Maximizing Maximal Angles for Plane Straight-Line Graphs*

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Abstract. Let $G = (S, E)$ be a plane straight-line graph on a finite point set $S \subset \mathbb{R}^2$ in general position. For a point $p \in S$ let the *maximum incident angle* of p in G be the maximum angle between any two edges of G that appear consecutively in the circular order of the edges incident to p. A plane straight-line graph is called φ*-open* if each vertex has an incident angle of size at least φ. In this paper we study the following type of question: What is the maximum angle φ such that for any finite set $S \subset \mathbb{R}^2$ of points in general position we can find a graph from a certain class of graphs on S that is φ-open? In particular, we consider the classes of triangulations, spanning trees, and paths on S and give tight bounds in most cases.

1 Introduction

Conditions on angles in plane straight-line graphs have been studied extensively in discrete and computational geometry. It is well known that Delaunay triangulations maximize the minimum angle over all triangulations, and that in a (Euclidean) minimum weight spanning tree each angle is at least $\frac{\pi}{3}$. In this paper we address the fundamental combinatorial question, what is the maximum value

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α such that for each finite point set in general position there exists a (certain type of) plane straight-line graph where each vertex has an incident angle of size at least α. In other words, we consider $\min - \max - \min - \max$ problems, where we minimize over all finite point sets S in general position in the plane, the maximum over all plane straight-line graphs G (of the considered type), of the minimum over all $p \in S$, of the maximum angle incident to p in G. We present bounds on α for three classes of graphs: spanning paths, (general and bounded degree) spanning trees, and triangulations. Most of the bounds we give are tight. In order to show that, we describe families of point sets for which no graph from the respective class can achieve a greater incident angle at each vertex.

Background. Our motivation for this research stems from the investigation of "pseudo-triangulations", a straight-line framework which apart from deep combinatorial properties has applications in motion planning, collision detection, ray shooting and visibility; see [3, 12, 13, 15, 16] and references therein. Pseudo-triangulations with a minimum number of pseudo-triangles (among all pseudo-triangulations for a given point set) are called *minimum* (or *pointed*) pseudo-triangulations. They can be characterized as plane straight-line graphs where each vertex has an incident angle greater than π. Furthermore, the number of edges in a minimum pseudo-triangulation is maximal, in the sense that the addition of any edge produces an edge-crossing or negates the angle condition.

In comparison to these properties, we consider connected plane straight-line graphs where each vertex has an incident angle α—to be maximized—and the number of edges is minimal (spanning trees) and the vertex degree is bounded (spanning trees of bounded degree and spanning paths). We further show that any planar point set has a triangulation in which each vertex has an incident angle which is at least $\frac{2\pi}{3}$. Observe that perfect matchings can be described as plane straight-line graphs where each vertex has an incident angle of 2π and the number of edges is maximal.

Related Work. There is a vast literature on triangulations that are optimal according to certain criteria, cf. [2]. Similar to Delaunay triangulations which maximize the smallest angle over all triangulations for a point set, farthest point Delaunay triangulations minimize the smallest angle over all triangulations for a convex polygon [9]. If all angles in a triangulation are $\geq \frac{\pi}{6}$ then it contains the relative neighborhood graph as a subgraph [14]. The relative neighborhood graph for a point set connects any pair of points which are mutually closest to each other (among all points from the set). Edelsbrunner et al. [10] showed how to construct a triangulation that minimizes the maximum angle among all triangulations for a set of n points in $O(n^2 \log n)$ time.

In applications where small angles have to be avoided by all means, a Delaunay triangulation may not be sufficient in spite of its optimality because even there arbitrarily small angles can occur. By adding so-called Steiner points one can construct a triangulation on a superset of the original points in which there is some absolute lower bound on the size of the smallest angle [7]. Dai et al. [8] describe several heuristics to construct minimum weight triangulations (triangulations which minimize the total sum of edge lengths) subject to absolute lower or upper bounds on the occurring angles.

Spanning cycles with angle constraints can be regarded as a variation of the traveling salesman problem. Fekete and Woeginger [11] showed that if the cycle may cross itself then any set of at least five points admits a locally convex tour, that is, a tour in which the angle between any three consecutive points is positive. Arkin et al. [5] consider as a measure for (non-)convexity of a point set S the minimum number of (interior) reflex angles (angles $> \pi$) among all plane spanning cycles for S. Aggarwal et al. [4] prove that finding a spanning cycle for a point set which has minimal total angle cost is NP-hard, where the angle cost is defined as the sum of direction changes at the points. Regarding spanning paths, it has been conjectured that each planar point set admits a spanning path with minimum angle at least $\frac{\pi}{6}$ [11]; recently, a lower bound of $\frac{\pi}{9}$ has been presented [6].

Definitions and Notation. Let $S \subset \mathbb{R}^2$ be a finite set of points in general position, that is, no three points of S are collinear. In this paper we consider plane straight-line graphs $G = (S, E)$ on S. The vertices of G are precisely the points in S, the edges of G are straight-line segments that connect two points in S, and two edges of G do not intersect except possibly at their endpoints. For a point $p \in S$ the *maximum incident angle* $\mathrm{op}_G(p)$ of p in G is the maximum angle between any two edges of G that appear consecutively in the circular order of the edges incident to p. For a point $p \in S$ of degree at most one we set $\mathrm{op}_G(p) = 2\pi$. We also refer to $\mathrm{op}_G(p)$ as the *openness* of p in G and call $p \in S$ *φ-open* in G for some angle φ if $\mathrm{op}_G(p) \geq \varphi$. Consider for example the graph depicted in Fig. 1. The point p has four incident edges of G and, therefore, four incident angles. Its openness is $\mathrm{op}_G(p) = \alpha$. The point q has only one incident angle and correspondingly $\mathrm{op}_G(q) = 2\pi$.

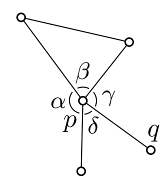

Fig. 1. The incident angles of p.

Similarly we define the *openness* of a plane straight-line graph $G = (S, E)$ as $\mathrm{op}(G) = \min_{p \in S} \mathrm{op}_G(p)$ and call G *φ-open* for some angle φ if $\mathrm{op}(G) \geq \varphi$. In other words, a graph is φ-open if and only if every vertex has an incident angle of size at least φ. The *openness* of a class $\mathcal{G}$ of graphs is the supremum over all angles φ such that for every finite point set $S \subset \mathbb{R}^2$ in general position there exists a φ-open connected plane straight-line graph G on S and G is an embedding of some graph from $\mathcal{G}$. For example, the openness of minimum pseudo-triangulations is π.

