Flips in combinatorial pointed pseudo-triangulations with face degree at most four (extended abstract)

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Abstract

In this paper we consider the flip operation for combinatorial pointed pseudo-triangulations where faces have size 3 or 4, so-called *combinatorial 4-PPTs*. We show that every combinatorial 4-PPT is stretchable to a geometric pseudo-triangulation, which in general is not the case if faces may have size larger than 4. Moreover, we prove that the flip graph of combinatorial 4-PPTs with triangular outer face is connected and has diameter $O(n^2)$.

1 Introduction

Given a graph of a certain class, a *flip* is the operation of removing one edge and inserting a different one such that the resulting graph is again of the same class. For the class of maximal planar (simple) graphs, any combinatorial embedding (clockwise order of edges around each vertex) has only faces of size 3 and hence is called a *combinatorial triangulation*. Flips in combinatorial triangulations remove the common edge of two triangular faces and replace it by the edge between the two vertices not shared by the faces, provided that these two vertices where not already joined by an edge. Combinatorial triangulations have a geometric counterpart in triangulations of point sets in the plane, which are maximal plane geometric (straight-line) graphs with predefined vertex positions. In this geometric setting there is also a flip operation, for which a different restriction applies: An edge can be flipped iff the two adjacent triangles form a convex quadrilateral (otherwise the new edge would create a crossing).

Flips in (combinatorial) triangulations have been thoroughly studied. See [4] for a survey. A prominent question about flips is to study the *flip graph*. This is an abstract graph whose vertices are the members of the same graph class having the same number of vertices, and in which two graphs are neighbors iff one can be transformed into the other by a single flip. For both, combinatorial triangulations and triangulations (with fixed vertex positions), the flip graph is connected. However, the different settings imply linear and quadratic diameter, respectively (see [4] for references).

Triangulations have a natural generalization in pseudo-triangulations. They have become a popular structure in Computational Geometry within the last two decades, with applications in, e.g., rigidity theory and motion planning. See [7] for a survey. A *pseudo-triangle* is a simple polygon in the plane with exactly three convex vertices (i.e., vertices whose interior angle is smaller than π). A *pseudo-triangulation* $\mathcal{T}$ of a finite point set S in the plane is a partition of the convex hull of S into pseudo-triangles such that the union of the vertices of the pseudo-triangles is exactly S. Triangulations are a particular type of pseudo-triangulations, actually the ones with the maximum number of edges. Those with the minimum number of edges are the so-called *pointed pseudo-triangulations*, in which every vertex is *pointed*, i.e., incident to a reflex angle (an angle larger than π).

Flips can also be defined for the class of pseudo-triangulations of point sets in the plane. The flip graph for general pseudo-triangulations is known to be connected, as well as the subgraph induced by pointed pseudo-triangulations. The currently best known bound on the diameter is $O(n \log n)$ for both

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flip graphs [2, 3].

In a pseudo-triangulation, the pseudo-triangles can have linear size. Hence, in contrast to triangulations, the flip operation can no longer be computed in constant time. One way to bypass this issue is to consider only pseudo-triangulations in which the size of the pseudo-triangles is bounded by a constant. Kettner et al. [5] showed that every point set admits a pointed pseudo-triangulation with face degree at most four (except, maybe, for the outer face). We call such a pseudo-triangulation a *4-PPT*.

On the one hand, 4-PPTs behave nicely for problems which are hard for general pseudo-triangulations. For instance, they are always properly 3-colorable, while 3-colorability is NP-complete to decide for general pseudo-triangulations [1]. On the other hand, known properties of general pseudo-triangulations remain open for 4-PPTs. For instance, it is not known whether the flip graph of 4-PPTs is connected, even for the basic case of a triangular convex hull.

The aim of this paper is to make a step towards answering this last question, by considering the combinatorial counterpart of 4-PPTs.

A *combinatorial pseudo-triangulation* [6] is a topological embedding of a planar simple graph together with an assignment of labels *reflex/convex* to its angles such that (1) every interior face has exactly three angles labeled convex, (2) all the angles of the outer face are labeled reflex, and (3) no vertex is incident to more than one reflex angle.

