LARGE BICHROMATIC POINT SETS ADMIT EMPTY MONOCHROMATIC 4-GONS*

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Abstract. We consider a variation of a problem stated by Erdős and Szekeres in 1935 about the existence of a number $f^{\mathrm{ES}}(k)$ such that any set S of at least $f^{\mathrm{ES}}(k)$ points in general position in the plane has a subset of k points that are the vertices of a convex k-gon. In our setting the points of S are colored, and we say that a (not necessarily convex) spanned polygon is monochromatic if all its vertices have the same color. Moreover, a polygon is called empty if it does not contain any points of S in its interior. We show that any sufficiently large bichromatic set of points in $\mathcal{R}^2$ in general position determines at least one empty, monochromatic quadrilateral (and thus linearly many).

Key words. Erdős Szekeres theorem, monochromatic k-gons, colored point sets, empty polygons

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1. Introduction. Throughout this paper all point sets in the plane are assumed to be in *general position*, that is, no three points in the set are collinear. When a subset T of a point set S is the vertex set of a polygon P, we say that S *contains* P, that T *determines* P, or that P is *spanned* by points in S. For the purpose of this paper we will only consider simple, that is, non self intersecting polygons.

Erdős and Szekeres [9] asked the following question. "What is the smallest integer $f^{\mathrm{ES}}(k)$ such that any set of $f^{\mathrm{ES}}(k)$ points contains at least one convex k-gon?" In the mathematical history this problem is also known as the "Happy End Problem", see e.g. [6, 14]. Exact values have been known for $k \leq 5$ and only very recently the case $k = 6$ has been settled by Szekeres and Peters [21] by an exhaustive computer search. For $k \geq 7$ upper bounds exist, but exact values are unknown. See also Erdős and Guy [8] for the related problem on the smallest number of convex k-gons determined by any set of n points in the plane.

In a variation raised by Erdős the k-gons are required to be *empty*, that is, to not contain any point of the point set in its interior. For $k \leq 5$ exact lower bounds on the number of points to guarantee the existence of an empty k-gon are known: As already observed by Esther Klein, every set of 5 points determines an empty convex 4-gon, and 10 points always contain an empty convex pentagon, a fact proved by Harborth [13]. However, Horton showed that there exist arbitrarily large sets of points which do not contain any empty heptagon [15]. Until recently the existence of empty hexagons remained open, but in 2005 Gerken [12] and independently Nicolás [19] proved that every sufficiently large point set contains a convex empty hexagon. Valtr [24] gives a simpler version of Gerken's proof, but requires more points. As for a lower bound it is known that at least 30 points are needed, that is, there exists a set of 29 points without empty convex hexagons [20].

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Several variations on the preceding problems when the points in S belong to different classes – that are usually described as *colors* – were introduced by Devillers et al. [7]. In particular, a (simple) polygon spanned by points in S is called *monochromatic* if all its vertices have the same color. It was proved in [7] that any bichromatic set of n points in the plane determines at least $\lceil \frac{n}{4} \rceil - 2$ monochromatic triangles with pairwise disjoint interiors, which is tight. Later it was shown in [2] that any bichromatic set of n points contains at least $\Omega(n^{5/4})$ empty monochromatic triangles (no disjointness is required), which recently has been improved to $\Omega(n^{4/3})$ [18]. It is conjectured [2] that any bichromatic set of n points in $\mathcal{R}^2$ in general position spans a quadratic number of empty monochromatic triangles. For values of k larger than 3, Devillers et al. show that for $k \geq 5$ and any n there are bichromatic sets of n points where no empty monochromatic convex k-gon exists (Theorem 3.4 in [7]).

It is natural to wonder whether similar results are possible when there are more than two colors. In [7] (Theorem 3.3) this question has been settled by showing that already for three colors there are sets not even spanning any empty monochromatic triangle.

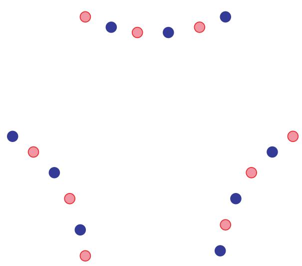

FIG. 1.1. *Example without empty monochromatic* 4*-gons*

Hence, the interesting remaining case is the existence of empty monochromatic (convex) quadrilaterals in bichromatic point sets. Figure 1.1 shows a set with 18 points which does not contain an empty monochromatic convex quadrilateral, and larger examples with 20 [5], 30 [10], 32 [25] and most recently 36 [16] points have been found. However, all these examples do contain non-convex empty monochromatic quadrilaterals, while the one in Figure 1.1 does not.