Observe that without the general position assumption many of the questions become trivial because for a set of collinear points the non-crossing spanning tree is unique—the path that connects them along the line—and its interior points have no incident angle greater than π.

The convex hull of a point set S is denoted with $CH(S)$. Points of S on $CH(S)$ are called vertices of $CH(S)$. Let a, b, and c be three points in the plane that are not collinear. With $\angle abc$ we denote the counterclockwise angle between the segment (b, a) and the segment (b, c) at b.

Triangulations	Trees	Trees with maxdeg. 3	Paths (convex sets)	Paths (general)
$\frac{2\pi}{3}$	$\frac{5\pi}{3}$	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{4}$

Table 1. Openness of several classes of plane straight-line graphs. All given values except for paths on point sets in general position are tight.

Results. In this paper we study the openness of several well-known classes of plane straight-line graphs, such as triangulations (Section 2), (general and bounded degree) trees (Section 3), and paths (Section 4). The results are summarized in Table 1 above.

2 Triangulations

Theorem 1. *Every finite point set in general position in the plane has a triangulation that is $\frac{2\pi}{3}$-open and this is the best possible bound.*

Proof. Consider a point set $S \subset \mathbb{R}^2$ in general position. Clearly, $\mathrm{op}_G(p) > \pi$ for every point $p \in \mathrm{CH}(S)$ and every plane straight-line graph G on S. We recursively construct a $\frac{2\pi}{3}$-open triangulation T of S by first triangulating $\mathrm{CH}(S)$; every recursive subproblem consists of a point set with a triangular convex hull.

Let S be a point set with a triangular convex hull and denote the three points of $\mathrm{CH}(S)$ with a, b, and c. If S has no interior points, then we are done. Otherwise, let a', b' and c' be (not necessarily distinct) interior points of S such that the triangles $\Delta a'bc$, $\Delta ab'c$ and $\Delta abc'$ are empty (see Fig. 2). Since the sum of the six exterior angles of the hexagon $ba'cb'ac'$ equals 8π, the sum of the three angels $\angle ac'b$, $\angle ba'c$, and $\angle cb'a$ is at least 2π. In particular, one of them, say $\angle cb'a$, is at least $2\pi/3$. We then recurse on the two subsets of S that have $\Delta b'bc$ and $\Delta b'ab$ as their respective convex hulls.

The upper bound is attained by a set S of n points as depicted in Fig. 3. S consists of a point p and of three sets S_a, S_b, and S_c that each contain $\frac{n-1}{3}$ points. S_a, S_b, and S_c are placed at the vertices of an equilateral triangle Δ and p is placed at the barycenter of Δ. Any triangulation T of S must connect p with at least one point of each of S_a, S_b, and S_c and hence $\mathrm{op}_T(p)$ approaches $\frac{2\pi}{3}$ arbitrarily close. □

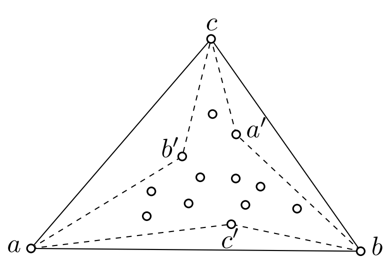

Fig. 2. Constructing a $\frac{2\pi}{3}$-open triangulation.

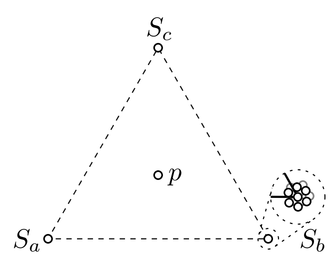

Fig. 3. The openness of triangulations of this point set approaches $\frac{2\pi}{3}$.

3 Spanning Trees

In this section we give tight bounds on the φ-openness of two basic types of spanning trees, namely general spanning trees and spanning trees with bounded vertex degree. Consider a point set $S \subset \mathbb{R}^2$ in general position and let p and q be two arbitrary points of S. Assume w.l.o.g. that p has smaller x-coordinate than q. Let l_p and l_q denote the lines through p and q that are perpendicular to the edge (p, q). We define the *orthogonal slab* of (p, q) to be the open region bounded by l_p and l_q.

Observation 1 *Assume that $r \in S \setminus \{p, q\}$ lies in the orthogonal slab of (p, q) and above (p, q). Then $\angle qpr \leq \frac{\pi}{2}$ and $\angle rqp \leq \frac{\pi}{2}$. A symmetric observation holds if r lies below (p, q).*

Recall that the diameter of a point set is the distance between a pair of points that are furthest away from each other. Let a and b define the diameter of S and assume w.l.o.g. that a has a smaller x-coordinate than b. Clearly, all points in $S \setminus \{a, b\}$ lie in the orthogonal slab of (a, b).

Observation 2 *Assume that $r \in S \setminus \{a, b\}$ lies above a diametrical segment (a, b) for S. Then $\angle arb \geq \frac{\pi}{3}$ and hence at least one of the angles $\angle bar$ and $\angle rba$ is at most $\frac{\pi}{3}$. A symmetric observation holds if r lies below (a, b).*

3.1 General Spanning Trees

Theorem 2. *Every finite point set in general position in the plane has a spanning tree that is $\frac{5\pi}{3}$-open and this is the best possible bound.*

The upper bound is attained by the point set depicted in Fig. 4. Each of the sets $S_i, i \in 1, 2, 3$ consists of $\frac{n}{3}$ points. If a point $p \in S_1$ is connected to any other point from $S_1 \cup S_2$, then it can only be connected to a point of S_3 forming an angle of at least $\frac{\pi}{3} - \varepsilon$. As the same argument holds for S_2 and S_3, respectively, any connected graph, and thus any spanning tree on S is at most $\frac{5\pi}{3}$-open.

The proof for the lower bound is deferred to the appendix.

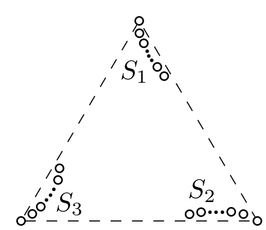

Fig. 4. Every spanning tree of S is at most $\frac{5\pi}{3}$-open.

