Triangulations Without Pointed Spanning Trees¹

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Abstract

Problem 50 in the Open Problems Project of the computational geometry community asks whether any triangulation on a point set in the plane contains a pointed spanning tree as a subgraph. We provide a counterexample. As a consequence we show that there exist triangulations which require a linear number of edge flips to become Hamiltonian.

 $Key\ words:$ triangulation, spanning tree, pointed pseudo-triangulation, Hamiltonian cycle, edge flip

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1 Introduction

Let S be a finite set of points in the plane in general position (no three points are on a common line), and let G be a straight-line graph (drawing in the plane) with vertex set S and edges E. A point $p \in S$ is *pointed* in G if there exists an angle less than π that contains all edges incident to p in G. The graph G is pointed if all its vertices are pointed.

A triangulation of S is a maximal planar straight-line graph on the point set S, in the sense that we can not add an edge without making it non-planar. A planar spanning tree of S is a connected, acyclic, plane straight-line graph with vertex set S. Several interesting relations between triangulations and spanning trees exist. For example it is well known that the Delaunay triangulation of S contains a minimum (minimizing the sum of Euclidean edge length) spanning tree of S as a subgraph. Another example is a result of Schnyder [6] who shows that every triangulation of a point set with three extreme vertices allows a partition of its interior edges into three trees.

In this note we disprove the following conjecture which was posed as Problem 10 at the First Gremo Workshop on Open Problems (Stels, Switzerland) in July 2003 (by Bettina Speckmann) and at the CCCG 2003 open-problem session (Halifax, Canada) in August 2003. It later on became Problem 50 in the Open Problems Project of the computational geometry community [3].

Conjecture 1 Every triangulation of a set of points in the plane (in general position) contains a pointed spanning tree as a subgraph.

This conjecture arose while proving sub-structure properties when investigating flips in pointed and non-pointed pseudo-triangulations [1]. *Pseudo*triangulations are a generalization of triangulations. A pseudo-triangle is a planar polygon with exactly three interior angles less than π . A pseudotriangulation of S is a partition of the convex hull of S into pseudo-triangles whose vertex set is S. Pseudo-triangulations have become a versatile data structure. Beside several applications in computational geometry, the rich combinatorial properties of pseudo-triangulations have stimulated much research, see e.g. [1,7] and references therein.

Obviously Conjecture 1 would be true if a triangulation always contained a Hamiltonian path or a pointed pseudo-triangulation as a subgraph. Several triangulations not containing these structures can be found in the literature [5], but for each example it is still easy to find a pointed spanning tree as a sub-



Fig. 1. Triangulation without a spanning path, but containing a pointed spanning tree.

graph. For an example see Figure 1, where the triangulation does not contain a spanning path, but a pointed spanning tree (bold edges). This observation supported the general belief that the conjecture should be true. However, in the next section we provide a (non-trivial) counterexample, constructed on a point set S with 124 points. In Section 3 we discuss some implications of this result, like a lower bound for the number of necessary edge flips to transform a given triangulation such that it contains a Hamiltonian cycle.

2 A Counterexample

Figure 2(a) shows the simplest example of a plane connected straight-line graph not containing a pointed spanning tree as a subgraph. We call this graph a 3-star and it is a spanning tree which is not pointed at its central point.



Fig. 2. Small connected plane straight-line graphs that do not contain a pointed spanning tree as a subgraph: (a) the 3-star (b) the bird graph

The graph defined on the points $a_1, ..., a_6$ in Figure 2(b) is called the *bird graph*. It can be seen as two 3-stars plus one edge. In the next lemma we show that the bird graph does not contain a pointed spanning tree either.

Lemma 2 The bird graph does not contain a pointed spanning tree as a subgraph.

PROOF. Assume that the bird graph contains a pointed spanning tree T as a subgraph. Because of connectivity, the edges a_1a_2 and a_5a_6 are in T. The edge a_2a_5 cannot be in T, as otherwise any edge incident to a_3 would violate pointedness in either a_2 or a_5 . Thus, either a_3 or a_4 has to be connected to both, a_2 and a_5 . This prevents the other vertex, a_4 respectively a_3 , to be connected anyhow. \Box

In a next step, we extend the bird graph by two additional points b_1, b_2 , see Figure 3. Intuitively speaking b_1 and b_2 , respectively, are connected by edges to each visible point of the bird graph. Moreover we add the edge b_1, b_2 . We call the resulting full triangulation of the triangle b_1, b_2, a_1 with interior points $a_2, ..., a_6$ the *cage graph*. All but the three edges forming the outer triangle b_1, b_2, a_1 are called interior edges of the cage graph. The following lemma shows that it will play a crucial role in the construction of a triangulation not containing a pointed spanning tree.



Fig. 3. The cage graph: any connected, pointed, spanning subgraph contains at least one interior edge incident to b_1 or b_2 , respectively.

Lemma 3 Any connected, pointed, spanning subgraph of the cage graph contains at least one interior edge incident to b_1 or b_2 .

PROOF. Let A be a connected, pointed, spanning subgraph of the cage graph. By Lemma 2, the subgraph B of A induced by the points $a_1, ..., a_6$ does not contain a pointed spanning tree. That is, B consists of at least two components. Because of connectivity, A has to include edges that connect these components. For this, at least one interior edge incident to b_1 or b_2 has to be in A. \Box

We are now ready to prove the main result of this note.

