Plane Graphs with Parity Constraints*

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Abstract. Let S be a set of n points in general position in the plane. Together with S we are given a set of parity constraints, that is, every point of S is labeled either even or odd. A graph G on S satisfies the parity constraint of a point $p \in S$, if the parity of the degree of p in G matches its label. In this paper we study how well various classes of planar graphs can satisfy arbitrary parity constraints. Specifically, we show that we can always find a plane tree, a two-connected outerplanar graph, or a pointed pseudo-triangulation which satisfy all but at most three parity constraints. With triangulations we can satisfy about 2/3 of all parity constraints. In contrast, for a given simple polygon H with polygonal holes on S, we show that it is NP-complete to decide whether there exists a triangulation of H that satisfies all parity constraints.

1 Introduction

Computing a simple graph that meets a given *degree sequence* is a classical problem in graph theory and theoretical computer science, dating back to the work of Erdös and Gallai [6]. A degree sequence is a vector $d = (d_1, \ldots, d_n)$ of n positive numbers. It is *realizable*, iff there exists a simple graph whose nodes have precisely this sequence of degrees. Erdös and Gallai gave necessary and sufficient conditions for a degree sequence to be realizable, and several algorithms have been developed that generate a corresponding abstract graph.

An extension of this problem prescribes not only a degree sequence d, but also gives a set $S \subset \mathbb{R}^2$ of n points in general position, where $p_i \in S$ is assigned

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degree d_i. It is well known that a degree sequence d is realizable as a tree if and only if $\sum_{i=1}^{n} d_i = 2n-2$. Tamura and Tamura [11] extended this result to plane (straight line) spanning trees, giving an $O(n^2 \log n)$ time embedding algorithm, which in turn was improved by Bose et al. [4] to optimal $O(n \log n)$ time.

In this paper we study a relaxation of this problem, where we replace exact degrees with degree parity: odd or even. Although parity constrains are significantly weaker than actual degree constrains, they still characterize certain (classes of) graphs. For example, Eulerian graphs are exactly those connected graphs where all vertices have even degree, and a classical theorem of Whitney states that a maximal planar graph is 3-colorable iff all vertices have even degree. A given graph might satisfy only a subset of the parity constraints. So we study how well various classes of planar graphs can satisfy arbitrary parity constraints.

Definitions and notation. Let $S \subset \mathbb{R}^2$ be a set of n points in general position. We denote the convex hull of S by $\mathrm{CH}(S)$. The points of S have parity constraints, that is, every point of S is labeled either *even* or *odd*; for ease of explanation we refer to even and odd points. We denote by n_e and n_o the number of even and odd points in S, respectively. Throughout the paper an even point is depicted by ⊖, an odd point by ⦶, and a point that can be either by ⊕. A graph G on S makes a point $p \in S$ *happy*, if the parity of $\deg_G(p)$ matches its label. If p is not happy, then it is *unhappy*. Throughout the paper a happy point is depicted by ○, an unhappy point by ●, and a point that can be either by ❓.

Results. Clearly, not every arbitrary set of parity constraints can be fulfilled. For example, in any graph the number of odd-degree vertices is even. Hence, the number of unhappy vertices has the same parity as n_o. For the class of plane trees, the aforementioned results on degree sequences immediately imply:

Theorem 1. *On every point set $S \subset \mathbb{R}^2$ with parity constraints, there exists a plane spanning tree that makes (i) all but two points happy if $n_o = 0$, (ii) all but one point happy if n_o is odd, and (iii) all points happy if $n_o \geq 2$ is even.*

We show that we can always find a two-connected outerplanar graph (which is a Hamiltonian cycle with additional edges in the interior, Theorem 2) and a pointed pseudo-triangulation (Theorem 3), which satisfy all but at most three parity constraints. For triangulations (Theorem 4), we can satisfy about 2/3 of the parity constraints. Our proofs are based on simple inductive constructions, but sometimes involve elaborate case distinctions. We also argue that for triangulations the number of unhappy vertices might grow linearly in n. Finally, in Section 5 we show that if we are given a simple polygon H with polygonal holes on S, it is NP-complete to decide whether there exists a triangulation of H that satisfies all parity constraints.