Note that this labeling fulfills the same properties as actual reflex/convex angles in a (geometric) pseudo-triangulation. This analogy with the geometric case goes on by calling *pointed* vertices in a combinatorial pseudo-triangulation those which, indeed, are incident to one angle labeled reflex. Then, *combinatorial pointed pseudo-triangulations* are those in which every vertex is pointed. Combinatorial pointed pseudo-triangulations with face degree at most four (except, maybe, for the outer face), will be called *combinatorial 4-PPTs.*

2 Properties

Lemma 1 *Let G be a combinatorial 4-PPT and H be a subgraph of G with $|V(H)| \geq 3$. Then H has at least 3 vertices whose reflex angle is contained in the outer face of H (corners of first type in [6]).*

Corollary 2 *In any combinatorial 4-PPT of the interior of a simple cycle with b vertices, of which c have the reflex angle inside the cycle, the number t of triangular faces is given by $t = b - 2c - 2$.*

A combinatorial pseudo-triangulation has the *generalized Laman property* if every subset of x non-pointed vertices and y pointed vertices, where $x+y \geq 2$, induces a subgraph with at most $3x+2y-3$ edges. Both this property and the number of reflex angles from Lemma 1 are related to the stretchability of a combinatorial pseudo-triangulation into a geometric one. A face of a combinatorial pseudo-triangulation is called *degenerate* if it contains edges which appear twice on the boundary of this face.

Proposition 3 *[6, Corollary 2] The following properties are equivalent for a combinatorial pseudo-triangulation G: (1) G can be stretched to become a pseudo-triangulation. (2) G has the generalized Laman property. (3) G has no degenerate faces and every subgraph of G with at least three vertices has at least three corners of first type.*

Since, by definition, combinatorial 4-PPTs have no degenerate faces, we can use Proposition 3 to conclude the following.

Theorem 4 *Every combinatorial 4-PPT can be stretched to become a 4-PPT with the given assignment of angles. Furthermore, combinatorial 4-PPTs have the generalized Laman property.*

Note that there exist non-stretchable combinatorial pointed pseudo-triangulations with faces of size at most 5. See Figure 1. There and in the forthcoming figures, arcs denote angles labeled as reflex.

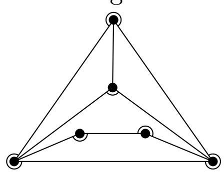

Figure 1: A non-stretchable combinatorial pointed pseudo-triangulation [6].

3 Flips

In the following we focus on combinatorial 4-PPTs with a fixed triangular outer face. For such a combinatorial 4-PPT, Corollary 2 implies that there is only one interior triangular face. Before defining flips between combinatorial 4-PPTs, we make some observations about their geometric counterpart.

Geometric 4-PPTs with triangular convex hull also have only one interior triangle. Furthermore, every edge of the triangle (except for those being part of the convex hull) is flippable [7]. Observe that the removal of the edge e to be flipped merges the triangle and the 4-face adjacent at e into a 5-face, which might be degenerate if the triangle and the 4-face share two edges. See Figure 2. Note that this is the only case in which the triangle and the 4-face can share three vertices, as there are no multiple edges in geometric graphs.

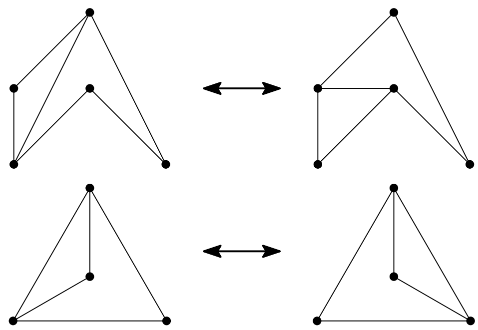

Figure 2: Geometric flip of an edge of a triangle. In the lower case, removal of the flipped edge gives a degenerate 5-face.

Similar to the geometric case, we consider flips of an edge e of the (unique) interior triangular face T in a combinatorial 4-PPT (with triangular outer face): Consider the 4-face F sharing e with T. A *flip* of e consists in replacing e by another edge e' such that (1) e' splits $(T \cup F) \setminus e$ into a triangular face T' and a 4-face F' and (2) the result is a combinatorial 4-PPT. In particular, and in contrast to the geometric case, in the combinatorial setting we have to explicitly avoid multiple edges and thus to ensure that the edge e' we insert is not already contained in the combinatorial 4-PPT (as an edge outside $T \cup F$). The following lemma shows that every interior edge of the interior triangular face can be flipped.

