Large bichromatic point sets admit empty monochromatic 4-gons*

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Abstract

We consider a variation of a problem stated by Erdős and Szekeres in 1935 about the existence of a number $f^{\mathrm{ES}}(k)$ such that any set S of at least $f^{\mathrm{ES}}(k)$ points in general position in the plane has a subset of k points that are the vertices of a convex k-gon. In our setting the points of S are colored, and we say that a (not necessarily convex) spanned polygon is monochromatic if all its vertices have the same color. Moreover, a polygon is called empty if it does not contain any points of S in its interior. We show that any bichromatic set of $n \geq 5044$ points in $\mathbb{R}^2$ in general position determines at least one empty, monochromatic quadrilateral (and thus linearly many).

1 Introduction

Throughout this paper all point sets in the plane are assumed to be in *general position*, i.e., no three points in the set are collinear. When a subset T of a point set S is the vertex set of a polygon P, we say that S *contains* P, that T *determines* P, or that P is *spanned* by points in S.

Erdős and Szekeres [6] asked the following question. "What is the smallest integer $f^{\mathrm{ES}}(k)$ such that any set of $f^{\mathrm{ES}}(k)$ points contains at least one convex k-gon?" In the mathematical history this problem is also known as the "Happy End Problem", see e.g. [3, 11]. Exact values have been known for $k \leq 5$ and only very recently the case $k = 6$ has been settled by Szekeres and Peters [17] by an exhaustive computer search. For $k \geq 7$ upper bounds exist, but exact values are unknown. See also Erdős and Guy [5] for the related problem on the smallest number of convex k-gons determined by any set of n points in the plane.

In a variation raised by Erdős the k-gons are required to be empty, i.e., to not contain any point of the point set in its interior. For $k \leq 5$ exact lower bounds on the number of points to guarantee the existence of an empty k-gon are known: As already observed by Esther Klein, every set of 5 points determines an empty convex 4-gon, and 10 points always contain an empty convex pentagon, a fact proved by Harborth [10]. However, Horton showed that there exist arbitrarily large sets of points which do not contain any empty heptagon [12]. Until recently the existence of empty hexagons remained open, but in 2005 Gerken [9] and independently Nicolás [15] proved that every sufficiently large point set contains a convex empty hexagon. Valtr [20] gives a simpler version of Gerken's proof, but requires more points. As for a lower bound it is known that at least 30 points are needed, that is, there exists a set of 29 points without empty convex hexagons [16].

Several variations on the preceding problems when the points in S belong to different classes – that are usually described as *colors* – were introduced by Devillers et al. [4]. In particular, a polygon spanned by points in S is called *monochromatic* if all its vertices have the same color, and it was proved in [4] that any bichromatic set of n points in the plane determines at least $\lceil \frac{n}{4} \rceil - 2$ monochromatic triangles with pairwise disjoint interiors, which is tight. Later it was shown in [1] that any bichromatic set of n points contains at least $\Omega(n^{5/4})$ empty monochromatic triangles (no disjointness is required), and they conjectured that any bichromatic set of n points in $\mathbb{R}^2$ in general position spans a quadratic number of empty monochromatic triangles. For values of k larger than 3, Devillers et al. showed that for $k \geq 5$ and any n there are bichromatic sets of n points where no empty monochromatic convex k-gon exists (Theorem 3.4 in [4]).

It is natural to wonder whether similar results are possible when there are more than two colors. In [4] (Theorem 3.3) this question has been settled by showing that already for three colors there are sets not even spanning any empty monochromatic triangle.

Hence, the interesting remaining case is the existence of empty monochromatic (convex) quadrilaterals in bichromatic point sets. Figure 1 shows a set with 18 points which does not contain an empty monochromatic quadrilateral, and larger examples with 20 [2], 30 [7], 32 [21] and most recently 36 [13] points have been found. However, all these examples

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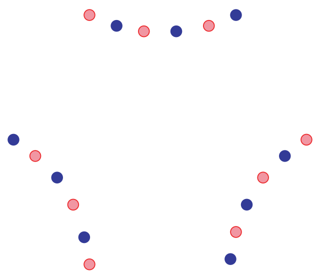

Figure 1: Example without empty monochromatic 4-gons

do contain non-convex empty monochromatic quadrilaterals, while the one in Figure 1 does not.

Notice that every point set that admits an empty convex heptagon will contain an empty monochromatic convex quadrilateral for any bicoloration, because at least four of the vertices of the heptagon will have the same color; however, it is known that for $n \geq 64$ any bichromatic Horton set contains empty monochromatic convex quadrilaterals. These facts led to Conjecture 3.1 in [4], which states that for sufficiently large n any bichromatic set contains at least one empty monochromatic convex quadrilateral.

To date this conjecture has not even been settled for quadrilaterals which are not required to be convex, a weaker conjecture that arose later [8, 14], as no progress in the original formulation had been obtained. In this paper we show that this relaxed version of the conjecture is true: if the cardinality of the bichromatic point set S is sufficiently large, there is always an empty (possibly non-convex) monochromatic quadrilateral spanned by S. To this goal, we prove several sufficient conditions and then show that for large point sets at least one of them must hold.