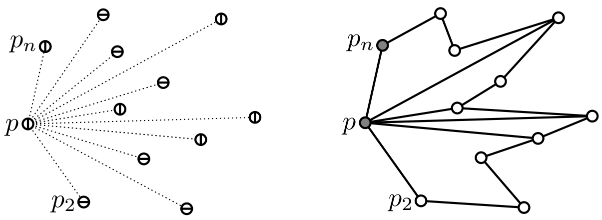
Related work. Many different types of degree restrictions for geometric graphs have been studied. For example, for a given set $S \subset \mathbb{R}^2$ of n points, are there planar graphs on S for which the maximum vertex degree is bounded? There clearly is a path, and hence a spanning tree, of maximum degree at most two. Furthermore, there is always a pointed pseudo-triangulation of maximum degree five [8], although there are point sets where every triangulation must have a vertex of degree $n-1$. Another related question is the following: we are given a set $S \subset \mathbb{R}^2$ of n points, together with a planar graph G on n vertices. Is there a plane straight-line embedding of G on S? Outerplanar graphs are the largest class of planar graphs for which this is always possible, in particular, Bose [3] showed how to compute such an embedding in $O(n \log^2 n)$ time.

One motivation for our work on parity restrictions stems from a bi-colored variation of a problem stated by Erdős and Szekeres in 1935: Is there a number $f^{\mathrm{ES}}(k)$ such that any set $S \subset \mathbb{R}^2$ of at least $f^{\mathrm{ES}}(k)$ bi-colored points in general position has a monochromatic subset of k points that form an empty convex k-gon (that is, a k-gon that does not contain any points of S in its interior)? It has been shown recently [1] that every bi-colored point set of at least 5044 points contains an empty (not necessarily convex) monochromatic quadrilateral. The proof uses, among others, a result that for any point set there exists a triangulation where at least half of the points have odd parity. Any increase in the guaranteed number of odd parity points translates into a lower minimum number of points required in the above statement. More specifically, Theorem 4 below shows that the above result holds for any set of at least 2760 points.

2 Outerplanar graphs

After trees as minimally connected graphs, a natural next step is to consider two-connected graphs. In particular, outerplanar graphs generalize trees both in terms of connectivity and with respect to treewidth. In this section we consider two-connected outerplanar graphs, which are the same as outerplanar graphs with a unique Hamiltonian cycle [5], in other words, simple polygons augmented with a set of pairwise non-crossing diagonals.

The following simple construction makes all but at most three points happy. Pick an arbitrary point p. Set $p_1 = p$ and denote by $p_2, \ldots, p_n$ the sequence of points from S, as encountered by a counterclockwise radial sweep around p, starting from some suitable direction (if p is on $\mathrm{CH}(S)$ towards its counterclockwise neighbor). The outerplanar graph G consists of the closed polygonal chain $P = (p_1, \ldots, p_n)$ plus an edge between p and every odd point in $p_3, \ldots, p_{n-1}$. All points are happy, with the possible exception of p, p_2, and p_n. The figure below shows an example of a point set S with parity constraints and an outerplanar graph on S such that all but two points are happy.



Theorem 2. *For every set $S \subset \mathbb{R}^2$ of n points with parity constraints, there exists an outerplanar graph on S that makes all but at most three points happy.*

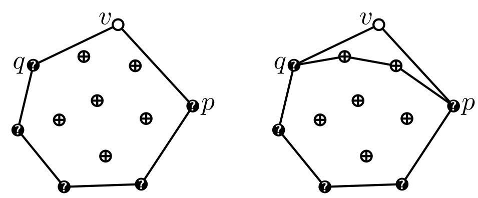
3 Pointed pseudo-triangulations

Pseudo-triangulations are related to triangulations and use *pseudo-triangles* in addition to triangles. A pseudo-triangle is a simple polygon with exactly three interior angles smaller than π. A pseudo-triangulation is called *pointed* if every vertex p has one incident region whose angle at p is greater than π. In the following we describe a recursive construction for a pointed pseudo-triangulation $\mathcal{P}$ on S that makes all but at most three points of S happy.

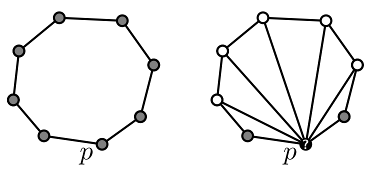
At any time in our construction we have only one recursive sub-problem to consider. This subproblem consists of a point set S^* whose convex hull edges have already been added to $\mathcal{P}$. The current set $\mathcal{P}$ is a pointed set of edges that subdivides the exterior of $\mathrm{CH}(S^*)$ into pseudo-triangles such that all points outside $\mathrm{CH}(S^*)$ are happy. $\mathcal{P}$ contains no edges inside $\mathrm{CH}(S^*)$. We say that S* is *hopeful* if at least one point on $\mathrm{CH}(S^*)$ is made happy by the current version of $\mathcal{P}$. Otherwise, we say that S^* is *unhappy.*

We initialize our construction by setting $S^* = S$ and adding $\mathrm{CH}(S)$ to $\mathcal{P}$. Now we distinguish four cases.