Notice that every point set that admits an empty convex heptagon will contain an empty monochromatic convex quadrilateral for any bicoloration, because at least four of the vertices of the heptagon will have the same color; however, it is known that for $n \geq 64$ any bichromatic Horton set contains empty monochromatic convex quadrilaterals. These facts led to Conjecture 3.1 in [7], which states that for sufficiently large n any bichromatic set contains at least one empty monochromatic convex quadrilateral.

To date this conjecture has not even been settled for quadrilaterals which are not required to be convex, a weaker version that arose later [11, 17], as no progress for the original question had been obtained[1]. As an important step towards solving the initial problem we show in this paper that this relaxed version of the conjecture is true: if the cardinality of the bichromatic point set S is sufficiently large, there is always an empty (possibly non-convex) monochromatic quadrilateral spanned by S. To this goal, we prove several sufficient conditions and then show that for large point

[1]This weaker form of the 4-gon conjecture was probably first proposed and popularized for many years by J. Pach.

sets at least one of them must hold.

As already mentioned, throughout this paper we assume S to be a set of n points in the plane in general position, that is, no three points of S lie on a common line. Also, for the sake for brevity, we will use the term 4-gon instead of quadrilateral for the rest of this paper. If $S = R\dot{\cup}B$ is a two-color partition of a point set, then we also write $S = (R, B)$, where R is the set of *red*, and B the set of *blue* points, respectively, with $r = |R|$, $b = |B|$, $r, b \geq 0$ and $n = r + b$.

With $CH(S)$ we denote the set of points of S on the boundary of the convex hull of S (the extreme points of S), with $h = |CH(S)|$ their number, and with $i = n - h$ the number of points in the interior of S. Similarly we define $CH(R)$, $h_r = |CH(R)|$, $i_r = r - h_r$, and $CH(B)$.

2. Preliminaries on uncolored point sets. Let us start with a result on triangulations for (uncolored) point sets which is of interest on its own.

LEMMA 2.1. *Let S be a set of n points in general position in the plane and let π be a fixed parity (even or odd). Then there exists a triangulation $T(S)$ of S such that the parity of the degrees in $T(S)$ of at least $2\lfloor\frac{n-1}{4}\rfloor$ points from S is π.*

Proof. Let us assume that all the points in S have different abscissa, which is always possible for a suitable choice of the coordinate system. Consider the points of S being sorted in x-order and group them into sets of 5 consecutive points such that two neighboring groups have one point in common. Each of the $\lfloor\frac{n-1}{4}\rfloor$ groups admits an empty convex 4-gon. Let Q be this set of 4-gons and note that two 4-gons in Q are interior disjoint and share at most one point of S, but no edge. Each of these 4-gons can either be isolated from the others, not sharing a point with any other 4-gon of Q, or connected to a chain of 4-gons from Q. Moreover, from the x-sorting of the groups it follows that chains of 4-gons in Q cannot close a cycle.

Draw the 4-gons of Q and complete them arbitrarily to an initial triangulation $T(S)$ of S by adding edges. For the remainder of the proof the parity of a point $p \in S$ always refers to the parity of the degree of p in the current triangulation $T(S)$, which we are updating whenever necessary. Our goal is to show that we can assign two points of S with parity π to each 4-gon in Q. First note that flipping the diagonal inside a convex 4-gon, i.e., exchanging it with the second diagonal, changes the parity of all four involved points; hence, if we had t, $0 \leq t \leq 4$, points of parity π before the flip then we get $4 - t$ afterwards.

