by convex sets. Hence, the truth for the lower bound in Theorem 4 of the cardinality n of the point sets lies somewhere in the range from 17 to 86.

4 Conclusion

In this paper we presented several results for a variant of a classical Erdős-Szekeres type problem for the case of 5-gons and 5-holes.

During the preparation of the full version of this paper we have been able to extend some of the presented results to k-gons and k-holes for $k > 5$. A preliminary version of these results has been presented at [4]. The thesis [21] summarizes all obtained results for $k \geq 4$.

Several questions remain unsettled, among which we specifically want to mention the following. Is there a super-linear lower bound for the number of convex 5-holes (cf. Theorem 2)? And does there exist a super-quadratic lower bound for the number of general 5-holes (cf. Theorem 3)?

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