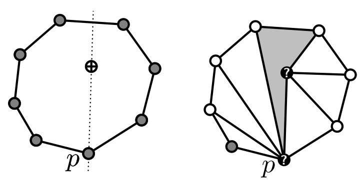
(1) S^* **is hopeful.** Let v be a point on $\mathrm{CH}(S^*)$ that is currently happy, let p and q be its neighbors, and let S' be the (possibly empty) set of points from S that lie in the interior of the triangle $\triangle_{qvp}$. Then $\mathrm{CH}(S' \cup \{p, q\})$ without the edge pq defines a convex chain C from p to q, in a way that C and v together form a pseudo-triangle. (If $S' = \emptyset$, then $C = pq$.) Remove v from consideration by adding C to $\mathcal{P}$. If $|S^*| \geq 5$, recurse on $S^* \setminus \{v\}$. Otherwise, there are at most three unhappy points.

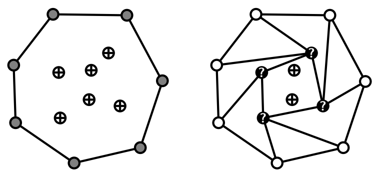


(2) S^* **is unhappy and has no interior points.** Choose one point p on $\mathrm{CH}(S^*)$ and triangulate $\mathrm{CH}(S^*)$ by adding edges from p. There are at most three unhappy points, namely p and its two neighbors.



(3) S^* **is unhappy and has exactly one interior point,** p_i**.** Pick an arbitrary point p on $\mathrm{CH}(S^*)$ and draw a line through p and p_i. This line intersects exactly one edge e of $\mathrm{CH}(S^*)$, and e, p, and p_i together define a pseudo-triangle ∇. Add ∇ to $\mathcal{P}$, which splits $\mathrm{CH}(S^*)$ into two sub-polygons. Triangulate the sub-polygon which contains p_i by adding edges from p_i to all other vertices, except to its neighbors. Similarly, triangulate the other sub-polygon by adding edges from p. There are at most three unhappy points: p, p_i, and a neighbor of p.





(4) **S^* is unhappy and has more than one interior point.** Let S_i be the set of interior points. First add the edges of $\mathrm{CH}(S_i)$ to $\mathcal{P}$. Then connect each point on $\mathrm{CH}(S^*)$ tangentially to $\mathrm{CH}(S_i)$ in clockwise direction, thereby creating a "lens shutter" pattern. Each point on $\mathrm{CH}(S^*)$ is now happy. If $|S_i| > 3$, then recurse on S_i. Otherwise, there are at most three unhappy points.

Theorem 3. *For every point set $S \subset \mathbb{R}^2$ with parity constraints, there exists a pointed pseudo-triangulation on S that makes all but at most three points of S happy.*

4 Triangulations

The final class of planar graphs which we consider are triangulations. If the point set S lies in convex position, then all pseudo-triangulations of S are in fact triangulations. Thus we obtain the following as a consequence of Theorem 3:

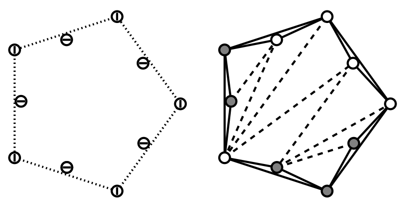
Corollary 1. *For every point set $S \subset \mathbb{R}^2$ in convex position with parity constraints, and any three points p, q, r that are consecutive along $\mathrm{CH}(S)$, there exists a triangulation on S that makes all points of S happy, with the possible exception of p, q, and r.*

The following simple observation will prove to be useful.

Observation 1. *For every set $S \subset \mathbb{R}^2$ of four points in convex position with parity constraints and every $p \in S$ there exists a triangulation on S that makes at least two of the points from $S \setminus \{p\}$ happy.*

For point sets of small cardinality we can investigate the number of happy vertices with the help of the order type data base [2]. For any set of 11 points with parity constraints we can always find a triangulation which makes at least 7 vertices happy. This immediately implies that there is always a triangulation that makes at least $7n/11 \approx 0.63n$ vertices happy.

The figure below shows a double circle for 10 points with parity constraints, such that at most 5 points can be made happy. This is in fact the only point configuration for $n = 10$ (out of 14 309 547) with this property.