Lemma 5 *In a combinatorial 4-PPT, every edge e of an interior triangular face that is not an edge of the outer face is flippable. Furthermore: (1) If the removal of e results in a degenerate 5-face, then there is a unique valid flip for e. (2) If removing e results in a non-degenerate 5-face, then there are at least two valid flips for e.*

Observe that, given a combinatorial flip between two combinatorial 4-PPTs, by Theorem 4 we know that both of them can be stretched into geometric 4-PPTs with straight edges. However, it might not be possible to use the same geometric embedding for the vertices in both of them.

4 Flip graph connectivity

Lemma 6 *For a given combinatorial 4-PPT with triangular outer face and for any edge b of this outer face, there is a sequence of flips resulting in a combinatorial 4-PPT whose interior triangular face is incident to b.*

Once the interior triangular face is incident to an edge b of the outer face, the next step will be flipping away interior edges incident to one endpoint of b.

Lemma 7 *Given a combinatorial 4-PPT with triangular outer face, in which the interior triangular face T is incident to the edge b of the outer face, there is a sequence of flips resulting in a combinatorial 4-PPT in which the endpoint v of $b = uv$ has no interior incident edges.*

Proof. We describe a flip sequence that flips all inner edges incident to v. This flip sequence can be partitioned into two phases and some cases. Let the vertices neighbored to the vertex v be ordered radially around v, starting with u. In each case, let the vertices in that order be $u = w_0, \ldots, w_k$.

Phase 1: During this phase, the inner triangular face T has uv as a side, i.e., $T = vuw_1$. We distinguish three different cases:

Case 1: vw_1 is the only inner edge incident to v, i.e., $k = 2$. If T is incident to only one 4-face F (i.e., $T \cup F$ is degenerate), we can flip the edge vw_1 and are done. Otherwise, let the 4-face F incident to vw_1 be vw_1sw_2. See Figure 3. The reflex angle inside F is either at s or w_1. If it is at s, we flip vw_1 to w_0s, obtaining the 4-face vw_0sw_2. Otherwise, the reflex angle is at w_1 and we flip vw_1 to w_1w_2, obtaining the 4-face $vw_0w_1w_2$. Either way, the degree of v is 2 and we are done.

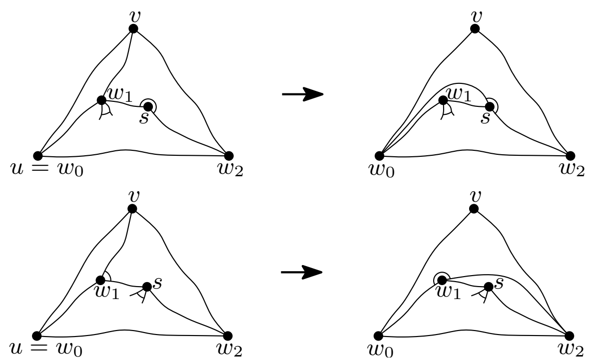

Figure 3: Phase 1, Case 1: Only one interior edge is incident to v.

Case 2: at least two inner edges are incident to v and there does not exist an edge w_0w_2. See Figure 4. Since the reflex angle of v is at the outer face we can replace the edge vw_1 by w_0w_2. This reduces the degree of v by one. The inner triangular face is again adjacent to w_0v, and we remain in Phase 1.

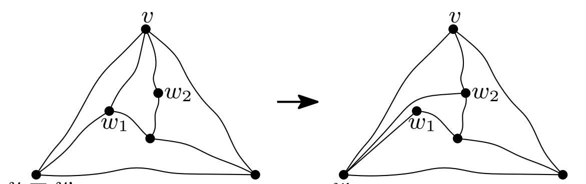

Figure 4: Phase 1, Case 2: Several interior edges are incident to v and w_0w_2 does not exist.

