Flip Graphs of Bounded-Degree Triangulations

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Abstract

We study flip graphs of triangulations whose maximum vertex degree is bounded by a constant k. Specifically, we consider triangulations of sets of n points in convex position in the plane and prove that their flip graph is connected if and only if $k > 6$; the diameter of the flip graph is $O(n^2)$. We also show that for general point sets, flip graphs of triangulations with degree $\leq k$ can be disconnected for any k.

Keywords: triangulation, flip graph, degree bound, flipping distance.

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1 Introduction

An edge flip is a local and constant-size operation that transforms one triangulation into another. It exchanges a diagonal of a convex quadrilateral, formed by two triangles, with its counterpart. The *flip graph* $\mathcal{F}_T(S)$ of triangulations of a planar point set S has a node for every triangulation of S, and two nodes are adjacent if there is a flip that transforms the corresponding triangulations into each other. One of the first and most fundamental results concerning edge flips in triangulations is the fact that flips can be used repeatedly to convert any triangulation into the Delaunay triangulation [6,8]. Hence $\mathcal{F}_T(S)$ is connected for any planar point set S.

The *flip distance* between two triangulations is the minimum number of flips needed to convert one triangulation into the other. The diameter of $\mathcal{F}_T(S)$ is an upper bound on the flip distance. For a set S of n points in the plane it is known that the maximum diameter of $\mathcal{F}_T(S)$ is $\Theta(n)$ if S is in convex position, and $\Theta(n^2)$ if S is in general position. However, the computational complexity of computing the flip distance between two particular triangulations is not known, even for convex sets [5]. In higher dimensions the flip graph does not even have to be connected [7].

Of interest are also subgraphs of flip graphs which correspond to particular classes of triangulations. Houle et al. [4] consider triangulations which contain a perfect matching of the underlying point set. They show that this class of triangulations is connected via flips, that is, the corresponding subgraph of the flip graph is connected. Related results exist for order-k Delaunay graphs: for general point sets the graph of order-k Delaunay graphs is connected via edge flips for $k \leq 1$, but there exist examples for $k \geq 2$ that cannot be converted into each other without leaving this class [2]. If the underlying point set is in convex position, then [2] also shows that the resulting flip graph is connected for any $k \geq 0$. The flip operation has been extended to other planar graphs, see [3] for a very recent and extensive survey.

There are point sets for which every triangulation has a vertex of degree $n-1$. However sets of points in convex position always have triangulations of maximum degree 4. So the question arises if the flip graphs of triangulations whose maximum degree is bounded by a constant k are connected for certain values of k. For sets of points in convex position (for sufficiently large n) we prove that the flip graphs of triangulations of maximum degree $k = 4$, $k = 5$, and $k = 6$ are not connected, and that they are connected for any $k > 6$. For general point sets we show that flip graphs of triangulations with degree $\leq k$ can be disconnected for any k. See [1] for a full version of the paper.

2 Point sets in convex position

First we study the flip graphs of bounded-degree triangulations of a set S of n points in convex position in the plane. These are the same as the (bounded-degree) triangulations of a convex n-gon, or the maximal outerplanar graphs with n vertices. Let k denote the maximum vertex degree of a triangulation T on S. If S has $n \geq 5$ points, then k must be at least 4. We call a triangulation with $k \leq 4$ a *zigzag triangulation*, as shown in Fig. 1(a). Every point set S in convex position has $\Theta(n)$ different zigzag triangulations. For $n \geq 7$ one cannot flip even a single edge in such a zigzag triangulation without exceeding a vertex degree of 4.

For $k = 5$ consider the triangulation depicted in Fig. 1(b). Only the dashed edges can be flipped, but there are $\Theta(n)$ rotationally symmetric versions of these triangulations, none of which can be reached from any other without exceeding a vertex degree of 5. For $k = 6$ consider the triangulation depicted in Fig. 1(c). No edge of this triangulation can be flipped but again there are $\Theta(n)$ rotationally symmetric versions of this triangulation, none of which can be reached from any other without exceeding a vertex degree of 6.