Based on the double circle we have been able to construct large examples with a repeating parity pattern (starting at an extreme vertex) $\sigma = \langle (ee(oe)^3ee(oe)^7ee(oe)^5)^3 \rangle$ of length 108, where e denotes even, and o odd parity. It can be shown by inductive arguments that for such configurations for any triangulation we get at least $n/108 + 2$ unhappy vertices. Triangulating the interior of the double circle is equivalent to triangulating a simple polygon, as the inner vertices are connected by *unavoidable edges*, that is, edges that have to be in any triangulation of the set. Hence, all base cases (over 46000) for the required induction can be checked using dynamic programming, see the full version of the paper and [10] for details. Open Problem 1 in [1] asks which is the maximum constant c such that for any point set there always exists a triangulation where $cn - o(n)$ points have odd degree. While for the question as stated we still believe that $c = 1$ is possible, the above construction shows that for general parity constraints we have $c \leq \frac{107}{108}$.

Theorem 4. *For every set $S \subset \mathbb{R}^2$ of n points with parity constraints, there exists a triangulation on S that makes at least $\lceil 2(n-1)/3 \rceil - 6$ points of S happy.*

Proof. Pick an arbitrary point p on CH(S), set $p_1 = p$, and denote by $p_2, \ldots, p_n$ the sequence of points from S, as encountered by a counterclockwise radial sweep around p. Consider the closed polygonal chain $P = (p_1, \ldots, p_n)$ and observe that P describes the boundary of a simple polygon (Fig. 1). With $\angle pqr$ denote the counterclockwise angle between the edges pq and qr around q. A point p_i, $2 \leq i < n$, is *reflex* if the interior angle of P at p_i is reflex, that is, $\angle p_{i-1}p_ip_{i+1} > \pi$; otherwise, p_i is *convex*. Thus, p_1, p_2, and p_n are convex.

We construct a triangulation T on S as follows. As a start, we take the edges of CH(S) and all edges of P, and denote the resulting graph by T_0. If P is convex then T_0 forms a convex polygon. Otherwise CH(S) is partitioned into two or more faces by the edges of P. Thinking of p as a light source and of P as opaque, we call the face of T_0 that contains p the *light face* and the other faces of T_0 *dark faces*. Dark faces are shown gray in figures.

In a next step, we insert further edges to ensure that all faces are convex. The light face is made convex by adding all edges pp_i where p_i is reflex. Hence the light face of T_0 might be split into a number of faces, all of which we refer to as light faces in the following. We partition the dark faces into convex faces as follows. First, we add all edges to connect the subsequence of P that consists of all convex points by a polygonal path. Note that some of those edges may be edges of P or CH(S) and, hence, already be present. Next, we triangulate those dark faces that are not convex. For now, let us say that these faces are triangulated arbitrarily. Later, we add a little twist.

Our construction is based on choosing particular triangulations for those faces that share at least two consecutive edges with P. Let us refer to these

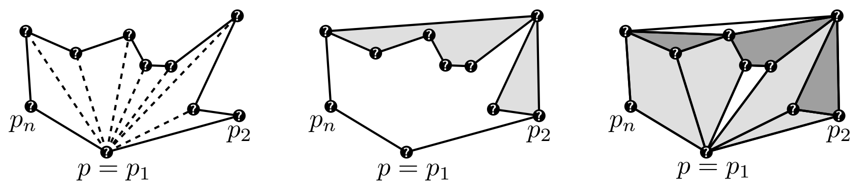


Fig. 1. The simple polygon bounded by P, the initial graph T_0 (with dark faces shown gray), and the graph T_1 in which all faces are convex (interesting light and dark faces shown light gray and dark gray, respectively).

faces as *interesting*, while the remaining ones are called *uninteresting*. The interesting faces can be ordered linearly along P, such that any two successive faces share exactly one edge. We denote this order by $f_1, \ldots, f_m$. Note that f_i is light for i odd and dark for i even, and that both f_1 and f_m are light. Also observe that p is a vertex of every light face; therefore, any interesting light face other than f_1 and f_m has at least four vertices and all uninteresting light faces are triangles. On the dark side, however, there may be both interesting triangles and uninteresting faces with more than three vertices. Similar to above, we triangulate all uninteresting dark faces, for now, arbitrarily (a little twist will come later). We denote the resulting graph by T_1.