Theorem 4 There exist triangulations on a point set in the plane in general position that do not contain a pointed spanning tree as a subgraph.



Fig. 4. A wing graph containing five cage graphs.

PROOF. As indicated in Figure 4 we connect three points a, b, c pairwise by cage graphs (shaded triangles), such that for each cage graph the two connected vertices correspond to b_1 and b_2 in Figure 3. Furthermore, we add four points near point c. These four points form a 3-star and are connected to a, b and c as shown in the figure. Next we add a copy of the whole construction, except the cage graph connecting a to b, mirrored along the line a, b. We call the resulting graph a wing graph, cf. Figure 4. Note that all edges incident to c form three wedges in an obvious way. If we group the edges of a wedge together the resulting graph corresponds to a 3-star with c as its center. The same holds for b and its incident edges. To complete our construction we finally form another 3-star-like graph with center a' by joining three wing graphs at their a-vertices, see Figure 5. Let us denote the resulting graph by \mathcal{G} and its vertex set by \mathcal{S} .

Let G be any planar straight-line graph drawing on S which contains G as a subgraph. Note that G might be a complete triangulation of S. Assume that there exists a pointed spanning tree T as a subgraph of G. Using the same argument as for the 3-star, T does not contain any edge incident to a' in at least one of the three wing graphs. Let W be this wing graph and consider its b and c vertices, see Figure 5. Applying Lemma 3, T has to contain at least one edge in the interior of the cage graph between a' and b incident to b. From the property of the 3-star we can find a cage graph B incident to b such that T does not contain any edge incident to b in B. W.l.o.g. let c be the vertex at



Fig. 5. The counterexample: A graph consisting of three wing graphs.

the other end of B. Again by Lemma 3, T contains at least one edge incident to c in B. The same holds for the cage graph between a' and c. But now, the four additional points near point c cannot be connected by any edge to b or cwithout destroying pointedness at one of these points, and connection to a' is excluded because of the initial choice of W. Therefore, the three bold edges in Figure 4 have to be in T to make T connected. However, the resulting graph is not pointed any more, as the three bold edges form a 3-star. Thus, there exists no pointed spanning tree as a subgraph for G. Adding more points to S in the outer face of G and completing the extended graph with edges to a full triangulation still gives a triangulation that does not contain a pointed spanning tree, as our argumentation is solely based on the interior of fully triangulated areas. \Box

3 Some Implications

As a consequence we get a lower bound on the minimum number of *edge flips* that might be necessary to transform a triangulation such that it contains a pointed spanning tree as a subgraph. An edge e of a triangulation T is called 'flippable' if it is contained in the boundary of two triangles whose union forms

a convex quadrilateral C. By flipping e we mean the operation of removing e from T and replacing it by the other diagonal of C [4]. We combine a linear number of disjoint copies of the 124-point example. After completing the graph on the resulting point set to a full triangulation, at least one flip has to be executed in each copy.

Corollary 5 There exist triangulations on a set of n points in the plane in general position that require $\Omega(n)$ edge flips to contain a pointed spanning tree as a subgraph.

From the Delaunay flip algorithm it follows that a quadratic number of flips is always sufficient to obtain a pointed spanning tree as a subgraph. So far no better upper bound on this flip distance is known.

Of particular interest is the investigation of Hamiltonicity of triangulations. A triangulation is called Hamiltonian if it contains a cycle visiting all vertices exactly once. Let T be a non-Hamiltonian triangulation. What is the minimum number of edge flips that is sufficient to come from T to a Hamiltonian triangulation T'? Note that Hamiltonicity implies the existence of a pointed spanning tree, whereas the reverse is not true in general. Therefore, we conclude from Corollary 5:

Corollary 6 There exist non-Hamiltonian triangulations on a set of n points in the plane in general position that require $\Omega(n)$ edge flips to become Hamiltonian.

The last statement can also be shown in a more direct way. Let a point set S with |S| = n > 5 be given such that the convex hull of S contains 3 points. Then any triangulation on S has 2n - 5 triangles. We place one additional point into each of these triangles and connect it by edges to the corners of the triangle. The set A of inserted points is independent, meaning that there is no edge between any two points of A in the resulting triangulation T on $S \cup A$. Assume there exists a sequence δ of vertices forming a Hamiltonian cycle. Between any two vertices of A in δ there must be at least one vertex of S because A is an independent set. Since |A| > |S|, T cannot be Hamiltonian, i.e., δ cannot exist. We want T to become Hamiltonian by performing a sequence of flips. Any flip connects at most two subgraphs induced by points from A and therefore reduces the number of such subgraphs by at most 1. Hence, a linear number of flips is necessary because |A| is about twice the cardinality of S.

4 Open Problems

There are several related open questions. First, is there a smaller (in the number of points) counterexample to Conjecture 1? What is the smallest such example? Moreover, how fast can we decide whether a given triangulation contains a pointed spanning tree as a subgraph? And if the answer is positive, how fast can we compute this tree? Regarding flipping, what are tight bounds on the required number of edge flips to transform a triangulation such that it contains a pointed spanning tree or a Hamiltonian cycle, respectively?

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