On Pseudo-Convex Decompositions, Partitions, and Coverings

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Abstract

We introduce pseudo-convex decompositions, partitions, and coverings for planar point sets. They are natural extensions of their convex counterparts that use both convex polygons and pseudo-triangles. We discuss some of their basic combinatorial properties and establish upper and lower bounds on their complexity.

1 Introduction

Let S be a set of n points in general position in the plane. The *convex cover number* of S, $\kappa_c(S)$, is the minimum number of convex polygons spanned by S and covering all points of S. The study of convex cover numbers is rooted in the classical work of Erdös and Szekeres [3, 4] who showed that any set of n points contains a convex subset of size $O(\log n)$. More recent results include the work by Urabe [9].

Together with convex coverings also *convex partitions* and *convex decompositions* have received much recent attention [9, 7, 5, 8, 10]. Here the *convex partition number* of S, $\kappa_p(S)$, is the minimum number of *disjoint* convex polygons spanned by S and covering all vertices of S; the *convex decomposition number* of S, $\kappa_d(S)$, is the minimum number of faces in a subdivision of the convex hull of S into convex polygons whose vertex set is exactly S.

Whether a chain of points is considered convex or reflex depends only on the point of view. Therefore, when studying convex chains and polygons contained in a set of points one might also consider reflex chains or polygons. See for example the work by Arkin et al. [2] who study questions related to convex coverings and partitions by examining the reflexivity of point sets. The 'most reflex' polygon possible is the *pseudo-triangle* which has exactly three convex vertices with internal angles less than π. A pseudo-triangle is the natural complement of convex polygons.

In this paper we introduce pseudo-convex decompositions, partitions, and coverings which use both convex polygons and pseudo-triangles. Pseudo-convex decompositions and partitions are significantly sparser than their convex counterparts while pseudo-convex and convex coverings have asymptotically the same complexity.

Definitions. A *pseudo-triangulation* for S is a partition of the convex hull of S into pseudo-triangles whose vertex set is exactly S. A vertex is called *pointed* if it has an adjacent angle greater than π. A planar straight line graph is pointed if every vertex is pointed.

The *pseudo-convex cover number* $\psi_c(S)$ of S is the minimum number of convex polygons and/or pseudo-triangles spanned by S and covering all points of S. The pseudo-convex cover number for all sets of fixed size n is $\psi_c(n) := max_S \psi_c(S)$.

The *pseudo-convex partition number* $\psi_p(S)$ of S is the minimum number of *disjoint* convex polygons and/or pseudo-triangles spanned by S and covering all vertices of S. The pseudo-convex partition number for all sets of fixed size n is $\psi_p(n) := max_S \psi_p(S)$. Note that disjoint here implies empty (of points): neither a convex nor a pseudo-convex partition contains nested polygons.

A pseudo-convex decomposition of S is a partition of the convex hull of S into convex polygons and/or pseudo-triangles spanned by S. For instance every triangulation or pseudo-triangulations of S is a pseudo-convex decomposition. The minimum number of polygons needed for a pseudo-convex decomposition of S is the *pseudo-convex decomposition number* $\psi_d(S)$. The pseudo-convex decomposition number for all sets of fixed size n is $\psi_d(n) := max_S \psi_d(S)$.

We denote the convex cover number (and equiva-

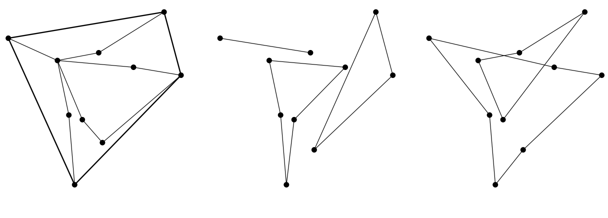

Figure 1: From left to right: Pseudo-convex decomposition, partition, and covering.

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lently the convex partition and decomposition number) for all sets of fixed size n with $\kappa_c(n) := max_S \kappa_c(S)$.