As a final step, we triangulate the interesting faces $f_1, \ldots, f_m$ of T_1 in this order to obtain a triangulation on S with the desired happiness ratio. We always treat a light face f_i and the following dark face f_{i+1} together. The vertices that do not occur in any of the remaining faces are *removed*, and the goal is to choose a local triangulation for f_i and f_{i+1} that makes a large fraction of those vertices happy. The progress is measured by the *happiness ratio* h/t, if h vertices among t removed vertices are happy. Note that these ratios are similar to fractions. But in order to determine the collective happiness ratio of two successive steps, the corresponding ratios have to be added component-wise. In that view, for instance, $2/2$ is different from $3/3$.

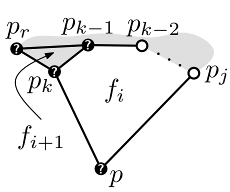
We say that some set of points can be made happy "using a face f", if f can be triangulated—for instance using Corollary 1 or Observation 1—such that all these points are happy. Two vertices are *aligned*, if either both are currently happy or both are currently unhappy. Two vertices that are not aligned are *contrary*. Denote the boundary of a face f by ∂f, and let $\partial f_i = (p, p_j, \ldots, p_k)$, for some $k \geq j + 2$, and $\partial f_{i+1} = (p_{k-1}, \ldots, p_r)$, for some $r \geq k + 1$.

After treating f_i and f_{i+1}, we have removed all vertices up to, but not including, the last two vertices p_{r-1} and p_r of f_{i+1}, which coincide with the first two vertices of the next face f_{i+2}. Sometimes, the treatment of f_i and f_{i+1} leaves the freedom to vary the parity of the vertex p_{r-1} while maintaining the desired happiness ratio as well as the parity of p_r. This means that the future treatment of f_{i+2} and f_{i+3} does not need to take care of the parity of p_{r-1}. By adjusting the triangulation of f_i and f_{i+1} we can always guarantee that p_{r-1} is happy.

Therefore, we distinguish two different settings regarding the treatment of a face pair: no choice (the default setting with no additional help from outside) and 1$^{\text{st}}$ choice (we can flip the parity of the first vertex p_j of the face and, thus, always make it happy).

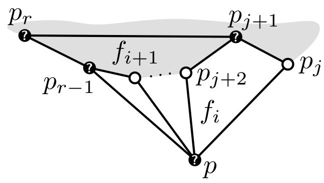
No choice. We distinguish cases according to the number of vertices in f_i.

(1.1) $k \geq j + 3$**,** that is, f_i has at least five vertices. Then $p_j, \ldots, p_{k-2}$ can be made happy using f_i, and $p_{k-1}, \ldots, p_{r-3}$ can be made happy using f_{i+1}. Out of the $r - j - 1$ points removed, at least $(k - 2 - j + 1) + (r - 3 - (k - 1) + 1) = r - j - 2$ are happy. As $r - j \geq 4$, this yields a happiness ratio of at least $2/3$. The figure to the right shows the case $r = k + 1$ as an example.

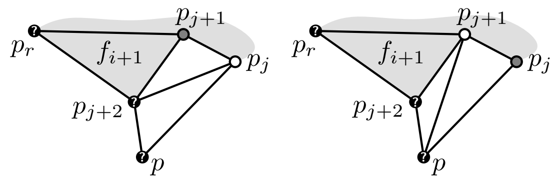


(1.2) $k = j + 2$**,** that is, f_i is a convex quadrilateral. We distinguish subcases according to the number of vertices in f_{i+1}.

(1.2.1) $r \geq j + 4$**,** that is, f_{i+1} has at least four vertices. Using f_{i+1}, all of $p_{j+3}, \ldots, p_{r-2}$ can be made happy. Then at least two out of $p_j, \ldots, p_{j+2}$ can be made happy using f_i. Overall, at least $r - 2 - (j + 3) + 1 + 2 = r - j - 2$ out of $r - j - 1$ removed points are happy. As $r - j \geq 4$, the happiness ratio is at least $2/3$.



(1.2.2) $r = j + 3$**,** that is, f_{i+1} is a triangle. If both p_j and p_{j+1} can be made happy using f_i, the happiness ratio is $2/2$. Otherwise, regardless of how f_i is triangulated exactly one of p_j and p_{j+1} is happy, see the figure to the right. This yields a ratio of $1/2$ and 1st choice for f_{i+2}.