For isolated 4-gons it is straightforward to obtain at least 2 points of parity π. Thus consider a chain of 4-gons in Q. By processing the 4-gons in the chain from left to right we assign to each 4-gon q two points of parity π not using the rightmost point of q. We will call this rightmost point the "connecting" point of the 4-gon. If, after a possible diagonal flip in q, the number of points with parity π in q is at least 3, we choose two non-connecting points. Otherwise we can always flip the diagonal of q in a way such that the two points with parity π do not include the connecting point. As the connecting point is the only element that two 4-gons in the chain might share, the given order allows us to consider the 4-gons independently, meaning without having to take care of restrictions imposed by previous assignments. Therefore we have assigned two different points of parity π to each 4-gon in Q, and we obtain a total of at least $2\lfloor\frac{n-1}{4}\rfloor$ points with parity π. ◻

For odd parity the preceding result can be slightly strengthened to a lower bound of $\frac{n-1}{2}$, by considering unused interior points and a case analysis of small sets. As the improvement is marginal and we believe that a much better result is possible, we skip the details and formulate the following problem instead:

OPEN PROBLEM 1. *Which is the maximum value of a constant c such that for any set S of n points in general position there exists a triangulation $T(S)$ in which at least $cn - o(n)$ points of S have odd (or even) degree?*

Remark: During the preparation of the final version of this paper the previous bound has been improved to guarantee roughly $\frac{2n}{3}$ points with odd degree, see [3]. As this improvement does not change our principle approach and in order to keep the paper self-containing and the proofs as simple as possible, the calculations in the following sections will still use Lemma 2.1. See Section 5 for a brief discussion on how the improved result of [3] influences the required cardinality of the point sets.

Next we consider a fixed triangulation $T(S)$ of S and give lower bounds of how many triangles of $T(S)$ have to be "pierced" (meaning that they contain an obstacle in their interior) so that $T(S)$ does not contain any unpierced (i.e., empty) 4-gon. We will see that the number $\#_{odd}$ of points of S with odd degree in $T(S)$ will play a central role.

LEMMA 2.2. *If for a triangulation $T(S)$ the number of pierced triangles is less than $n + \frac{\#_{odd}-4h-6}{6}$, then there exists an unpierced 4-gon in $T(S)$.*

Proof. Any two adjacent triangles form a 4-gon and at least one of these triangles has to be pierced to prevent unpierced 4-gons. Thus if we consider for a point $p \in S$ all triangles of $T(S)$ incident to p in cyclic order, then every other of these triangles has to be pierced. So if p is an interior point of S then we need to pierce at least $\left\lceil \frac{\delta(p)}{2} \right\rceil$ incident triangles of p, and if p is an extremal point of S then at least $\left\lceil \frac{\delta(p)}{2} \right\rceil - 1$, where $\delta(p)$ is the degree of p in $T(S)$. Note that here points with odd parity contribute a bigger share, as, for example, for inner points two adjacent triangles have to get pierced.

Using this observation and summing up over all points of S we overcount each piercing at most three times, once for each corner of a triangle. So let S_E be the set of points with even edge degree in $T(S)$ and S_O the set of points with odd edge degree in $T(S)$, respectively. Assuming that there is no unpierced 4-gon in $T(S)$ we get as a lower bound for the number of piercings:

$$\frac{\sum\limits_{p\in S}\left\lceil \frac{\delta(p)}{2}\right\rceil - h}{3} = \frac{\sum\limits_{p\in S_E}\frac{\delta(p)}{2} + \sum\limits_{p\in S_O}\frac{\delta(p)+1}{2} - h}{3} = \frac{\sum\limits_{p\in S}\frac{\delta(p)}{2} + \frac{\#_{odd}}{2} - h}{3} =$$

$$n + \frac{\#_{odd} - 4h - 6}{6}.$$

The last equality stems from the fact that $\sum\limits_{p\in S}\frac{\delta(p)}{2}$ is the number of edges in $T(S)$ and thus $3n - h - 3$ by Euler's formula. $\square$

3. Bichromatic sets with small convex hulls. Let now $S = (R, B)$ be a bichromatic set. We will triangulate R and use the results of the previous section to get bounds for piercing the resulting red quadrilaterals with blue points from B. From Lemma 2.2 we immediately get:

LEMMA 3.1. *Let $S = (R, B)$ be a bichromatic point set and $T(R)$ a triangulation of R. With $\#_{odd(R)}$ we denote the number of points of R with odd edge degree in $T(R)$.*

If $b < r + \frac{\#_{odd(R)} - 4h_r - 6}{6}$ then there exists at least one red empty 4-gon consisting of two adjacent triangles in $T(R)$.

