# Monotone Simultaneous Embedding of Directed Paths<sup>\*</sup>

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Abstract

We consider a variant of monotone simultaneous embeddings (MSEs) of directed graphs where all graphs are directed paths and have distinct directions of monotonicity. In contrast to the known result that any two directed paths admit an MSE, there exist examples of three paths that do not admit such an embedding for any possible choice of directions of monotonicity. We prove that if an MSE of three paths exists then it also exists for any possible choice of directions of monotonicity. We provide a polynomial-time algorithm that, given three paths, decides whether an MSE exists. Finally, we provide a polynomial-time algorithm that answers the existence question for any given number of paths and predefined directions of monotonicity.

# 1 Introduction

Let  $\{G_i = (V, E_i) | 1 \leq i \leq k\}$  be a set of k distinct planar graphs sharing the same vertex set. A simultaneous embedding of these graphs is a set of their planar drawings  $\{\Gamma_i | 1 \leq i \leq k\}$  such that each vertex of V is represented by the same point in the plane in each of the drawings. Simultaneous embeddings were introduced as a model for visual comparison of different relations of the same object set, as well as for a visualization of dynamic changes of a single relation. Bläsius, Kobourov, and Rutter [3] give an overview of known results for simultaneous embeddings with various restrictions on how edges are embedded.

Simultaneous embeddings were also studied for upward planar digraphs. A directed graph (*digraph*, for short) is called *upward planar* if it admits a planar drawing where the edges are represented by curves, monotonically increasing in a common direction (called upward). Upward drawings are motivated by a desire for a clearer expression of a hierarchy among a set of objects. An *upward simultaneous embedding* of k upward planar digraphs with the same vertex set is a simultaneous embedding of the graphs such that each graph is drawn upward planar. Giordano, Liotta, and Whitesides [6] gave a characterization of upward simultaneous embeddable digraphs with respect to the same direction. Giordano et al. [5] showed that any two upward planar digraphs admit an upward simultaneous embedding, where the directions of upwardness differ by 90°.

In this paper we study upward simultaneous embeddings for more than two graphs and different directions of upwardness. Let  $\vec{v}$  be a vector in  $\mathbb{R}^2$ . A drawing of a directed graph is called  $\vec{v}$ -monotone, if it is an upward drawing with  $\vec{v}$  as the direction of upwardness. Let  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$  (k > 1) be a sequence of vectors and let  $\langle G_1, G_2, \ldots, G_k \rangle$  be a sequence of k distinct upward planar digraphs sharing the same vertex set. A  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -monotone simultaneous embedding  $(\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE, for short) of  $\langle G_1, G_2, \ldots, G_k \rangle$  is an upward simultaneous embedding of  $\langle G_1, G_2, \ldots, G_k \rangle$  such that the embedding  $\Gamma_i$  of  $G_i$  is  $\vec{v_i}$ -monotone. A monotone simultaneous embedding (MSE, for short) of  $\langle G_1, G_2, \ldots, G_k \rangle$ is a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE for some sequence of vectors  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ . Note that MSEs require no special restriction on the shape of the edges (despite that they respect their direction of monotonicity). We study two closely related problems. First, given a sequence  $\langle G_1, G_2, \ldots, G_k \rangle$  of upward planar digraphs, we ask whether it admits an MSE. Second, given also vectors  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ , we study whether  $\langle G_1, G_2, \ldots, G_k \rangle$ admits a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE.

While any number of planar graphs admits a (general) simultaneous embedding [9], this is not the case for MSEs, and existence of an MSE strongly depends on the choice of the directions of monotonicity; see Section 3. Intuitively, it is clear that the choice of such directions becomes more restricted as the order among the vertices of the graphs becomes more strict. We look at the core of this problem by assuming that each of our graphs is a simple directed spanning path (i.e., directed from one end of the path to the other) of the common vertex set V. We remark that our results imply a series of results on general upward planar digraphs, assuming that an order in which the vertices of each graph appear in the orthogonal pro-

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jection on the direction of its monotonicity is fixed. For non-fixed directions, Pampel [10, p. 71] shows that given a set of paths whose vertices are elements of a common vertex set V, it is NP-complete to decide whether there exists an embedding such that each path is monotone in some direction.

# 2 The Dual Setting

We use the standard dual transform where a point  $s = (x_s, y_s)$  is mapped to a non-vertical line  $\ell : y = x_s x - y_s$  and vice versa. See [4] for properties of this transform. In this section we briefly recall some properties which are used in this paper; see Figure 1 for an example. In the primal setting, we denote with S a set of points; single points are denoted with s, and lines are denoted with  $\ell$ . Notation-wise, we do not distinguish between the primal and the dual setting. For example, s is a point in the primal and a line in the dual. As a special case, we denote vertical lines in the dual with v.