3.2 Spanning Trees of Bounded Vertex Degree

Theorem 3. *Let $S \subset \mathbb{R}^2$ be a set of n points in general position. There exists a $\frac{3\pi}{2}$-open spanning tree T of S such that every point from S has vertex degree at most three in T. The angle bound is best possible, even for the much broader class of spanning trees of vertex degree at most $n - 2$.*

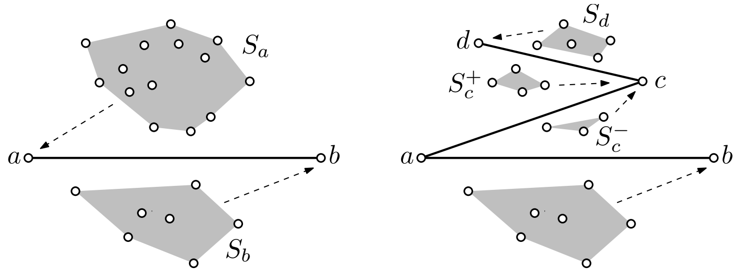

Fig. 5. Constructing a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree four.

Proof. We show in fact that S has a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree three. To do so, we first describe a recursive construction that results in a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree four. We then refine our construction to yield a spanning tree of maximum vertex degree three.

Let a and b define the diameter of S. W.l.o.g. a has a smaller x-coordinate than b. The edge (a, b) partitions $S \setminus \{a, b\}$ into two (possibly empty) subsets: the set S_a of the points above (a, b) and the set S_b of the points below (a, b). We assign S_a to a and S_b to b (see Fig 5). Since all points of $S \setminus \{a, b\}$ lie in the orthogonal slab of (a, b) we can connect any point $p \in S_a$ to a and any point of $q \in S_b$ to b and by this obtain a $\frac{3\pi}{2}$-open path $P = \langle p, a, b, q \rangle$. Based on this observation we recursively construct a spanning tree of vertex degree at most four.

If S_a is empty, then we proceed with S_b. If S_a contains only one point p then we connect p to a. Otherwise consider a diametrical segment (c, d) for S_a. W.l.o.g. d has a smaller x-coordinate than c and d lies above (a, c). Either $\angle adc$ or $\angle dca$ must be less than $\frac{\pi}{2}$. W.l.o.g. assume that $\angle dca < \frac{\pi}{2}$. Hence we can connect d via c to a and obtain a $\frac{3\pi}{2}$-open path $P = \langle d, c, a, b \rangle$. The edge (d, c) partitions S_a into two (possibly empty) subsets: the set S_d of the points above (d, c) and the set S_c of the points below (d, c). The set S_c is again partitioned by the edge (a, c) into a set S_c^+ of points that lie above (a, c) and a set S_c^- of points that lie below (a, c). We assign S_d to d and both S_c^+ and S_c^- to c and proceed recursively.

The algorithm maintains the following two invariants: (i) at most two sets are assigned to any point of S, and (ii) if a set S_p is assigned to a point p then p can be connected to any point of S_p and $\text{op}_T(p) \geq \frac{3\pi}{2}$ for any resulting tree T.

We now refine our construction to obtain a $\frac{3\pi}{2}$-open spanning tree of maximum vertex degree three. If S_c^+ is empty then we assign S_c^- to c, and vice

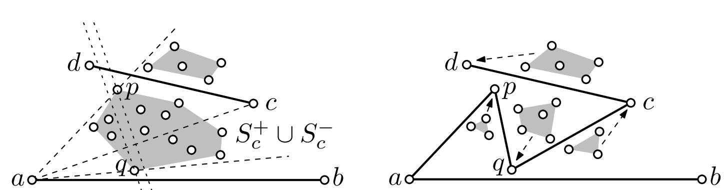

Fig. 6. Constructing a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree three.

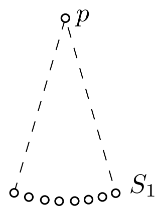

Fig. 7. Every spanning tree of this point set with vertex degree at most $n-2$ is at most $\frac{3\pi}{2}$-open.

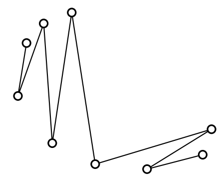

Fig. 8. A zigzag path.

versa. Otherwise, consider the tangents from a to S_c and denote the points of tangency with p and q (see Fig. 6). Let l_p and l_q denote the lines through p and q that are perpendicular to (a, c). W.l.o.g. l_q is closer to a than l_p. We replace the edge (a, c) by the three edges (a, p), (p, q), and (q, c). The resulting path is $\frac{3\pi}{2}$-open and partitions S_c into three sets which can be assigned to p, q, and c while maintaining invariant (*ii*). The refined recursive construction assigns at most one set to every point of S and hence constructs a $\frac{3\pi}{2}$-open spanning tree with maximum vertex degree three.

The upper bound is attained by the set S of n points depicted in Fig. 7. S consists of $n-1$ near-collinear points close together and one point p far away. In order to construct any connected graph with maximum degree $n-2$, one point of S_1 has to be connected to another point of S_1 and to p. Thus any spanning tree on S with maximum degree at most $n-2$ is at most $\frac{3\pi}{2}$-open. □

4 Spanning Paths

Spanning paths can be regarded as spanning trees with maximum vertex degree two. Therefore, the upper bound construction from Fig. 7 applies to paths as well. We will show below that the resulting bound of $\frac{3\pi}{2}$ is tight for points in convex position, even in a very strong sense: There exists a $\frac{3\pi}{2}$-open spanning path starting from any point.

4.1 Point Sets in Convex Position

Consider a set $S \subset \mathbb{R}^2$ of n points in convex position. We can construct a spanning path for S by starting at an arbitrary point $p \in S$ and recursively taking one of the tangents from p to $\mathrm{CH}(S \setminus \{p\})$. As long as $|S| > 2$, there are two tangents from p to $\mathrm{CH}(S \setminus \{p\})$: the left tangent is the oriented line t_ℓ through p and a point $p_\ell \in S \setminus \{p\}$ (oriented in direction from p to p_ℓ) such that no point from S is to the left of t_ℓ. Similarly, the right tangent is the oriented line t_r through p and a point $p_r \in S \setminus \{p\}$ (oriented in direction from p to p_r) such that no point from S is to the right of t_r. If we take the left and the right tangent alternately, see Fig. 8, we call the resulting path a *zigzag* path for S.