Case 3: at least two inner edges are incident to v and there exists an edge w_0w_2. See Figure 5. If the two inner edges of T are incident to a single 4-face, we have a degenerate case; we flip the edge w_0w_1 to w_1w_2, making vw_1w_2 the inner triangular face. Otherwise, let the 4-face F incident to vw_1 be vw_1sw_2; we flip vw_1 to vs (this is possible since if vs already existed, it would have to cross the cycle uw_2sw_1). Either way, the flip does not reduce the degree of v, but the inner triangular face is now inside the 3-cycle vw_0w_2. We switch to Phase 2.

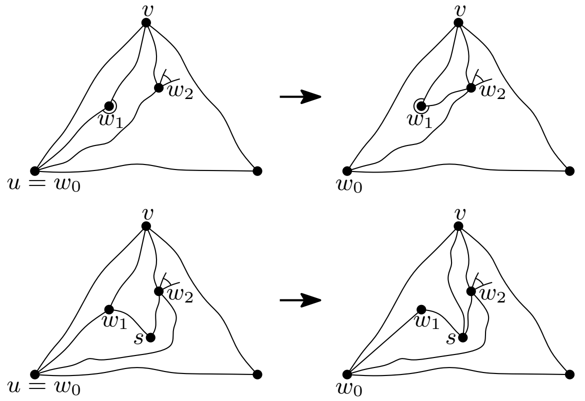

Figure 5: Phase 1, Case 3: The possible transitions to Phase 2.

Phase 2: During this phase, the inner triangular face is vw_1w_2, and w_1 stays fixed for the whole phase. Further, we know that w_1 was enclosed by a 3-cycle (at the transition to this phase), which implies that there are no edges from w_1 to w_i for any $i \geq 2$. We decrease the degree of v in the following manner.
Case 1: there is a 4-face F incident to vw_2. There cannot be an edge w_1w_3 since w_1 was enclosed by a 3-cycle. Further, the reflex angle of F is not at v. Hence, we can flip vw_2 to w_1w_3, which reduces the degree of v and we remain in Phase 2, with vw_1w_3 being the new inner triangular face.
Case 2: there is no 4-face incident to vw_2, i.e., $k = 2$. This case is symmetric to Case 1 of Phase 1. The edge vw_1 is flipped in one of the two described ways, reducing the degree of v to 2 and thus ending the process. □

Theorem 8 *The graph of flips in combinatorial 4-PPTs with n vertices and triangular outer face is connected with diameter $O(n^2)$.*

Proof. Given such a combinatorial 4-PPT, follow the steps in Lemmas 6 and 7, then use induction for the combinatorial 4-PPT obtained by removing v. This leads to the unique *canonical* combinatorial 4-PPT with triangular outer face, where two of the vertices in the outer face are adjacent to all other vertices, while the third one has degree 2. See Figure 6.

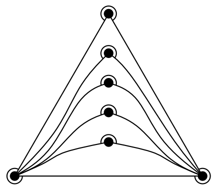

Figure 6: A canonical combinatorial 4-PPT.

Furthermore, the number of flips needed in Lemmas 6 and 7 is at most linear in the number of vertices of the combinatorial 4-PPT. □

The presented basic case of combinatorial 4-PPTs with triangular outer face is extendible to an arbitrarily sized outer face, to labeled vertices, and also to the general case of combinatorial 4-PPTs with an arbitrarily sized outer face on labeled vertices. Elaborating on these extensions would go beyond the scope of this extended abstract, though. Details (and omitted proofs) can be found in a forthcoming full version.

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A Omitted proofs

Lemma 1 *Let G be a combinatorial 4-PPT and H be a subgraph of G with $|V(H)| \geq 3$. Then H has at least 3 vertices whose reflex angle is contained in the outer face of H (corners of first type in [6]).*

Proof. W.l.o.g., we may assume that H consists of a single connected component. Let H' be the maximal subgraph of G that has the same outer face as H. Hence, if the claim holds for H' it also holds for H, and we only need to consider inner faces of size 3 or 4. For the subgraph H', let us denote with n the number of vertices, e the number of edges, t the number of inner faces of size 3, q the number of inner faces of size 4, b the number of boundary angles and c the number of convex boundary angles in the outer face of H'. Note that $b \geq 3$ and that $b > n$ is possible.