Figure 1: A point set (left) and its dual arrangement of lines (right). The two vectors to the left are represented by the two vertical lines in the dual.

A well-known property is that a primal point s is below a primal line  $\ell$  if and only if the dual point  $\ell$  is below the dual line s. Furthermore, the infinite set of dual points on a vertical line  $v : x = \alpha$  corresponds to the set of lines with slope  $\alpha$ . Let S be a primal set of npoints and let  $\pi$  be the order in which S is traversed by translating a line  $\ell$  of slope  $\alpha$  below S to a line above S. Then in the dual,  $\pi$  is the order in which lines of S intersect the vertical line  $v : x = \alpha$  in the negative y-direction. Consider now the lines that are normal to  $\ell$ , i.e., the ones of slope  $(-1/\alpha)$ , and let  $\pi'$  be the order in which the dual lines intersect the vertical line  $v' : x = (-1/\alpha)$ . Then  $\pi'$  also corresponds to the order in which the points of S are projected on the original line  $\ell$ .

Note that for any problem instance with a path  $P_i$ and a vector  $\vec{v_i}$ , there exists a solution if and only if there is a solution where  $P_i$  is replaced by its reverse and  $\vec{v_i}$  by a vector in the opposite direction. Therefore, we can assume that all vectors point in the positive y-direction. Further, we assume that the vectors are sorted by the slope of their normals. We call such a sequence of vectors *adjusted*. **Observation 1** Consider a point set S, a path P containing the points of S, and a vector  $\vec{v}$  of slope  $\alpha$ . Then there is a  $\vec{v}$ -monotone embedding of P on S if and only if the dual lines of S intersect the vertical line  $v : x = (-1/\alpha)$  in the same order as the points of S appear along P.

Even though the vertical line v is not the exact dual of the vector  $\vec{v}$ , they correspond to each other. Hence, constructing a set of points that allows an MSE for a given sequence of vectors and paths is equivalent to finding a set of n lines (dual to the points) that intersect a set of vertical lines (corresponding to the vectors) in a predefined order (given by the paths).

Our problem is strongly related to the *circular se*quence of a point set (see Goodman and Pollack [7, 8]for details). Orthogonally projecting the points of a point set  $S = \{s_1, \ldots, s_n\}$  onto some line  $\ell$  gives a sequence of the indices of the points. When continuously rotating  $\ell$ , two indices change their position every time a supporting line of two points becomes normal to  $\ell$ . After having rotated  $\ell$  by 180°, the sequence of indices is reversed and every two indices have changed their relative position exactly once. This sequence of index permutations defines the circular sequence of a point set<sup>1</sup>. A sequence of index permutations which fulfills these latter properties (i.e., every pair changes its relative position exactly once and at the end the initial index sequence is reversed) is called an allowable sequence. Hence, a circular sequence is an allowable sequence that stems from a projection of a point set onto a rotating line. When initially  $\ell$  has slope 0, then the circular sequence corresponds to the sequence of intersections of the dual line set S as encountered by sweeping the dual plane with a vertical line. Thus, our problem is equivalent to the problem of constructing a point set from some given snapshots of its circular sequence. We note that circular sequences were also used in related work, e.g., by Giordano, Liotta, and Whitesides [6]. Further, during the reviewing process of this extended abstract, we have been made aware of related work by Asinowski [2], which partly overlaps with our results. He introduced suballowable sequences, which are subsequences of allowable sequences, and investigated their properties; see also the remarks in Section 3.

### 3 Monotone Simultaneous Embeddings in the Dual

It is well known that given any sequence of two paths  $\langle P_1, P_2 \rangle$  and two vectors  $\langle \vec{v_1}, \vec{v_2} \rangle$ , there always exists a  $\langle \vec{v_1}, \vec{v_2} \rangle$ -MSE of  $\langle P_1, P_2 \rangle$ ; see for example [3, 5]. To give some intuition for the more complex cases of

<sup>&</sup>lt;sup>1</sup>Strictly speaking, the circular sequence is infinite, however, a half-period, which corresponds to a rotation of  $\ell$  by 180°, completely determines the sequence.

three or more paths, we present the following algorithm which utilizes the dual setting.