Theorem 4. *Every finite point set in convex position in the plane admits a spanning path that is $\frac{3\pi}{2}$-open and this is the best possible bound.*

Proof. As a zigzag path is completely determined by one of its endpoints and the direction of the incident edge, there are exactly n zigzag paths for S. (Count directed zigzag paths: There are n choices for the startpoint and two possible directions to continue in each case, that is, $2n$ directed zigzag paths and, therefore, n (undirected) zigzag paths.)

Now consider a point $p \in S$ and sort all other points of S radially around p, starting with one of the neighbors of p along $\mathrm{CH}(S)$. Any angle that occurs at p in some zigzag path for S is spanned by two points that are consecutive in this radial order. Moreover, any such angle occurs in exactly one zigzag path because it determines the zigzag path completely. Since the sum of all these angles at p is less than π, for each point p at most one angle can be $\geq \frac{\pi}{2}$. Furthermore, if p is an endpoint of a diametrical segment for S then all angles at p are $< \frac{\pi}{2}$. Since there is at least one diametrical segment for S, there are at most $n-2$ angles $> \frac{\pi}{2}$ in all zigzag paths together. Thus, there exist at least two spanning zigzag paths that have no angle $> \frac{\pi}{2}$, that is, they are $\frac{3\pi}{2}$-open.

To see that the bound of $\frac{3\pi}{2}$ is tight, consider again the point set shown in Fig. 7. □

A constructive proof for Theorem 4 is deferred to the appendix. There we also proof the following stronger statement.

Corollary 1 *For any finite set $S \subset \mathbb{R}^2$ of points in convex position and any $p \in S$ there exists a $\frac{3\pi}{2}$-open spanning path for S which has p as an endpoint.*

4.2 General Point Sets

The main result of this section is the following theorem about spanning paths of general point sets.

Theorem 5. *Every finite point set in general position in the plane has a $\frac{5\pi}{4}$-open spanning path.*

Let $S \subset \mathbb{R}^2$ be a set of n points in general position. For a suitable labeling of the points of S we denote a spanning path for (a subset of k points of) S with $\langle p_1, \ldots, p_k \rangle$, where we call p_1 the starting point of the path. Then Theorem 5 directly follows from the following, stronger result.

Theorem 6. *Let S be a finite point set in general position in the plane. Then*

(1) For every vertex q of the convex hull of S, there exists a $\frac{5\pi}{4}$-open spanning path $\langle q, p_1, \ldots, p_k \rangle$ on S starting at q.
(2) For every edge $\overline{q_1 q_2}$ of the convex hull of S there exists a $\frac{5\pi}{4}$-open spanning path starting at either q_1 or q_2 and using the edge $\overline{q_1 q_2}$, that is, a spanning path $\langle q_1, q_2, p_1, \ldots, p_k \rangle$ or $\langle q_2, q_1, p_1, \ldots, p_k \rangle$.

Proof. For each vertex p in a path G the maximum incident angle $\mathrm{op}_G(p)$ is the larger of the two incident angles (except for start- and endpoint of the path). To simplify the case analysis we will consider the smaller angle at each point and prove that we can construct a spanning path such that it is at most $\frac{3\pi}{4}$. We denote with (q, S) a spanning path for S starting at q, and with $(\overline{q_1 q_2}, S)$ a spanning path for S starting with the edge connecting q_1 and q_2. The *outer normal cone* of a vertex y of a convex polygon is the region between two half-lines that start at y, are respectively perpendicular to the two edges incident at y, and are both in the exterior of the polygon.

We prove the statements (1) and (2) of Theorem 6 by induction on $|S|$. The base cases $|S| = 3$ are obviously true.

Induction for (1): Let $\mathcal{K} = CH(S \setminus \{q\})$.

Case 1.1 q lies between the outer normal cones of two consecutive vertices y and z of $\mathcal{K}$, where z lies to the right of the ray $\overrightarrow{qy}$.

Induction on $(\overline{yz}, S \setminus \{q\})$ results in a $\frac{5\pi}{4}$-open spanning path $\langle y, z, p_1, \ldots, p_k \rangle$ or $\langle z, y, p_1, \ldots, p_k \rangle$ of $S \setminus \{q\}$. Obviously $\angle qyz \leq \frac{\pi}{2} < \frac{3\pi}{4}$ and $\angle yzq \leq \frac{\pi}{2} < \frac{3\pi}{4}$, and thus we get a $\frac{5\pi}{4}$-open spanning path $\langle q, y, z, p_1, \ldots, p_k \rangle$ or $\langle q, z, y, p_1, \ldots, p_k \rangle$ for S (see Fig. 9).

Case 1.2 q lies in the outer normal cone of a vertex of $\mathcal{K}$.

Let p be that vertex and let y and z be the two vertices of $\mathcal{K}$ adjacent to p, z being to the right of the ray $\overrightarrow{py}$. The three angles $\angle qpz$, $\angle zpy$ and $\angle ypq$ around p obviously add up to 2π. We consider subcases according to which of the three angles is the smallest, the cases of $\angle qpz$ and $\angle ypq$ being symmetric (see Fig. 10).

Case 1.2.1 $\angle zpy$ is the smallest of the three angles.

Then, in particular, $\angle zpy < \frac{3\pi}{4}$. Assume without loss of generality that $\angle qpz$ is smaller than $\angle ypq$ and, in particular, that it is smaller than π. Since q is in the normal cone of p, $\angle qpz$ is at least $\frac{\pi}{2}$, hence $\angle pzq$ is at most $\frac{\pi}{2} < \frac{3\pi}{4}$. Let $S' = S \setminus \{q, z\}$ and consider the path that starts with q and z followed by (p, S'), that is $\langle q, z, p, p_1, \ldots, p_k \rangle$. Note that $\angle zpp_1 \leq \angle zpy$.

Case 1.2.2 $\angle ypq$ is the smallest of the three angles.