Previous work and results. The convex decomposition number $\kappa_d(n)$ is bounded by

$$n - 3 + \lfloor \sqrt{2(n-3)} \rfloor \leq \kappa_d(n) \leq \frac{10n - 18}{7}$$

(left: García-López et al. [5], right: Neumann-Lara et al. [8]). We show that the pseudo-convex decomposition number is bounded by

$$\frac{3}{5}n \leq \psi_d(n) \leq \frac{7}{10}n \ .$$

The convex partition number $\kappa_p(n)$ is bounded by

$$\left\lceil \frac{n-1}{4} \right\rceil \leq \kappa_p(n) \leq \left\lceil \frac{5n}{18} \right\rceil$$

(left: Urabe [9], right: Hosono and Urabe [7]). We show that the pseudo-convex partition number $\psi_p(n)$ is bounded by

$$\left\lfloor \frac{n}{6} \right\rfloor + 1 \leq \psi_p(n) \leq \frac{n}{4} \ .$$

The convex cover number $\kappa_c(n)$ is bounded by

$$\frac{n}{\log_2 n + 2} < \kappa_c(n) < \frac{2n}{\log_2 n - \log_2 e},$$

for $n \geq 3$ [9]. There is an easy connection between the pseudo-convex cover number and the convex cover number, namely $\psi_c(n) \leq \kappa_c(n) \leq 3\psi_c(n)$ (all points which can be covered by a pseudo-triangle can be covered by at most three convex sets). Thus both numbers have the same asymptotic behavior, which implies $\psi_c(n) \in \Theta(\frac{n}{\log n})$.

The upper bound construction for $\psi_d(n)$ depends on exact results for small point sets. These are related to a combinatorial geometry problem posed by Erdös. For $n(k) \geq 3$ find the smallest integer $n(k)$ such that any set S of $n(k)$ points contains the vertex set of a convex k-gon whose interior does not contain any points of S. Klein [3] showed that every set of 5 points contains an empty convex quadrilateral, that is $n(4) = 5$. Urabe proved in [9] that every set of 7 points can be partitioned into a triangle and a disjoint convex quadrilateral. Hosono and Urabe [7] showed that every set of 9 points contains two disjoint empty convex quadrilaterals. Harborth [6] proved that every set of 10 points contains an empty convex pentagon, that is $n(5) = 10$. We prove the following two Ramsey-type results:

Theorem 1 *Every set of* 8 *points in general position contains either an empty convex pentagon or two disjoint empty convex quadrilaterals.*

Theorem 2 *Every set of* 11 *points in general position contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral.*

Both results were established with the help of the order type data base [1]. In the full paper we also provide a surprisingly intuitive geometric proof of Theorem 1 that requires only a moderate number of case distinctions.

2 Basic Properties

Our first (trivial) observation is that $\psi_d(n) \leq \kappa_d(n)$, $\psi_p(n) \leq \kappa_p(n)$, and $\psi_c(n) \leq \kappa_c(n)$. It is well known that $\kappa_c(n) \leq \kappa_p(n) \leq \kappa_d(n)$. For pseudo-convex faces we trivially have $\psi_c(n) \leq \psi_p(n)$. $\psi_p(n) \leq \psi_d(n)$ follows from the bounds given in the previous section.

Next we observe that $\psi_d(n+1) \leq \psi_d(n)+1$, $\psi_p(n+1) \leq \psi_p(n) + 1$, and $\psi_c(n+1) \leq \psi_c(n) + 1$. This follows by induction when inserting the points in x-sorted order. For covering and partitioning the last inserted vertex is a singleton, for decomposing it forms a corner of a pseudo-triangle similar to the last step in a Henneberg construction.

The following Lemma establishes an interesting connection between the convex partition number and the pseudo-convex decomposition number.

Lemma 3 *For any point set S $\psi_d(S) \leq 3\kappa_p(S) - 2$ and thus $\psi_d(n) \leq 3\kappa_p(n) - 2$.*

Table 1 shows the exact values of $\psi_c(n)$, $\psi_p(n)$, and $\psi_d(n)$ for small sets of points. There is one intriguing open case: $\psi_p(13) \in \{3, 4\}$: $\psi_p(13) = 3$ would imply an improved upper bound of $\psi_p(n) \leq 3n/13$.

The pseudo-convex decomposition, partition, and covering numbers for a particular point set S are not necessarily monotone. Consider the examples in Figure 2: (left) A set S with 9 points and $\psi_d(S) = 3$. Removing the bottom most point of S results in a set S' with 8 points and $\psi_d(S') = 4$. (right) A set S with 6 points and $\psi_c(S) = \psi_p(S) = 1$. Removing the topmost point of S results in a set S' with 5 points and $\psi_c(S') = \psi_p(S') = 2$.