Let  $v_1$  and  $v_2$  be the two vertical lines along which the order for the paths is defined in the dual. Let the dual lines be labeled in increasing order of appearance in  $P_1$ , and let  $I(i, P_2)$  be the index of  $s_i$  in  $P_2$  (the function I gives us the order  $\pi$  used before in the form of indices). Let the line  $s_i$  pass through the point  $(x_{v_1}, -i)$  and  $(x_{v_2}, -I(i, P_2))$ . This gives us a primal point set allowing a  $\langle \vec{v_1}, \vec{v_2} \rangle$ -MSE of  $\langle P_1, P_2 \rangle$ .

In contrast to two paths, it is not always possible to find an MSE for three paths. Consider a sequence  $\mathcal{P}$  of at least three paths on a vertex set V for an adjusted sequence of vectors. If for a triple of paths  $P_a$ ,  $P_b$ ,  $P_c \in \mathcal{P}$ , where a < b < c, there exists a pair of vertices  $s_i, s_j \in V$  with  $I(i, P_a) < I(j, P_a), I(i, P_b) > I(j, P_b),$ and  $I(i, P_c) < I(j, P_c)$ , we say that the path sequence  $\mathcal{P}$  is *non-allowable*. From the dual line arrangement it is easy to see that there is no point set that allows a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE of a non-allowable path sequence  $\mathcal{P}$ , as the dual lines  $s_i$  and  $s_j$  would have to cross between  $v_a$  and  $v_b$  and also between  $v_b$  and  $v_c$ . In the more general setting of allowable sequences, the three paths cannot be snapshots of any allowable sequence; see [2, Observation 2]. For self-containment, we re-state this observation in the terminology of the work at hand.

**Observation 2** Let  $\mathcal{P} = \langle P_1, \ldots, P_k \rangle$  be a sequence of paths that is non-allowable. Then, for any sequence of adjusted vectors  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ , the sequence  $\mathcal{P}$  does not admit a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE.

Now consider an (unordered) set  $\mathcal{Q}$  of paths and assume that there exists a point set S, a sequence of vectors  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ , and a sequence  $\mathcal{P}$  of the elements of  $\mathcal{Q}$  such that  $\mathcal{P}$  allows a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE. We rotate  ${\cal S}$  and the vectors such that the supporting line of two arbitrary points  $s_a, s_b \in S$  is vertical. Adjusting the vectors and reversing the corresponding paths, we obtain an MSE with an adjusted sequence  $\mathcal{P}'$  of vectors where the relative order of  $s_a$  and  $s_b$  is the same in each path of  $\mathcal{P}'$ . We call a sequence or set of paths for which there exists such a pair  $(s_a, s_b)$  adjusted. If an adjusted sequence  $\mathcal{P}'$  has a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE for some sequence of vectors  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ , these vectors are all directed towards the same half-plane and are ordered radially (and thus, adjusted after a rotation). Therefore, if for a general path sequence  $\mathcal{P}$  its adjusted path sequence  $\mathcal{P}'$  is non-allowable, we can apply Observation 2 to  $\mathcal{P}'$ , by this obtaining that  $\mathcal{P}$  does not allow a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE for any sequence  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ of vectors. The following proposition generalizes this observation to sets of adjusted paths. This result has also been discussed in [2, page 4751].

**Proposition 3** Let Q be a set of k adjusted paths on the same vertex set. If an ordering  $\langle P_1, \ldots, P_k \rangle$ of the paths in Q forms an allowable path sequence then this ordering is unique. Moreover, if the sequence  $\langle P_1, \ldots, P_k \rangle$  admits a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE, then the relative order of the adjusted vectors  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$  is fixed up to rotation and reflection.

Observation 2 and Proposition 3 imply necessary conditions for sequences and sets of paths to admit an MSE. However, these conditions are not sufficient. The proof of the following theorem uses a classical result by Ringel [11] on the stretchability of pseudoline arrangements based on Pappus' Theorem.

**Theorem 4** ([2, Proposition 8]) There are allowable path sets of  $k \ge 3$  paths that do not admit an MSE.

On the positive side, an MSE for three paths does not strongly depend on the choice of the vectors:

**Theorem 5** If for an adjusted sequence of three paths there exists a sequence of three adjusted vectors that admits an MSE of these paths, then any sequence of three adjusted vectors admits such an MSE.