Let us double-count the edges. On the one hand, the number of angles equals twice the number of edges; since there are n reflex angles and $3t + 3q + c$ convex angles, we get that $2e = 3t + 3q + c + n$. On the other hand, from Euler's formula we have $e = n + t + q - 1$. Eliminating e from these two equations, we get that the number of reflex angles is $n = t+q+2+c$. Now we can express the number n of reflex angles as $b-c+q$, and obtain $b-c = t+2+c$, which is at least 3 if $c > 0$. Either in this case or if $c = 0$, we get that $b - c \geq 3$, as desired. □

Corollary 2 *In any combinatorial 4-PPT of the interior of a simple cycle with b vertices, of which c have the reflex angle inside the cycle, the number t of triangular faces is given by $t = b - 2c - 2$.*

Proof. Consider again the proof of Lemma 1. If the subgraph H is a simple cycle, then every vertex of H has exactly one boundary angle. Hence the number of vertices which have the reflex angle inside the cycle is equal to the number c of convex boundary angles. □

Lemma 5 *In a combinatorial 4-PPT, every edge e of an interior triangular face that is not an edge of the outer face is flippable. Furthermore: (1) If the removal of e results in a degenerate 5-face, then there is a unique valid flip for e. (2) If removing e results in a non-degenerate 5-face, then there are at least two valid flips for e.*

Proof. We have to distinguish two cases. The first case is when the removal of e results in a degenerate 5-face $(T\cup F)\backslash e$. Then, there is only one choice of e' in order to split $(T\cup F)\backslash e$ as required. Furthermore, the corresponding edge e' could not be already an edge, since it was not in the interior of $T\cup F$ and it cannot go through the exterior of $T\cup F$ due to planarity. See Figure 7. Hence, this choice is always valid.

The second case is that of $(T\cup F)\backslash e$ being non-degenerate. We show that flipping towards an edge

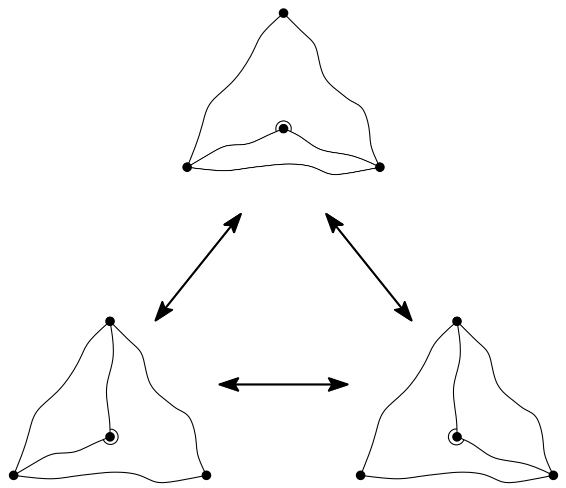

Figure 7: Flip operation for an edge of a triangular face when the 5-face is degenerate.

which has the reflex vertex as an endpoint is always valid. See Figure 9. Denote by $v_1, \ldots, v_5$ the boundary vertices of $(T\cup F)\backslash e$, in counterclockwise order and with v_1 being the reflex vertex. The edge e' we intend to insert is then either v_1v_3 or v_1v_4. Let us focus on the first case, the other one being handled analogously. If $e' = v_1v_3$ is not valid, there has to be already an edge between v_1 and v_3 in the exterior of $T\cup F$. But then at most two vertices of the 3-cycle $v_1v_2v_3$ have their reflex angle outside that cycle, contradicting Lemma 1. See Figure 8.

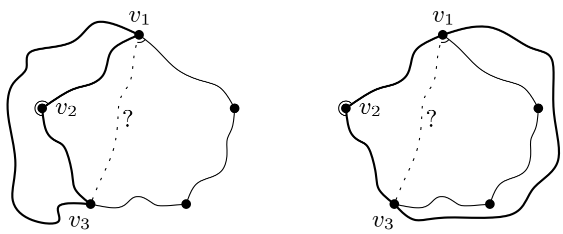

Figure 8: Flipping towards an edge incident to the reflex vertex is always valid.

It remains to prove that in the non-degenerate case there are at least two valid flips for e. Figure 9 shows the possible flips when $T\cup F$ is non-degenerate. Solid arrows indicate flips which are always valid, while dotted arrows indicate flips which might be valid or not.