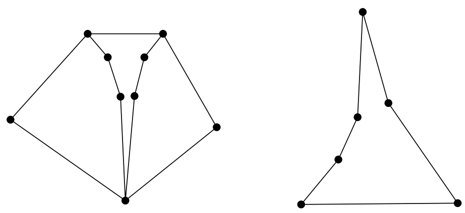

Figure 2: Sets with non-monotone behavior.

3 Pseudo-Convex Decompositions

We first give a formula for the number of faces in a pseudo-convex decomposition:

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$\psi_c(n)$	1	1	2	2	2	2	2	2..3	2..3	2..3	2..3	2..3	2..4
$\psi_p(n)$	1	1	2	2	2	2	3	3	3	3	3..4	3..4	4
$\psi_d(n)$	1	2	2	3	4	4	5	6	6	7	8	8..9	8..9

Table 1: Bounds on the pseudo-convex cover number $\psi_c(n)$, partition number $\psi_p(n)$, and decomposition number $\psi_d(n)$ for small point sets.

Lemma 4 *Let S be a set of n points in general position. Let P be a pseudo-convex decomposition of S, n_k the number of convex k-gons in P, and p the number of pointed vertices. Then the number of faces of P is*

$$|P| = 2n - p - 2 - \sum_{k=4}^{n} n_k(k-3)$$

Corollary 5 *The number of faces in a pointed pseudo-convex decomposition is*

$$|P| = n - 2 - \sum_{k=4}^{n} n_k(k-3)$$

Although the pseudo-convex decomposition number for a particular point set S might not be monotone (recall Figure 2), $\psi_d(n)$ nevertheless increases monotonically with n.

Theorem 6 *The pseudo-convex decomposition number increases monotonically with the number of points.*

Proof. We have to show that $\psi_d(n) \leq \psi_d(n+1)$ which is equivalent to show that for all point sets S, $|S| = n$, $\psi_d(S) \leq \psi_d(n+1)$ holds. So let S be some point set with n vertices and let $q \in S$ be an extreme point of S. We place a new vertex q^+ arbitrarily close to q to get the set $S^+ = S \cup q^+$ such that both, q and q^+, are extreme vertices of S^+. Note that $S^+ \setminus q$ has the same order type as S, that is, for any two points $p_1, p_2 \in S \setminus q$ the triples p_1, p_2, q and p_1, p_2, q^+ have the same orientation.

As S^+ has $n+1$ points it can be pseudo-decomposed with at most $\psi_d(n+1)$ faces. Let D^+ be such a decomposition. Note that the face F of D^+ which contains the edge qq^+ has to be convex, as otherwise q and q^+ would lie on different sides of at least one edge of the pseudo-triangle F. Now contract the edge qq^+ until q and q^+ coincide. By this transformation the face F loses one edge, but all other faces of D^+ remain combinatorially unchanged, that is, either convex polygons or valid pseudo-triangles. Thus we obtain a pseudo-decomposition D of S which has either the same number of faces as D^+ or, in the case that F was a triangle, one less. Therefore $\psi_d(S) \leq \psi_d(S^+) \leq \psi_d(n+1)$. $\square$

The general lower bound construction as well as a detailed analysis of upper and lower bounds for small point sets can be found in the full paper.

3.1 Upper Bound

Our upper bound construction is based on exact bounds for small point sets. Assume that we are given a set S with n points and that we know the value of $\psi_d(k)$ for some $k < n$. We choose an extremal point p. Now we take a line through p that has the whole point set on one side and perform a circular sweep around p, splitting off point sets of size $k-1$. Together with p each of these *petals* contains k points. We have a total of $\frac{n}{k-1}$ petals which each can be decomposed into at most $\psi_d(k)$ faces.

Two adjacent petals can be combined with a pseudo-triangle into one larger convex set. We apply this method until all of them are combined and so obtain an upper bound of

$$\psi_d(n) \leq \frac{\psi_d(k)+1}{k-1} n \ .$$

The best current upper bound can be achieved by combining Theorem 2 with Corollary 5. We construct a decomposition for $k = 11$ points by pseudo-triangulating in a pointed way around the convex polygons guaranteed by Theorem 2. Then Corollary 5 states that $\psi_d(11) = 11 - 2 - 3 = 6$ if the point set contains an empty convex hexagon and $\psi_d(11) = 11 - 2 - 1 - 2 = 6$ if the point set contains an empty convex pentagon and a disjoint empty convex quadrilateral.