Then $\angle ypq < \frac{3\pi}{4}$. Moreover, in this case all three angles $\angle qpz$, $\angle ypq$ and $\angle zpy$ are at least $\frac{\pi}{2}$, the first two because q lies in the normal cone of p, the latter because it is is not the smallest of the three angles. We have $\angle qyp < \frac{\pi}{2}$ because this angle lies in the triangle containing $\angle ypq \geq \frac{\pi}{2}$, and $\angle ypq < \frac{3\pi}{4}$ by assumption. We iterate on $(\overline{py}, S \setminus \{q\})$ and get a $\frac{5\pi}{4}$-open spanning path

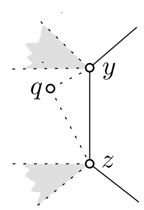

Fig. 9. Case 1.1

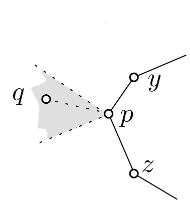

Fig. 10. Case 1.2

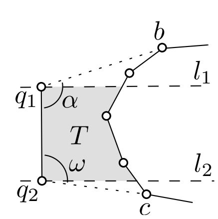

Fig. 11. Case 2

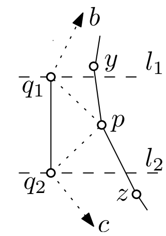

Fig. 12. Case 2.2.1

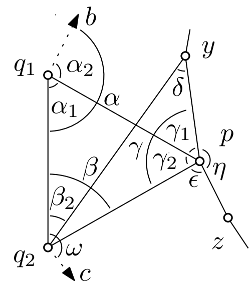

Fig. 13. Case 2.2.1.[1,2]

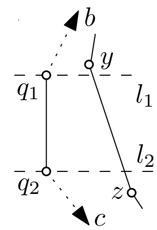

Fig. 14. Case 2.2.2

on $S \setminus \{q\}$ by induction, which can be extended to a $\frac{5\pi}{4}$-open spanning path on S, $\langle q, p, y, p_1, \ldots, p_k\rangle$ or $\langle q, y, p, p_1, \ldots, p_k\rangle$, respectively.

Induction for (2): Let b and c be the neighboring vertices of q_1 and q_2 on $CH(S)$, such that $CH(S)$ reads $\ldots, c, q_2, q_1, b, \ldots$ in clockwise order (see Fig. 11).

Case 2.1 $\alpha < \frac{3\pi}{4}$ or $\omega < \frac{3\pi}{4}$ (see Fig. 11).
Without loss of generality assume that $\alpha < \frac{3\pi}{4}$. By induction on $(q_1, S\setminus\{q_2\})$ we get a $\frac{5\pi}{4}$-open spanning path $\langle q_1, p_1, \ldots, p_k\rangle$ on $S \setminus \{q_2\}$. As $\angle q_2q_1p_1 \leq \alpha < \frac{3\pi}{4}$ we get a $\frac{5\pi}{4}$-open spanning path $\langle q_2, q_1, p_1, \ldots, p_k\rangle$ on S.

Case 2.2 Both α and ω are at least $\frac{3\pi}{4}$.
Let l_1 and l_2 be the lines through q_1 and q_2, respectively, and orthogonal to $\overline{q_1q_2}$. Further let $\mathcal{K} = CH(S \setminus \{q_1, q_2\})$ and with T we denote the region bounded by $\overline{q_1q_2}$, l_1, l_2 and $\mathcal{K}$ (see Fig. 11).

Case 2.2.1 At least one vertex p of $\mathcal{K}$ exists in T.
If there exist several vertices of $\mathcal{K}$ in T, then we choose p as the one with smallest distance to $\overline{q_1q_2}$. Obviously the edges $\overline{q_1p}$ and $\overline{q_2p}$ intersect $\mathcal{K}$ only in p and the angles α_1 and β are each at most $\frac{\pi}{2}$ (see Fig. 12).

Case 2.2.1.1 $\gamma_2 > \frac{\pi}{2}$ (see Fig.13).
By induction on $(p, S\setminus\{q_1, q_2\})$ we get a $\frac{5\pi}{4}$-open spanning path $\langle p, p_1, \ldots, p_k\rangle$ for $S \setminus \{q_1, q_2\}$. Moreover the smaller of $\angle q_2pp_1$ and $\angle p_1pq_1$ is at most $\frac{2\pi - \frac{\pi}{2}}{2} = \frac{3\pi}{4}$. Thus we get a $\frac{5\pi}{4}$-open spanning path $\langle q_1, q_2, p, p_1, \ldots, p_k\rangle$ or $\langle q_2, q_1, p, p_1, \ldots, p_k\rangle$ for S.

Case 2.2.1.2 $\gamma_2 \leq \frac{\pi}{2}$ (see Fig.13).
Let y and z be vertices of $\mathcal{K}$, with y being the clock-wise neighbor of p and z being the counterclockwise one (b might equal y and c might equal z). At least one of α_1 or β is $\geq \frac{\pi}{4}$. Without loss of generality assume that $\beta \geq \frac{\pi}{4}$, the other case is symmetric. Then q_1, q_2, p, y form a convex four-gon because $\alpha \geq \frac{3\pi}{4}$ and $\beta \geq \frac{\pi}{4}$ imply that $\angle bpq_2$ in the four-gon b, q_1, q_2, p is less than π. Therefore also $\gamma \leq \angle bpq_2 < \pi$. We will show that all four angles α_1, γ_1, β_2 and δ are at most $\frac{3\pi}{4}$. Then we apply induction on $(\overline{py}, S \setminus \{q_1, q_2\})$ and get a $\frac{5\pi}{4}$-open spanning path on $S\setminus\{q_1, q_2\}$, which can be completed to a $\frac{5\pi}{4}$-open spanning path for S, $\langle q_2, q_1, p, y, p_1, \ldots, p_k\rangle$ or $\langle q_1, q_2, y, p, p_1, \ldots, p_k\rangle$, respectively.

- Both α_1 and $\beta_2 < \beta$ are clearly smaller than $\frac{\pi}{2}$, hence smaller than $\frac{3\pi}{4}$.

– For γ_1, observe that the supporting line of $\overline{yp}$ must cross the segment $\overline{q_1b}$, so that we have $\alpha_2 + \gamma_1 < \pi$ (they are two angles of a triangle). Also, $\alpha_2 = \alpha - \alpha_1 \geq \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$, so $\gamma_1 < \frac{3\pi}{4}$.
– Finally, for δ we look at the angles around vertex p. By the same arguments used in γ_1, we conclude $\varepsilon < \frac{3\pi}{4}$. Since $\eta < \pi$ as the three vertices z, p and y are on $\mathcal{K}$, we have $\gamma > 2\pi - \pi - \frac{3\pi}{4} = \frac{\pi}{4}$. Considering the angles inside the triangle q_2yp we get $\delta < \pi - \gamma < \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

Case 2.2.2 No vertex of $\mathcal{K}$ exists in T.