If e is not incident to the reflex vertex, then there are two valid flips towards edges incident to that reflex vertex. If e is incident to the reflex vertex, there is always a valid flip towards the other diagonal e' incident to that vertex. For a second valid flip, we show that the two remaining diagonals cannot simultaneously give invalid flips. Let the edge to flip be $e = v_1v_3$ (the other case is analogous). In order for both v_2v_4 and v_3v_5 to give invalid flips, the combina-

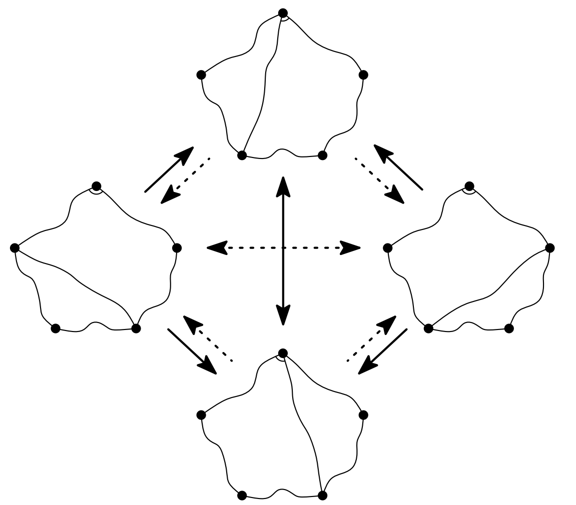

Figure 9: Flip operation for an edge of a triangular face in a combinatorial 4-PPT in the non-degenerate case.

torial 4-PPT must have both edges v_2v_4 and v_3v_5 in the exterior of the 5-face $v_1, \ldots, v_5$. This is impossible

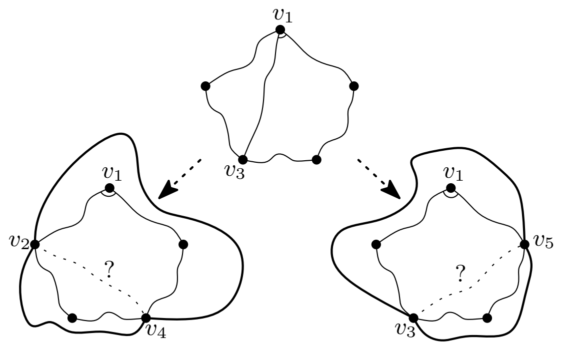

Figure 10: In a non-degenerate 5-face, the two diagonals not incident to the reflex vertex cannot both give invalid flips.

since it would imply a crossing. See Figure 10. $\square$

Lemma 6 *For a given combinatorial 4-PPT with triangular outer face and for any edge b of this outer face, there is a sequence of flips resulting in a combinatorial 4-PPT whose interior triangular face is incident to b.*

Proof. Consider the dual of the original combinatorial 4-PPT and choose a path from the interior triangular face to the outer face, $F_0 = T \rightarrow F_1 \rightarrow F_2 \rightarrow \cdots \rightarrow F_k = F_{\text{outer}}$, such that it reaches the outer face through the edge b in the statement. That is, if e_i is the edge separating faces F_i and F_{i+1} in the path, then $e_{k-1} = b$.

We define the sequence of flips in a way that, after the i-th flip, the interior triangular face T is incident to e_i, which then separates T from F_{i+1}. Thus, after $k-1$ flips T will be incident to $F_k = F_{\text{outer}}$ through $e_{k-1} = b$, as required.

At the i-th flip we consider $T \cup F_i$ and we have to flip an edge e shared by T and F_i, inserting a valid edge e' such that the new triangular face T' is incident to e_i. Two cases arise:

First, if $T \cup F_i$ is degenerate, then there are two edges shared by T and F_i and for each of them there is a unique valid flip, by Lemma 5. In this case, flipping the edge which does not share a vertex with e_i yields the desired result. See again Figure 7.

Second, if $T \cup F$ is non-degenerate, then there is only one edge shared by T and F_i, for which there are two valid flips, by Lemma 5. See Figure 9. For any instance of $T \cup F$ and for any choice of e_i, at least one of the two valid flips makes the new triangular face T' being incident to e_i. $\square$