This implies that

$$\psi_d(n) \leq \frac{\psi_d(11)+1}{11-1} n = \frac{6+1}{10} n = \frac{7}{10} n \ .$$

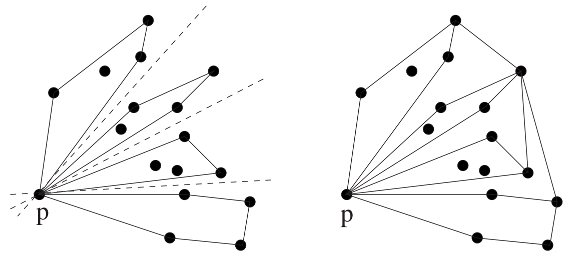

Figure 3: Petals of size 5.

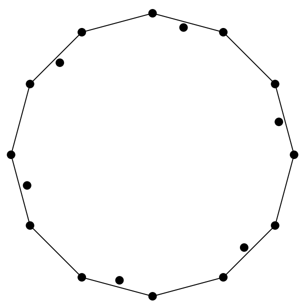

Figure 4: A point set S with $\psi_p(S) = \frac{n}{6} + 1$.

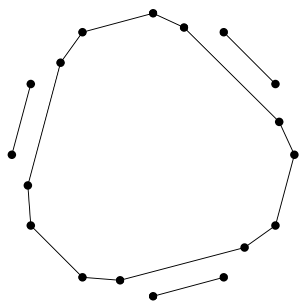

Figure 5: A partition of S consisting of $\frac{n}{6}+1$ faces.

4 Pseudo-Convex Partitions

An upper bound of $\psi_p(n) \leq n/4$ can be easily established: Any four points form either a pseudo-triangle or a convex quadrilateral and grouping them in x-sorted order guarantees disjointness.

4.1 Lower Bound

Lemma 7 *Let S be set of points in convex position with $|S| = 2m$, $m \geq 1$. We partition S into m pairs of consecutive points (along the convex hull). Let $\psi'_p(S)$ denote the minimum number of faces in a pseudo-convex partition of S in which no face contains a pair. Then $\psi'_p(S) = \lceil \frac{m}{2} \rceil + 1$.*

Theorem 8 $\psi_p(n) \geq \lfloor \frac{n}{6} \rfloor + 1$, $n \geq 3$.

Proof. We consider the point set S shown in Figure 4. S contains $\lfloor \frac{n}{3} \rfloor$ interior points which are placed very close to every second convex hull edge. Let P be a pseudo-convex partition of S using the minimum number $\psi_p(S)$ of faces. We say that two consecutive points p and q of the convex hull form a *pair* if there is an interior point close to the edge pq. There are $\lfloor \frac{n}{3} \rfloor$ such pairs. We partition the faces of P into two classes: Class A denotes faces containing at least one pair. Class B consists of faces of P containing no pair. Observe that a face of class A contains points of at most two pairs. Thus, when drawing the faces of A, there remain at least $\lfloor \frac{n}{3} \rfloor - 2|A|$ unused pairs. There might also remain additional interior points and other convex hull points. Since all faces of B are convex removing these additional points from the optimal partition P only can decrease the number of faces of B. Hence, the number of faces of B is at least the number of faces needed for the $\lfloor \frac{n}{3} \rfloor - 2|A|$ remainng unused pairs. By Lemma 7 we need at least $(\lfloor \frac{n}{3} \rfloor - 2|A|)/2+1$ faces for these pairs. Hence, $|B| \geq (\lfloor \frac{n}{3} \rfloor - 2|A|)/2+1$, and $\psi_p(S) = |A|+|B| \geq |A| + \lfloor \frac{n}{6} \rfloor - |A| + 1 \geq \lfloor \frac{n}{6} \rfloor + 1$. A partition of S consisting of $\frac{n}{6} + 1$ faces is shown in Figure 5. $\square$

Note: In the proof of Theorem 8 we did not use ceiling and floor functions to their utmost limit to simplify the resulting formula. For example with $n = 9$ we get from Lemma 7 a lower bound of $2 + |A|$ and thus $\psi_p(9) \geq 3$. Similar we get $\psi_p(15) \geq 4$.

Acknowledgements. The first two authors want to thank Ferran Hurtado and Hannes Krasser for valuable discussions on the presented subject.

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