Theorem 2.1 *Let S be a set of n points in convex position and let T be a triangulation of S with maximum vertex degree $k > 6$. Then T can be flipped into any given zigzag triangulation of S in $O(n^2)$ flips while at no time exceeding a vertex degree of k.*

Proof (Sketch) We first introduce some notation. Let D be the dual graph of T, without a vertex for the exterior face; clearly, D is a tree. We distinguish three different types of triangles in T: *ears*, which have two edges on the convex hull of S, *path triangles*, which have one edge on the convex hull of S, and *inner triangles*, which have no edge on the convex hull of S. The ears of T are dual to the *leaves* of D and inner triangles of T are dual to *branching vertices* of degree three. A *path* in D is any connected sub-graph of D that

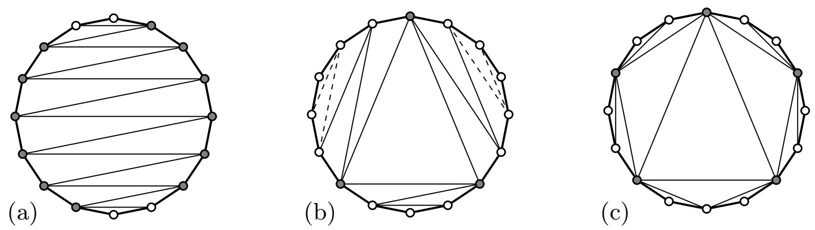

Fig. 1. Triangulations with maximum degree $k = 4$ (a), $k = 5$ (b), and $k = 6$ (c). Maximal degree vertices are shown in grey.

consists only of vertices of degree two; any vertex of a path is dual to a path triangle. A path that is adjacent to at least one leaf is called a *leaf path*. An inner triangle that is adjacent to at least two leaf paths is referred to as *merge triangle*. Consider a path in D of length at least two. If the convex hull edges of its dual path triangles are adjacent on the convex hull of S then they form a *fan*, if the convex hull edges of every second path triangle are adjacent on the convex hull of S then they form a *zigzag*. Flipping every second edge of a zigzag is called an *inversion*. A *fringe triangulation* is a triangulation which has no fans that consist of more than five triangles, where each fan is adjacent to an inner triangle, and where every leaf path is dual to a zigzag.

We first convert T into a fringe triangulation with $O(n)$ flips. If T is not a fringe triangulation, then it has at least one fan. We can convert this fan into an inner triangle and a zigzag which ends in an ear. Particular care must be taken if the fan is adjacent to another fan or to an inner triangle. In the latter case we cannot completely remove the fan, but we can ensure that it consists of at most four triangles. If such a small fan is adjacent to a zigzag we can remove it by inverting and extending the zigzag at most three times.

Each fringe triangulation has a *light* merge triangle, that is, a merge triangle with two vertices of degree $< k$. To prove this, we create a graph D' by removing all leafs and leaf paths from D. Each triangle $\triangle'$ which is dual to a leaf vertex of D' is a merge triangle in T. We now create a second graph D'' by removing each leaf of D'. Since D is a tree, both D' and D'' are trees as well. Hence D'' has at least two leaves. Let $\triangle''$ be dual to a leaf of D''. Any child of $\triangle''$ in D' is a light merge triangle. We remove such a light merge triangle by merging its adjacent zigzags with $O(n)$ flips, resulting in a fringe triangulation that has one fewer inner triangles. After repeating this step $O(n)$ times we have converted T into a zigzag triangulation. Finally, T can be "rotated" into any other zigzag triangulation of S, again with $O(n)$ flips. □

3 Point sets in general position

Let S now be a set of points in general position. Consider two triangulations T_1 and T_2 of S, both with maximum vertex degree at most k: is it possible to flip from T_1 to T_2 while at no time exceeding a vertex degree of k? Consider the example depicted in Fig. 2. The shaded parts indicate zigzag triangulations and the dark vertices have degree 8, thus $k = 8$ in this example. In the left triangulation only edges of the zigzags can be flipped without exceeding vertex degree 8, hence it is impossible to reach the triangulation on the right. This example can be easily modified for any constant $k > 3$. Therefore, for any

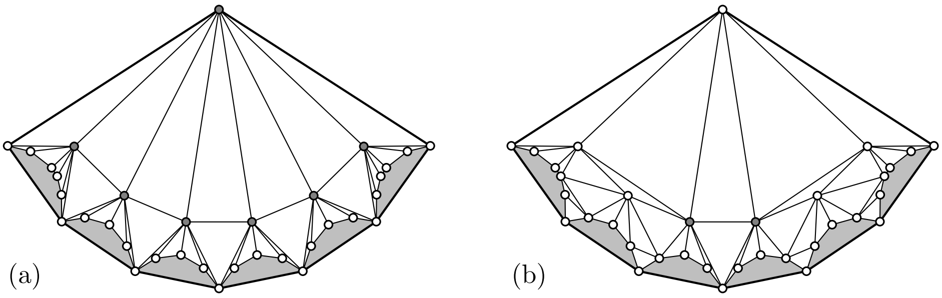

Fig. 2. Two triangulations which cannot be flipped into each other.

k there exist point sets where the flip graph of triangulations, respecting the maximal vertex degree, is not connected.

Concluding remarks. Our concept of constant maximum degree transformation can clearly be extended to other classes of geometric graphs. Some first results for pseudo-triangulations can be found in the full paper [1].

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