A consequence of Lemma 3.1 is a relation between the number of points in R of odd degree in $T(R)$ and the size of the convex hull of R. Namely, if $\#_{odd(R)} > 4h_r + 6 - 6(r-b)$ then there exists at least one empty red 4-gon in the triangulation $T(R)$. We can now combine this fact with the choice of an appropriate triangulation $T(R)$, with $\#_{odd(R)} \geq 2\left\lfloor\frac{r-1}{4}\right\rfloor$, whose existence has been proved in Lemma 2.1, and we get:

PROPOSITION 3.2. *Let $S = (R, B)$ be a bichromatic point set. If $h_r < \frac{\left\lfloor\frac{r-1}{4}\right\rfloor - 3}{2} + \frac{3}{2}(r-b)$, then S contains at least one red empty 4-gon.*

Note that for this result the role of R and B can of course be switched. Proposition 3.2 also shows that the worst case occurs if R and B have the same cardinality. In this case, or more generally for $r \geq b$, we can simplify the bound to $h_r < \frac{r}{8} - 2$. In particular, this proves that if the convex hull of the larger subset has sub-linear size, we immediately get an empty monochromatic 4-gon.

4. Bichromatic sets with large discrepancy. In this section we consider the case that the cardinalities of the red and the blue set differ significantly. As a first step we generalize a result of Sakai and Urrutia [22] on convex empty, monochromatic 4-gons to simple, but not necessarily convex, 4-gons.

LEMMA 4.1. *If in a bichromatic point set $r \geq \frac{3}{2}b + 4$ then there exists at least one empty red 4-gon in R.*

Proof. The proof is based on induction over b and follows the lines given in [22].

Induction base: The case $b = 0$ with $r \geq 4$ is trivially true. For $b = 1$ we have $r \geq 6$ as $r \in \mathbb{N}$. Fix one extremal point $p \in R$ and sort $R \setminus \{p\}$ around p. Connecting the points of $R \setminus \{p\}$ in their order around p and to p results in at least 4 red triangles. As only one of these triangles can be pierced by the only blue point, there exist at least two neighboring unpierced triangles and thus at least one empty red 4-gon in R.

Induction step: Let $b \geq 2$: Consider the supporting line l through an edge of $CH(B)$. Exactly 2 points of B lie on l. The remaining $b' = b - 2$ points of B lie on one side of l, say to the right. If more than 3 points of R lie to the left of l, then they span at least one empty red 4-gon in R. Otherwise we can apply induction on the $b' + r'$ points to the right of l because $r' \geq r - 3 \geq \frac{3}{2}b + 4 - 3 = \frac{3}{2}(b' + 2) + 4 - 3 = \frac{3}{2}b' + 4$. □

Let us recall that for a point set with an even number of extreme points, a quadrangulation is a maximal planar bipartite graph. If the size of the convex hull is odd, we allow one triangle. The number of 4-gons in a quadrangulation of a point set is given in the next observation in terms of n and h, a fact that will be used in Lemma 4.2. For more details and a proof see e.g. [1].

OBSERVATION 1. *A quadrangulation $Q(S)$ on a point set S with n points and h extreme points contains $n - \left\lceil\frac{h}{2}\right\rceil - 1$ (empty) 4-gons.*

Notice that Lemma 4.1 can be rephrased in the form that $\frac{b}{r-4} > \frac{2}{3}$ is a necessary condition for S to not contain any empty monochromatic 4-gons. In combination with Observation 1 this leads to an interesting iterative relation between the size of a set $S = (R, B)$ not containing an empty monochromatic 4-gon and the maximum discrepancy between R and B.

LEMMA 4.2. *Let $k \geq 4$ be a constant, $f(k) = \frac{4}{3}k(2k-1)$, and $g(k) = \frac{k}{k+2}$. If for* *every set $S' = (R', B')$ with $|R'| = r' \geq f(k)$, $|B'| = b'$ the inequality*

$$\frac{b'}{r'-4} > g(k) \tag{4.1}$$

is a necessary condition for S' to not contain an empty monochromatic 4-gon, then for every set $S = (R, B)$ with $r \geq f(k+1)$ the inequality

$$\frac{b}{r-4} > g(k+1) \tag{4.2}$$

is a necessary condition for S to not contain an empty monochromatic 4-gon.