As already mentioned, throughout this paper we assume S to be a set of n points in the plane in general position, that is, no three points of S lie on a common line. Also, for the sake for brevity, we will use the term 4-gon instead of quadrilateral for the rest of this paper. If $S = R\dot{\cup}B$ is a two-color partition of a point set, then we also write $S = (R, B)$, where R is the set of *red*, and B the set of *blue* points, respectively, with $r = |R|$, $b = |B|$, $r, b \geq 0$ and $n = r + b$.

With $CH(S)$ we denote the set of points of S on the boundary of the convex hull of S (the extreme points of S), with $h = |CH(S)|$ their number and with $i = n - h$ the number of points in the interior of S. Similarly we define $CH(R)$, $h_r = |CH(R)|$, and $i_r = r - h_r$.

2 Preliminaries on uncolored point sets

Let us start with a result on triangulations for (uncolored) point sets which is of interest on its own.

Lemma 1 *Let S be a set of n points in general position in the plane and let π be a fixed parity (even or odd). Then there exists a triangulation $T(S)$ of S such that the parity of the degrees in $T(S)$ of at least $2\lfloor \frac{n-1}{4} \rfloor$ points from S is π.*

Proof. Omitted. □

For odd parity the preceding result can be slightly strengthened to a lower bound of $\frac{n-1}{2}$. As the improvement is marginal and we believe that a much better result is possible, we skip the details and formulate the following problem instead:

Open problem 1 *Which is the maximum value of a constant c such that for any set S of n points in general position there exists a triangulation $T(S)$ in which at least $cn - o(n)$ points of S have odd degree?*

Next we consider a fixed triangulation $T(S)$ of S and give lower bounds of how many triangles of $T(S)$ have to be "pierced" (by placing an obstacle in the interior) so that $T(S)$ does not contain any unpierced (that is, empty) 4-gon. The number $\#_{odd}$ of points of S with odd degree in $T(S)$ plays a central role, as for each interior point with degree δ we need to pierce at least $\lceil \frac{\delta}{2} \rceil$ incident triangles. Summing over all points leads to the following lemma.

Lemma 2 *If for a triangulation $T(S)$ the number of pierced triangles is less than $n + \frac{\#_{odd} - 4h - 6}{6}$, then there exists an unpierced 4-gon in $T(S)$.*

Proof. Omitted. □

3 Bichromatic sets with small convex hulls

Now let $S = (R, B)$ be a bichromatic set. We will triangulate R and use the results of the previous section to get bounds for piercing the resulting red quadrilaterals with blue points from B. From Lemma 2 we immediately get:

Lemma 3 *Let $S = (R, B)$ be a bichromatic point set and $T(R)$ a triangulation of R. With $\#_{odd(R)}$ we denote the number of points of R with odd edge degree in $T(R)$. If $b < r + \frac{\#_{odd(R)} - 4h_r - 6}{6}$ then there exists at least one red empty 4-gon consisting of two adjacent triangles in $T(R)$.*

A consequence of Lemma 3 is a relation between the number of points in R of odd degree in $T(R)$ and the size of the convex hull of R, namely that if $\#_{odd(R)} > 4h_r + 6 - 6(r - b)$ then there exists at least one empty red 4-gon in the triangulation $T(R)$. We can now combine this fact with the choice of an appropriate triangulation $T(R)$, with $\#_{odd(R)} \geq 2\left\lfloor \frac{r-1}{4} \right\rfloor$, whose existence has been proved in Lemma 1, and we get:

Proposition 4 *Let $S = (R, B)$ be a bichromatic point set. If $h_r < \frac{\lfloor \frac{r-1}{4} \rfloor - 3}{2} + \frac{3}{2}(r - b)$, then S contains at least one red empty 4-gon.*

Note that for this result the role of R and B can of course be switched. Proposition 4 also shows that the worst case occurs if R and B have the same cardinality. In this case, or more generally for $r \geq b$, we can simplify the bound to $h_r < \frac{r}{8} - 2$. In particular, this proves that if the convex hull of the larger subset has sub-linear size, we immediately get an empty monochromatic 4-gon.

4 Bichromatic sets with large discrepancy

In this section we consider the case that the cardinalities of the red and the blue set differ significantly. As a first step we generalize a result of Sakai and Urrutia [18] on convex empty, monochromatic 4-gons to simple, but not necessarily convex, 4-gons.

Lemma 5 *If in a bichromatic point set $r \geq \frac{3}{2}b + 4$ then there exists at least one empty red 4-gon in R.*

Proof. The proof is based on induction over b and follows the lines given in [18]. Details are omitted. □

Let us recall that for a point set with an even number of extreme points, a quadrangulation is a maximal planar bipartite graph. If the size of the convex hull is odd, we allow one triangle. The number of 4-gons in a quadrangulation of a point set is given in the next observation in terms of n and h, a fact that will be used in Lemma 6.