Both, l_1 and l_2, intersect the same edge $\overline{yz}$ of $\mathcal{K}$ (in T), with y closer to l_1 than to l_2 (see fig. 14). We will show that the four angles $\angle yzq_1$, $\angle q_2q_1z$, $\angle yq_2q_1$ and $\angle q_2yz$ are all smaller than $\frac{3\pi}{4}$. Then induction on $(\overline{yz}, S\backslash\{q_1, q_2\})$ yields a path that can be extended to a $\frac{5\pi}{4}$-open path $\langle q_2, q_1, z, y, p_1, \ldots, p_k\rangle$ or $\langle q_1, q_2, y, z, p_1, \ldots, p_k\rangle$. Clearly, the angles $\angle q_2q_1z$ and $\angle yq_2q_1$ are both smaller than $\frac{\pi}{2}$. The sum of $\angle q_2yz + \angle cq_2y$ is smaller than π because the supporting line of $\overline{yz}$ intersects the segment $\overline{q_2c}$. Now, $\angle cq_2y$ is at least $\frac{\pi}{4}$ by the assumption that $\angle cq_2q_1 \geq \frac{3\pi}{4}$. So, $\angle q_2yz < \frac{3\pi}{4}$. The symmetric argument shows that $\angle yzq_1 < \frac{3\pi}{4}$. □

Note that for Theorem 6 it is essential that the predefined starting point of a $\frac{5\pi}{4}$-open path is an extreme point of S, as an equivalent result is in general not true for interior points. As a counter example consider a regular n-gon with an additional point in its center. It is easy to see that for sufficiently large n starting at the central point causes a path to be at most $\pi + \varepsilon$-open for a small constant ε. Similar, non-symmetric examples already exist for $n \geq 6$ points, and analogously, if we require an interior edge to be part of the path, there exist examples bounding the openness by $\frac{4\pi}{3} + \varepsilon$ [17]. Despite these examples we conclude this section with the following conjecture.

Conjecture 1 *Every finite point set in general position in the plane has a $\frac{3\pi}{2}$-open spanning path.*

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5 Appendix

5.1 Proof of Theorem 2

We say that an angle φ is *large* if $\varphi > \frac{\pi}{3}$. Correspondingly, if $\varphi \le \frac{\pi}{3}$ then we say that φ is *small*.

Theorem 2. *Every finite point set in general position in the plane has a spanning tree that is $\frac{5\pi}{3}$-open and this is the best possible bound.*

Proof. Consider a point set $S \subset \mathbb{R}^2$ in general position and let a and b define the diameter of S. W.l.o.g. a has a smaller x-coordinate than b. Let $c \in S \setminus \{a, b\}$ be the point above (a, b) that is furthest away from (a, b) and let $d \in S \setminus \{a, b\}$ be the point below (a, b) that is furthest away from (a, b). (The special case that (a, b) is an edge of the convex hull of S and hence either c or d does not exist is handled at the end of the proof.) All points of S lie within the bounding box defined by the orthogonal slab of (a, b) and two lines through c and d parallel to (a, b).

To construct a $\frac{5\pi}{3}$-open spanning tree, we first construct a special $\frac{5\pi}{3}$-open path P whose endpoints are either a and b or c and d. P has the additional property that the smaller angle at its endpoints between the path and the bounding box is also small. We extend P to a spanning tree in the following manner. Every point p_i of P has a small incident angle. Consider the cone C_i with apex p_i defined by the edges of P (and the bounding box if p_i is an endpoint) enclosing the small angle at p_i. When constructing P we ensure that every point p of $S \setminus P$ is contained in exactly one cone C_i. We assemble the spanning tree by connecting each point in $S \setminus P$ to the apex of its containing wedge (see Fig. 15).

It remains to show that we can always find a path P with the properties described above. We will prove this through a case distinction on the size of the angles that are depicted in Fig. 16.

Since (a, b) is diametrical for S, Observation 2 implies that $\gamma \ge \frac{\pi}{3}$ and $\delta \ge \frac{\pi}{3}$. Furthermore, at least one of α_1 and β_1 and one of α_2 and β_2 is small.

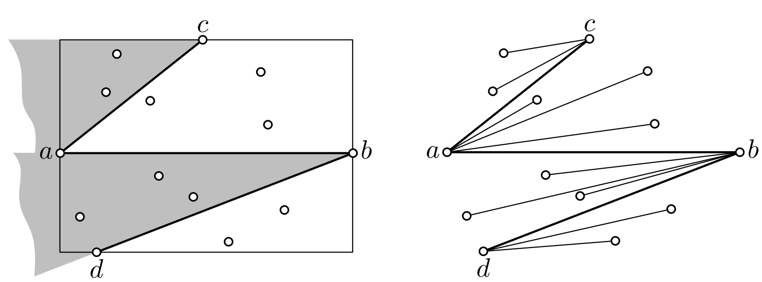


Fig. 15. The path P (thick edges), the cones of c and b (left), the spanning tree constructed from P (right).

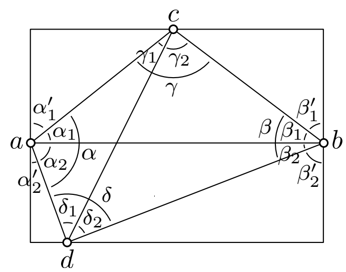


Fig. 16. The bounding box of S with all relevant angles labeled.

Case 1 Neither at a nor at b both angles (α_1 and α_2 or β_1 and β_2, respectively) are large.

This means that either α_1 and β_2 or α_2 and β_1 are small. If α_1 and β_2 are small, then we choose $P = \langle c, a, b, d\rangle$. P is $\frac{5\pi}{3}$-open and the smaller angles at c and d between P and the bounding box are at most $\frac{\pi}{3}$. Furthermore, P partitions $S \setminus \{a, b, c, d\}$ into four subsets and each subset is contained in exactly one of the four cones with apex a, b, c, and d. Symmetrically, if α_2 and β_1 are small, then $P = \langle c, b, a, d\rangle$.

Case 2 Either at a or at b both angles are large.

W.l.o.g. assume that α_1 and α_2 are large and hence β_1 and β_2 are both small.

Case 2.1 $\beta = \beta_1 + \beta_2$ is small.