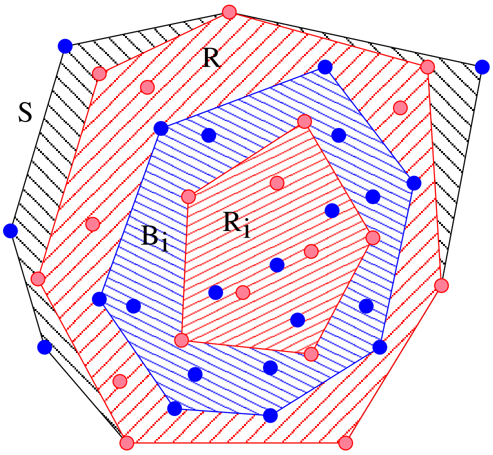

FIG. 4.1. *Red and blue layers in the proof of Lemma 4.2.*

Proof. Consider a set $S = (R, B)$ with $r \geq f(k+1)$ and assume that S does not contain an empty monochromatic 4-gon. Let b_i be the cardinality of the set $B_i \subseteq B$ of blue points in the interior of $CH(R)$, and let r_i be the cardinality of the set $R_i \subseteq R$ of red points in the interior of $CH(B_i)$, see Figure 4.1.

As $r \geq f(k+1) > f(k)$ we can apply (4.1) to the set (R, B_i). Note that blue points of $B \setminus B_i$ would not interfere with empty 4-gons in (R, B_i) as they are outside $CH(R)$. Thus we get the bound

$$\begin{aligned} b_i > \frac{k}{k+2}(r-4) &\geq \frac{k}{k+2}(f(k+1)-4) \\ &= \frac{k}{k+2}\left(\frac{4}{3}(k+1)(2k+1)-4\right) \\ &= \ldots = \frac{4}{3}k(2k-1) = f(k). \end{aligned}$$

Repeating the argument we can thus apply (4.1) to the set (B_i, R_i) and get

$$r_i > \frac{k}{k+2}(b_i - 4). \tag{4.3}$$

Let $\alpha = \frac{b_i}{r-4}$ and thus $b_i = \alpha(r-4)$. By inserting this relation for b_i into (4.3) this inequality rewrites to

$$r_i > \frac{k}{k+2}(\alpha(r-4)-4) = \alpha\frac{k(r-4)}{k+2} - \frac{4k}{k+2}. \tag{4.4}$$

Putting a quadrangulation on R we use Observation 1 to obtain a necessary condition on $\alpha(r-4)$ for S not to contain an empty red 4-gon:

$$\alpha(r-4) = b_i \geq r - \left\lceil \frac{r-i_r}{2} \right\rceil - 1 \geq \frac{r}{2} + \frac{i_r}{2} - 2 \geq \frac{r}{2} + \frac{r_i}{2} - 2. \tag{4.5}$$

Inserting the lower bound (4.4) for r_i into (4.5) we get

$$\alpha(r-4) > \frac{r}{2} + \frac{1}{2}\left(\alpha\frac{k(r-4)}{(k+2)} - \frac{4k}{(k+2)}\right) - 2\,.$$

Using that $k \geq 4$ and thus $r \geq 60$ standard manipulation shows that this is equivalent to

$$\alpha > \frac{k+2}{k+4} - \frac{4k}{(k+4)(r-4)}$$

as a necessary condition such that S does not contain an empty red 4-gon. As $r \geq f(k+1) \geq 2(k+1)(k+2)$ this implies that

$$\alpha > \frac{k+1}{k+3} \tag{4.6}$$

has to hold. Relation (4.6) holds for any fixed set S with $r \geq f(k+1)$ and as $B \supseteq B_i$ we have $b \geq b_i$ which consequently implies Statement (4.2):

$$\frac{b}{r-4} \geq \frac{b_i}{r-4} = \alpha > \frac{k+1}{k+3} = g(k+1)\,.$$

$\square$

We are now ready to show that for sets with sufficiently large cardinality the factor of discrepancy has to be arbitrary small in order to avoid empty monochromatic 4-gons.

PROPOSITION 4.3. *For $l \in \mathbb{N}$, $l \geq 4$, a set $S = (R, B)$ with $r \geq \frac{4}{3}l(2l-1)$ and $b \leq \frac{l}{l+2}(r-4)$ contains an empty monochromatic 4-gon.*

Proof. For $l = 4$ the statement follows directly from Lemma 4.1. For $l > 4$ we iteratively apply Lemma 4.2 for $k = 4 \ldots l-1$, where the precondition for the first iteration is given by Lemma 4.1. Proposition 4.3 then follows from the result of the last step. $\square$

5. Putting things together. As a consequence of Proposition 4.3 we derive a lower bound on the number of extreme points of the red set, which guarantees the existence of empty 4-gons.