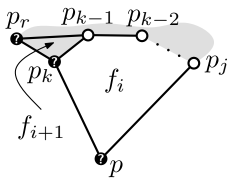


First choice. Denote by f' the other (than f_i) face incident to the edge $p_j p_{j+1}$ in the current graph. As all of $f_1, \ldots, f_{i-1}$ are triangulated already, f' is a triangle whose third vertex (other than p_j and p_{j+1}) we denote by p'. Recall that in the 1st choice setting we assume that, regardless of how f_i is triangulated, p_j can be made happy. More precisely, we assume the following in a 1st choice scenario with a face pair f_i, f_{i+1} to be triangulated: By adjusting the triangulations of $f_1, \ldots, f_{i-1}$, we can synchronously flip the parity of both p_j and p', such that

(C1) All faces $f_i, f_{i+1}, \ldots, f_m$ as well as f' remain unchanged,
(C2) the degree of all of $p_{j+1}, \ldots, p_n$ remains unchanged, and
(C3) the number of happy vertices among $p_2, \ldots, p_{j-1}$ does not decrease.

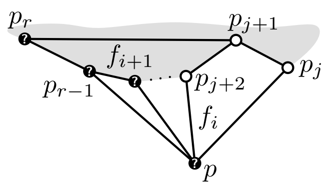
Observe that these conditions hold after Case 1.2.2. Using this 1st *choice flip*, we may suppose that p' is happy. Then by (C3) the number of happy vertices among $\{p_2, \ldots, p_{j-1}\} \setminus \{p'\}$ does not decrease, in case we do the 1st choice flip (again) when processing f_i, f_{i+1}. We distinguish cases according to the number of vertices in f_i.

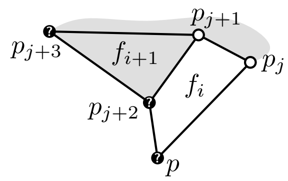
(2.1) $k \geq j + 3$**,** that is, f_i has at least five vertices. Then $p_{j+1}, \ldots, p_{k-1}$ can be made happy using f_i. If f_{i+1} is a triangle (as shown in the figure to the right), this yields a ratio of at least $3/3$. Otherwise ($r \geq k + 2$), apart from keeping p_{k-1} happy, f_{i+1} can be used to make all of $p_k, \ldots, p_{r-3}$ happy. At least $r - j - 2$ out of $r - j - 1$ vertices removed are happy, for a happiness ratio of at least $3/4$.



(2.2) $k = j + 2$**,** that is, f_i is a convex quadrilateral. We distinguish subcases according to the size of f_{i+1}.

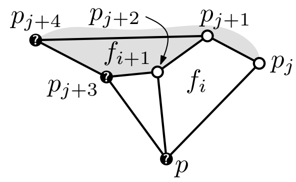
(2.2.1) $r \geq j+5$**,** that is, f_{i+1} has at least five vertices. Triangulate f_i arbitrarily and use f_{i+1} to make all of $p_{j+1}, \ldots, p_{r-3}$ happy. At least $r-j-2$ out of $r-j-1$ vertices removed are happy, for a happiness ratio of at least 3/4.



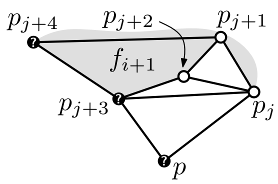


(2.2.2) $r = j+3$**,** that is, f_{i+1} is a triangle. Use f_i to make p_{j+1} happy for a perfect ratio of 2/2.

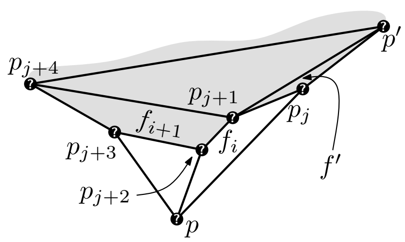
(2.2.3) $r = j+4$**,** that is, f_{i+1} is a convex quadrilateral. If p_{j+1} and p_{j+2} are aligned, then triangulating f_i arbitrarily makes them contrary. Using f_{i+1} both can be made happy, for a perfect 3/3 ratio overall. Thus, suppose that p_{j+1} and p_{j+2} are contrary. We make a further case distinction according to the position of p_j with respect to f_{i+1}.

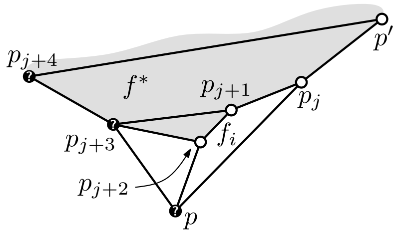


(2.2.3.1) $\angle p_{j+3}p_{j+2}p_j \leq \pi$**,** that is, p, p_j, p_{j+2}, p_{j+3} form a convex quadrilateral. Add edge p_jp_{j+2} and exchange edge pp_{j+2} with edge p_jp_{j+3}. In this way, p_{j+1} and p_{j+2} remain contrary. Hence, both p_{j+1} and p_{j+2} can be made happy using f_{i+1}, for a perfect ratio of 3/3 overall.