We choose $P = \langle c, b, d\rangle$ (see Fig. 17). P is $\frac{5\pi}{3}$-open and the smaller angles at c and d between P and the bounding box are at most $\frac{\pi}{3}$. Furthermore, P partitions $S \setminus \{b, c, d\}$ into three subsets and each subset is contained in exactly one of the three cones with apex b, c, and d.

Case 2.2 $\beta = \beta_1 + \beta_2$ is large.

Since α_1 and α_2 are both large it follows that both γ_1 and δ_1 are small (as even their sum is small). Additionally, both $\alpha'_1 = \frac{\pi}{2} - \alpha_1$ and $\alpha'_2 = \frac{\pi}{2} - \alpha_2$ are small. Furthermore, since $\beta = \beta_1 + \beta_2$ is large it follows that at least one of γ_2 and δ_2 and at least one of $\beta'_1 = \frac{\pi}{2} - \beta_1$ and $\beta'_2 = \frac{\pi}{2} - \beta_2$ is small.

Case 2.2.1 Either both β'_1 and γ_2 are small or both β'_2 and δ_2 are small.

If both β'_1 and γ_2 are small then we choose $P = \langle a, d, c, b\rangle$ (see Fig. 18). P is $\frac{5\pi}{3}$-open and partitions $S \setminus \{a, b, c, d\}$ into four subsets which each are contained in exactly one of the four cones with apex a, b, c, and d. Symmetrically, if both β'_2 and δ_2 are small, then $P = \langle a, c, d, b\rangle$.

Case 2.2.2 Either γ_2 is small and β'_1 is large or δ_2 is small and β'_2 is large.

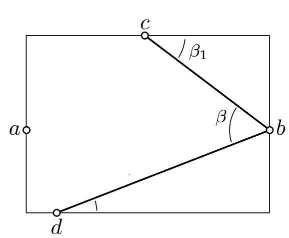


Fig. 17. Case 2.1

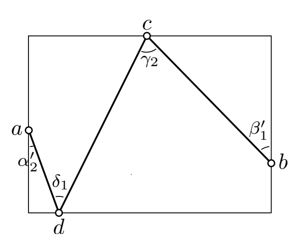


Fig. 18. Case 2.2.1

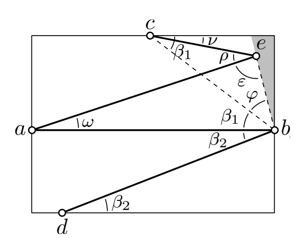


Fig. 19. Case 2.2.2.2

Case 2.2.2.1

If γ_2 is small and β'_1 is large, consider the subset S_b of S that consists of the points above (c, b). If the angle $\angle rbc$ is small for all points $r \in S_b$ then we can still use the construction from Case 2.2.1. If δ_2 is small and β'_2 is large, consider the subset S_b of S that consists of the points below (d, b). If the angle $\angle dbr$ is small for all points $r \in S_b$ then we can again use the construction from Case 2.2.1.

Case 2.2.2.2 γ_2 is small, β'_1 is large, and there is at least one point $p \in S_b$ such that the angle $\angle pbc$ is large.

Let $e \in S_b$ be the point such that $\varphi = \angle ebc$ is largest among the points in S_b. We choose $P = \langle c, e, a, b, d\rangle$ (see Fig. 19). The angle ν is small since it is smaller than β_1, and β_1 is small. Furthermore, φ is large by definition of e and Observation 2 implies that $\angle aeb = \varepsilon$ is at least $\frac{\pi}{3}$. Summing the angles within $\triangle cbe$ yields $\varrho + \beta_1 + \varphi + \varepsilon = \pi$ and therefore $\varrho + \beta_1$ is small. Similarly, the angle sum within $\triangle abe$ is $\omega + \beta_1 + \varphi + \varepsilon = \pi$ and therefore $\omega + \beta_1$ is small. In summary, all of β_2, ω, ϱ, and ν are small and hence P is $\frac{5\pi}{3}$-open. Since the gray-shaded region in Fig. 19 does not contain any points of S by construction, P partitions $S \setminus \{a, b, c, d, e\}$ into five subsets and each subset is contained in exactly one of the five cones with apex a, b, c, d, and e.

If δ_2 is small, β'_2 is large, and there is at least one point $r \in S_b$ such that the angle $\angle dbr$ is large, then P can be constructed similarly.

Finally, if (a, b) is an edge of the convex hull then either c or d does not exist. If c does not exist then we can choose either $P = \langle a, b, d\rangle$ or $P = \langle a, d, b\rangle$. A symmetric argument holds if d does not exist. □

5.2 Constructive Proof for Theorem 4 and Corollary 1

For two distinct points $p, r \in \mathbb{R}^2$ denote by $H^-(p, r)$ the set of points on or to the right of the ray $\overrightarrow{pr}$, that is, those $t \in \mathbb{R}^2$ for which $\angle prt \leq \pi$. Correspondingly, denote by $H^+(p, r)$ the set of points on or to the left of the ray $\overrightarrow{pr}$, that is, those $t \in \mathbb{R}^2$ for which $\angle prt \geq \pi$. Let $S^+(p, r) := S \cap H^+(p, r)$ and $S^-(p, r) := S \cap H^-(p, r)$.

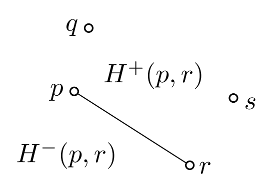


Fig. 20. (p, r) is expanding in direction "+".

Consider a directed segment (p, r), for some $p, r \in S$, and a direction $\tau \in \{+, -\}$. Denote by q and s the neighbors of p and r, respectively, along $\mathrm{CH}(S)$ that are in $S^\tau(p, r)$ (possibly, $q = s$ or even $q = r$ and $s = p$). We call (p, r) *expanding* in direction τ if the two rays $\overrightarrow{qp}$ and $\overrightarrow{sr}$ intersect outside $H^\tau(p, r)$; otherwise, (p, r) is called *non-expanding* in direction τ. Observe that if $|S^\tau(p, r)| \leq 3$ then (p, r) is non-expanding in direction τ.

Theorem 4. *Every finite point set in convex position in the plane admits a spanning path that is $\frac{3\pi}{2}$-open and this is the best possible bound.*

Proof (Constructive proof for Theorem 4). The proof is based on the following more general statement.