**Proof.** Suppose we have a set of lines and three vertical lines that correspond to an MSE. Recall the sphere model of the projective plane. Consider a plane and a sphere in  $E^3$  such that the center point is not in the plane. Every line that intersects the plane in one point and passes through the center of the sphere intersects the sphere in two antipodal points, and every line through the center of the sphere intersects the plane, except if the line is parallel to the plane. The union of the points on the sphere defined in this way by lines parallel to the plane represents the line at infinity  $\ell_{\infty}$ . Hence, we are given a bijective mapping from every point in the plane to two antipodal points on the sphere (not on  $\ell_{\infty}$ ). In this mapping, a line in the plane corresponds to a great circle on the sphere. Consider now a dual line arrangement corresponding to a valid MSE being drawn in the plane. If we apply the projective transformation that corresponds to rotating the sphere such that the great circle corresponding to  $v_1$  equals  $\ell_{\infty}$ , we get a different set of lines in the plane, with two vertical lines and such that the slopes of the lines have the same order as the lines along  $v_1$ . Equivalently,  $v_1$  is the vertical line  $v_1: x = -\infty$ . In the primal,  $v_1$  corresponds to a vector  $\vec{v_1}$  with slope 0. Now, we can scale and translate our transformed set of lines such that we have the vectors  $\vec{v_2}$  and  $\vec{v_3}$  in any position we want. 

**Theorem 6** Given a sequence  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$  of vectors and a sequence of paths, it can be decided in polynomial time (w.r.t. the input size) if a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE of the paths exists. If such an embedding exists, it can be constructed in polynomial time. **Proof.** The problem can be formulated as a linear program that can be solved in polynomial time w.r.t. the input size: Let  $y_{i,j}$  be the *y*-coordinate of the intersection of the line  $v_i$  with the line  $s_j$ . Then, for every path  $P_i$  and every pair  $(s_l, s_m)$  of neighbored elements in  $P_i$ , we have the constraint  $y_{i,l} > y_{i,m}$ , or, equivalently,  $y_{i,l} \ge y_{i,m} + 1$  (since any solution can be scaled along the *x*-axis). Further, let  $q_i$  be the distance between the vertical lines  $v_i$  and  $v_{i+1}$ . To produce straight lines, we have the constraint  $(y_{2,j} - y_{1,j})/q_1 = (y_{(i+1),j} - y_{i,j})/q_i)$  for all  $1 \le j \le n$  and all  $2 \le i < k$  (recall that k is the number of paths).

In [2], Asinowski asks whether deciding realizability of a suballowable sequence with three terms, which, in our terminology, is equivalent to the existence of an MSE for three paths, is a tractable problem. Combining Theorem 5 with Theorem 6, we can answer this question in the affirmative.

**Corollary 7** Given a set of three paths, it can be decided in polynomial time (w.r.t. the number of vertices) whether there exists an MSE of the paths. In case of existence, such an MSE can be constructed in polynomial time as well.

### 4 Conclusion

In this paper we considered MSEs of sequences of paths, with both predefined and arbitrary directions of monotonicity. We proved that if an MSE for three paths exists, then it also exists for an arbitrary choice of directions with the same circular order and presented a polynomial-time construction algorithm. Further, we showed that the existence question for an arbitrary number of paths, but with predefined directions, can be solved in polynomial time as well.

A full version of this paper can be found at ArXiv [1]. There, we also discuss MSEs for k > 3paths and arbitrary directions, as well as implications of the MSEs of directed paths for MSEs of upward planar digraphs. We show that even if an MSE of three given paths exists, it might require an exponential (in the number of vertices) ratio of the smallest and largest distance between points of the embedding. Further, we show that starting from k = 4, not only the relative circular order of the directions but also the actual choice of the slopes influences monotone simultaneous embeddability. We also consider the complexity of the problem for k > 3 paths and arbitrary directions. In contrast to Theorem 6, the construction problem gets hard for arbitrary directions, since the constructed embedding might require a representation using coordinates of exponential size. However, showing hardness of the decision question remains an open problem for k > 3.

It might also be interesting to look at a problem that is between the settings with predefined and arbitrary directions. Let  $\langle \alpha_1, \ldots, \alpha_k \rangle$  be a set of wedges centered at the origin. If a sequence of paths  $\langle P_1, \ldots, P_k \rangle$  admits a  $\langle \vec{v_1}, \ldots, \vec{v_k} \rangle$ -MSE such that  $\vec{v_i} \in \alpha_i$ , we say that  $\langle P_1, \ldots, P_k \rangle$  admits an  $\langle \alpha_1, \ldots, \alpha_k \rangle$ -MSE. What is the complexity of deciding whether  $\langle P_1, \ldots, P_k \rangle$  admits an  $\langle \alpha_1, \ldots, \alpha_k \rangle$ -MSE?

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