Observation 1 *A quadrangulation $Q(S)$ on a point set S with n points and h extreme points contains $n - \lceil \frac{h}{2} \rceil - 1$ empty 4-gons.*

Notice that Lemma 5 can be rephrased in the form that $\frac{b}{r-4} > \frac{2}{3}$ is a necessary condition for S to not contain any empty monochromatic 4-gons. In combination with Observation 1 this leads to an interesting iterative relation between the size of a set $S = (R, B)$ not containing an empty monochromatic 4-gon and the maximum discrepancy between R and B.

Lemma 6 *Let $k \geq 4$ be a constant, $f(k) = \frac{4}{3}k(2k - 1)$, and $g(k) = \frac{k}{k+2}$. If for every set $S' = (R', B')$ with $|R'| = r' \geq f(k)$, $|B'| = b'$ the inequality*

$$\frac{b'}{r' - 4} > g(k) \qquad (1)$$

is a necessary condition for S' to not contain an empty monochromatic 4-gon, then for every set $S = (R, B)$ with $r \geq f(k + 1)$ the inequality

$$\frac{b}{r - 4} > g(k + 1) \qquad (2)$$

is a necessary condition for S to not contain an empty monochromatic 4-gon.

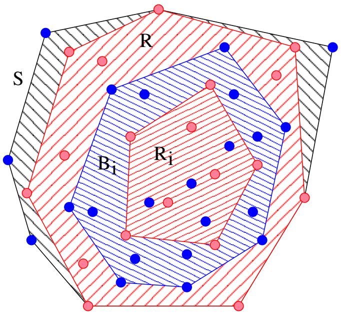

Figure 2: Proof of Lemma 6: red and blue layers.

Proof. The proof is based on the cardinality of nested red and blue convex layers as depicted in Figure 2. Details are omitted. □

We are now ready to show that for sets with sufficiently large cardinality the factor of discrepancy has to be arbitrary small in order to avoid empty monochromatic 4-gons.

Proposition 7 *For $l \in \mathbb{N}$, $l \geq 4$, a set $S = (R, B)$ with $r \geq \frac{4}{3}l(2l - 1)$ and $b \leq \frac{l}{l+2}(r - 4)$ contains an empty monochromatic 4-gon.*

Proof. For $l = 4$ the statement follows directly from Lemma 5. For $l > 4$ we iteratively apply Lemma 6 for $k = 4 \ldots l - 1$, where the precondition for the first iteration is given by Lemma 5. Proposition 7 then follows from the result of the last step. □

5 Putting things together

As a consequence of Proposition 7 we derive a lower bound on the number of extreme points of the red set, which guarantees the existence of empty 4-gons.

Lemma 8 *For $l \in \mathbb{N}$, $l \geq 4$ let $S = (R, B)$ be a set with $r \geq \frac{4}{3}(l+1)(2l+1)$. Then $h_r \geq r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3}$ implies that S contains an empty monochromatic 4-gon.*

Proof. The proof is based on Proposition 7 and on the cardinality of nested convex layers, similar to the proof of Lemma 6. Details are omitted. □

By combining the results of Proposition 4 and Lemma 8 we finally obtain our main result.

Theorem 9 *Every bichromatic set $S = (R, B)$ with $n \geq 5044$ contains an empty monochromatic 4-gon.*

Proof. Without loss of generality, assume that $r \geq b$. From Proposition 4 we know that $h_r < \frac{\lfloor \frac{r-1}{4} \rfloor - 3}{2} + \frac{3}{2}(r-b)$ is a sufficient condition to obtain an empty monochromatic 4-gon. This can be simplified to $h_r < \frac{r}{8} - 2$. On the other hand, Lemma 8 provides $h_r \geq r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3}$ for $l \geq 4$ and $r \geq \frac{4}{3}(l+1)(2l+1)$ as a second sufficient condition. So if

$$\frac{r}{8} - 2 \geq r\frac{2(2l+3)}{(l+2)(l+3)} + \frac{8l}{l+3} \qquad (3)$$

holds, then Theorem 9 follows. Using the inequality $r \geq (4/3)(l+1)(2l+1)$ it follows that inequality (3) is fulfilled for $l \geq 30$, and thus for any set with $r \geq 2522$; in other words, for any set with $n \geq 5044$ points. □

6 Open problems

The existence of convex empty monochromatic 4-gons in sufficiently large bichromatic point sets is still open. It seems that the techniques used in our approach cannot be generalized to the convex case, as convexity invalidates several of our lemmas and intermediate results. Another interesting open question are non-convex empty monochromatic k-gons for $k > 4$.

It would also be interesting to establish a 3D version of these results for hexahedra consisting of two tetrahedra sharing a face. Let us recall in this respect that Urrutia [19] proved that in any 4-colored point set in $\mathbb{R}^3$ in general position there is at least one empty monochromatic tetrahedron (in fact, a linear number of them).

Let us finally mention again Open Problem 1, which asks which is the maximum constant c such that for any point set there always exists a triangulation where $cn - o(n)$ points have odd degree.

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