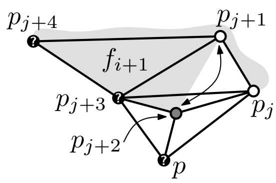


(2.2.3.2) $\angle p_jp_{j+1}p_{j+3} \leq \pi$**,** that is, the points $p_j, p_{j+4}, p_{j+3}, p_{j+1}$ form a convex quadrilateral. To conquer this case we need $p'p_{j+4}$ to be an edge of T_1. In order to ensure this, we apply the before mentioned little twist: before triangulating the non-convex dark faces, we scan through the sequence of dark faces for configurations of points like in this case. Call a dark quadrilateral f_i with $\partial f_i = (p_{j+1}, \ldots, p_{j+4})$ *delicate* if $\angle p_jp_{j+1}p_{j+3} \leq \pi$. For every delicate dark quadrilateral f_i in $f_4, f_6, \ldots, f_{m-1}$ such that f_{i-2} is not delicate, add the edge $p_{j+4}p_h$, where p_h is the first vertex of f_{i-2}. Observe that this is possible as $p_h, \ldots, p_{j+1}, p_{j+3}, p_{j+4}$ form a convex polygon f^*: $p_h, \ldots, p_{j+1}$ and $p_{j+1}, p_{j+3}, p_{j+4}$ form convex chains being vertices of f_{i-2} and f_i, respectively, and p_{j+1} is a convex vertex of f^* because $\angle p_jp_{j+1}p_{j+3} \leq \pi$. Then we triangulate the remaining non-convex and the uninteresting dark faces arbitrarily to get T_1.





To handle this case we join f_{i+1} with f' by removing the edges $p_{j+1}p_{j+4}$ and $p'p_{j+1}$ and adding the edge $p_{j+3}p_{j+1}$, which yields a convex pentagon $f^* = p_{j+4}, p_{j+3}, p_{j+1}, p_j, p'$. Observe that p_{j+1} and p_{j+2} are aligned now. Thus, making p_{j+2} happy using f_i leaves p_{j+1} unhappy. If p' and p_j are aligned, then triangulate f^* using a star from p', making p_{j+1} happy. As p' and p_j remain aligned, both can be made happy—possibly using the 1st choice flip—for a perfect 3/3 ratio. If, on the other hand, p' and p_j are contrary, then triangulate f^* using a star from p_{j+4}, making p_{j+1} happy. Now p' and p_j are aligned and both can made happy—possibly using the 1st choice flip—for a perfect 3/3 ratio.



(2.2.3.3) Neither of the previous two cases occurs and, thus, $p_j, p_{j+1}, p_{j+3}, p_{j+2}$ form a convex quadrilateral f^*. Remove $p_{j+1}p_{j+2}$ and add $p_{j+1}p_{j+3}$ and p_jp_{j+2}. Note that p_j is happy because of 1st choice for f_i, and p_{j+1} and p_{j+2} are still contrary. Therefore, independent of the triangulation of f^*, at least two vertices out of p_j, p_{j+1}, p_{j+2} are happy. Moreover, using f^* we can synchronously flip the parity of both p_{j+1} and p_{j+3} such that (C1)–(C3) hold. This gives us a ratio of 2/3 and 1st choice for f_{i+2}.

Putting things together. Recall that the first face f_1 and the last face f_m are the only light faces that may be triangles. In case that f_1 is a triangle, we just accept that p_2 may stay unhappy, and using f_2 the remaining vertices removed, if any, can be made happy. Similarly, from the last face f_m up to three vertices may remain unhappy. To the remaining faces $f_3, \ldots, f_{m-1}$ we apply the algorithm described above.

In order to analyze the overall happiness ratio, denote by $h_0(n)$ the minimum number of happy vertices obtained by applying the algorithm described above to a sequence $P = (p_1, \ldots, p_n)$ of $n \geq 3$ points in a no choice scenario. Similarly, denote by $h_1(n)$ the minimum number of happy vertices obtained by applying the algorithm described above to a sequence $P = (p_1, \ldots, p_n)$ of $n \geq 3$ points in a 1st choice scenario. From the case analysis given above we deduce the following recursive bounds.

a) $h_0(n) = 0$ and $h_1(n) = 1$, for $n \leq 4$.
b) $h_0(n) \geq \min\{2 + h_0(n-3), 1 + h_1(n-2)\}$.
c) $h_1(n) \geq \min\{3 + h_0(n-4), 2 + h_0(n-2), 2 + h_1(n-3)\}$.