LEMMA 5.1. *For $l \in \mathbb{N}$, $l \geq 4$ let $S = (R, B)$ be a set with $r \geq \frac{4}{3}(l+1)(2l+1)$. Then $h_r \geq r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3}$ implies that S contains an empty monochromatic 4-gon.*

Proof. Consider a set $S = (R, B)$ with $r \geq \frac{4}{3}(l+1)(2l+1)$. As in the proof of Lemma 4.2 let b_i be the cardinality of the set $B_i \subseteq B$ of blue points in the interior of $CH(R)$, and let r_i be the cardinality of the set $R_i \subseteq R$ of red points in the interior of $CH(B_i)$, cf. Figure 4.1.

We apply Proposition 4.3 to (R, B_i) and $l+1$. If $b_i \leq \frac{l+1}{l+3}(r-4)$ we have an empty monochromatic 4-gon and we are done. So assume $b_i > \frac{l+1}{l+3}(r-4)$ which, together with the lower bound on r, implies that $b_i \geq \frac{4}{3}l(2l-1)$. We can thus apply now Proposition 4.3 to (B_i, R_i) (switching colors) and l. This implies that $r_i > \frac{l}{l+2}(b_i - 4)$, as otherwise we again have an empty monochromatic 4-gon and are done.

Combining these two lower bounds for b_i and r_i, respectively, we obtain

$$r_i > \frac{l}{l+2}\left(\frac{l+1}{l+3}(r-4)-4\right) = r\frac{l(l+1)}{(l+2)(l+3)} - 8\frac{l}{l+3}\,. \tag{5.1}$$

As $h_r = r - i_r \leq r - r_i$ we can plug in relation 5.1 and get

$$h_r < r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3}$$

as a necessary condition such that S does not contain an empty monochromatic 4-gon, which proves the statement. ◻

By combining the results of Proposition 3.2 and Lemma 5.1 we finally obtain our main result.

THEOREM 5.2. *Every bichromatic set $S = (R, B)$ with $n \geq 5044$ contains an empty monochromatic 4-gon.*

Proof. Without loss of generality, assume that $r \geq b$. Moreover, from Proposition 3.2 we know that $h_r < \frac{\lfloor \frac{r-1}{4} \rfloor - 3}{2} + \frac{3}{2}(r-b)$ is sufficient to obtain an empty monochromatic 4-gon. This can be simplified to $h_r < \frac{r}{8} - 2$. On the other hand, Lemma 5.1 provides $h_r \geq r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3}$ for $l \geq 4$ and $r \geq \frac{4}{3}(l+1)(2l+1)$ as a second sufficient condition. So if

$$\frac{r}{8} - 2 \geq r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3} \tag{5.2}$$

holds, then Theorem 5.2 follows. Using the inequality $r \geq (4/3)(l+1)(2l+1)$ it follows that inequality (5.2) is fulfilled for $l \geq 30$, and thus for any set with $r \geq 2522$; in other words, for any set with $n \geq 5044$ points. ◻

As already mentioned in Section 2 there exists an improved lower bound of at least $\lceil 2(n-1)/3 \rceil - 6$ points we can guarantee to have odd degree in a triangulation of n points [3]. Plugging this result into the above framework reduces the required number of points in Theorem 5.2 to 2760. As this is only a marginal change of the statement we leave the details of the computations to the reader. Note that even if we would be able to find a triangulation which guarantees all points to have odd degree, the lower bound on the cardinality of the point set in Theorem 5.2 still would be 1160.

6. Open problems. Though our results show the existence of empty monochromatic 4-gons in sufficiently large bichromatic point sets, the initial conjecture for convex 4-gons remains open. It seems that the techniques used in our approach cannot be generalized to the convex case, as convexity invalidates several of our lemmas and intermediate results.

It would also be interesting to establish a 3D version of these results for hexahedra consisting of two tetrahedra sharing a face. Let us recall in this respect that Urrutia [23] proved that in any 4-colored point set in $\mathcal{R}^3$ in general position there is at least one empty monochromatic tetrahedron (in fact, a linear number of them).

Let us finally mention again Open Problem 1, which asks which is the maximum constant c such that for any point set there always exists a triangulation where $cn - o(n)$ points have odd (or even) degree. More generally, the question might be stated for any predefined parity assignment, see [3] for more details.

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