By induction on n we can show that $h_0(n) \geq \lceil (2n-8)/3 \rceil$ and $h_1(n) \geq \lceil (2n-7)/3 \rceil$. Taking the at most four unhappy vertices from f_1 and f_m into account yields the claimed overall happiness ratio. □

5 Triangulating polygons with holes

Theorem 5. *Let H be a polygon with holes and with parity constraints on the vertices. It is NP-complete to decide whether there exists a triangulation of H such that all vertices of H are happy.*

Proof. Following Jansen [7], we use a restricted version of the NP-complete *planar 3-SAT* problem [9], in which each clause contains at most three literals and each variable occurs in at most three clauses.

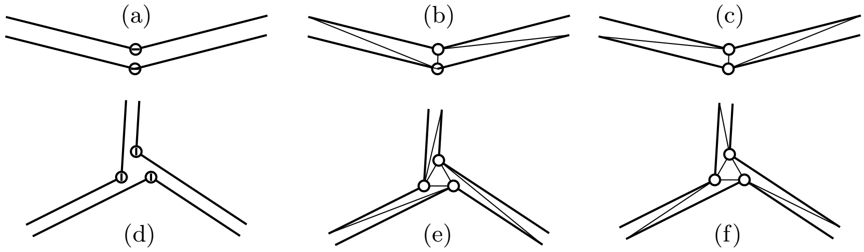


Fig. 2. Wire (a) that transfers TRUE (b), and FALSE (c). The short edge between the two vertices is in every triangulation. A variable (d) in TRUE (e) and FALSE (f) state.

The *edges* of the planar formula are represented by *wires* (Fig. 2(a)–(c)), narrow corridors which can be triangulated in two possible ways, and thereby transmit information between their ends. Negation can easily be achieved by swapping the labels of a single vertex pair in a wire. The construction of a *variable* (Fig. 2(d)–(f)) ensures that all wires emanating from it carry the same state, that is, their diagonals are oriented in the same direction.

To check clauses we use an OR-gate with two inputs and one output wire which we build by cascading two OR-gates and fixing the output of the second gate to true (Fig. 3(b)). The OR-gate is a convex 9-gon with three attached wires, and a *don't-care loop* (Fig. 3(a)) attached to the two top-most vertices. This loop has two triangulations and gives more freedom for the two vertices to which it is attached: they must have an even number of incident diagonals *in total*.

Fig. 4 shows triangulations for the four possible input configurations, where the output is FALSE iff both inputs are false. We have to ensure that the config-

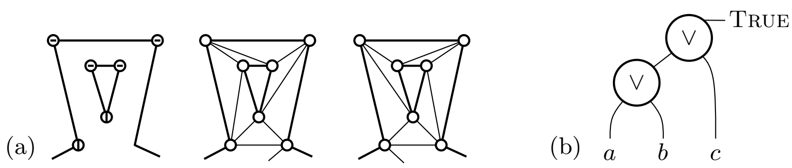


Fig. 3. A don't-care loop (a), checking a clause $a \vee b \vee c$ by joining two OR-gates (b).

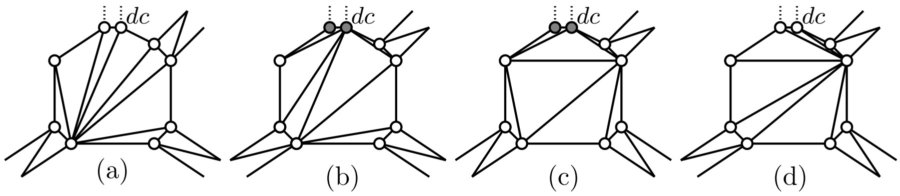


Fig. 4. An OR-gate with inputs FALSE, FALSE (a), TRUE, FALSE (b), FALSE, TRUE (c), and TRUE, TRUE (d). The two inputs are at the lower side and the output is at the upper right side. A don't-care loop *dc* is attached to the two top-most vertices.

uration where both inputs are FALSE and the output is TRUE is infeasible. This can be checked by an exhaustive search of the 429 triangulations of the convex 9-gon. (The output of an OR-gate can be FALSE even if only one input is FALSE; this does not affect the correctness of the clause gadget.)

To combine the constructed elements to a simple polygon H with holes representing a given Boolean formula ϕ is now straightforward. □

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