Claim 1 *Consider a directed segment (p, r), for some $p, r \in S$, and a direction $\tau \in \{+, -\}$. Denote by q and s the neighbors of p and r, respectively, along $\mathrm{CH}(S)$ that are in $S^\tau(p, r)$ (possibly, $q = s$ or even $q = r$ and $s = p$). Suppose that (p, r) is non-expanding in direction τ and that*

- *if $\tau = +$ then $\angle trp \leq \frac{\pi}{2}$ for all $t \in S^+(p, r) \setminus \{p, r\}$;*
- *if $\tau = -$ then $\angle prt \leq \frac{\pi}{2}$ for all $t \in S^-(p, r) \setminus \{p, r\}$.*

Then there is a $\frac{3\pi}{2}$-open spanning path for $S^\tau(p, r)$ that starts with $\langle p, r\rangle$.

Observe that the condition about the angles above states exactly that $\langle p, r\rangle$ can be extended to a $\frac{3\pi}{2}$-open path by any single point from $S^\tau(p, r) \setminus \{p, r\}$. In particular, all conditions from the claim are fulfilled by any diametrical segment (p, r) of S, for both of its two possible orientations. Therefore, applying the claim to both (p, r) and direction "+" as well as (r, p) and direction "+" yields Theorem 4. □

Proof (of Claim 1).

We use induction on $|S^\tau(p, r)|$. The statement is trivial if $|S^\tau(p, r)| \in \{2, 3\}$. Therefore let $|S^\tau(p, r)| \geq 4$ and consider the segment (q, s). Observe that by convexity of S the segment (q, s) is non-expanding in direction τ and $S^\tau(q, s) = S^\tau(p, r) \setminus \{p, r\}$. From now on, assume that $\tau = +$; the case $\tau = -$ is symmetric.

Case 1 $\angle qsr \geq \frac{\pi}{2}$**.**

Illustrated in Fig. 21a,b — (q, s) fulfills the angle condition, since for every $t \in S^+(q, s) \setminus \{q, s\}$

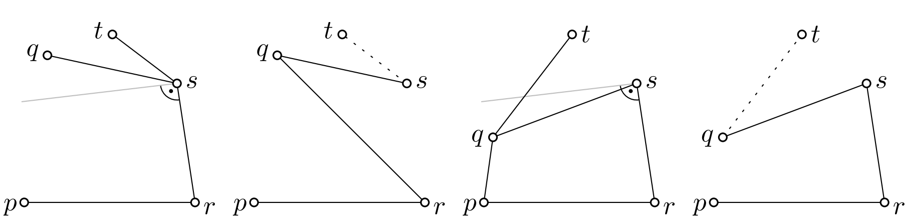
$$\angle tsq = \angle tsr - \angle qsr \leq \angle tsr - \frac{\pi}{2}$$

and $\angle tsr \leq \pi$ by convexity of S. Thus, we can extend $\langle q, s\rangle$ to a $\frac{3\pi}{2}$-open spanning path for $S^+(q, s)$ inductively and that path together with $\langle p, r, q\rangle$ forms a $\frac{3\pi}{2}$-open spanning path for S.

Case 2 $\angle qsr < \frac{\pi}{2}$.

Illustrated in Fig. 21c,d — as (p, r) is non-expanding in direction "+", we have $\angle srp + \angle rpq \leq \pi$. Summing the angles within the quadrilateral (p, r, s, q) yields

$$2\pi = \angle srp + \angle rpq + \angle pqs + \angle qsr < \frac{3\pi}{2} + \angle pqs\,,$$



(a) Case 1: Angles. (b) Case 1: Path. (c) Case 2: Angles. (d) Case 2: Path.

Fig. 21. Constructing a $\frac{3\pi}{2}$-open spanning path.

that is, $\angle pqs > \frac{\pi}{2}$. We conclude that for every $t \in S^-(s,q) \setminus \{q,s\}$

$$\angle sqt = \angle pqt - \angle pqs < \angle pqt - \frac{\pi}{2} \leq \frac{\pi}{2}$$

where $\angle pqt \leq \pi$ by convexity of S. Thus, we can extend $\langle s,q \rangle$ to a $\frac{3\pi}{2}$-open spanning path for $S^-(s,q)$ inductively and that path together with $\langle p,r,s \rangle$ forms a $\frac{3\pi}{2}$-open spanning path for S. □

Corollary 1. *For any finite set $S \subset \mathbb{R}^2$ of points in convex position and any $p \in S$ there exists a $\frac{3\pi}{2}$-open spanning path for S which has p as an endpoint.*

Proof. For $|S| \leq 3$ the statement is trivial. Hence suppose $|S| \geq 4$. Denote by $(p = p_0, p_1, \ldots, p_{n-1})$ the sequence of points along $\mathrm{CH}(S)$ in counterclockwise order and consider the sequence

$$(s_i = (p_i, p_{n-i}))_{i=1\ldots\lfloor (n-1)/2 \rfloor}$$

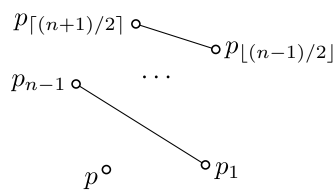


Fig. 22. Segments "parallel to" p.

of segments "parallel to p", as depicted in Fig. 22. Observe that $s_{\lfloor (n-1)/2 \rfloor}$ is non-expanding in direction "−" because there are no more than three points in $S^-(p_{\lfloor (n-1)/2 \rfloor}, p_{\lceil (n+1)/2 \rceil})$. Analogously, s_1 is non-expanding in direction "+". Therefore, the minimum index k, $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, for which s_k is non-expanding in direction "−" is well defined.

If $k = 1$ then s_1 is a segment that is non-expanding for both directions. Otherwise, by the minimality of k the segment s_{k-1} is expanding for direction "−". By definition if s_i is expanding in direction "−" then s_{i+1} is non-expanding in direction "+", for $1 \leq i < \lfloor (n-1)/2 \rfloor$. Thus, in any case s_k is a segment that is non-expanding for both directions.

Suppose there is a point $q \in S^-(p_k, p_{n-k}) \setminus \{p_k, p_{n-k}\}$ for which $\angle p_k p_{n-k} q > \frac{\pi}{2}$. Then the convexity of S implies $\angle r p_{n-k} p_k < \frac{\pi}{2}$ for all $r \in S^+(p_k, p_{n-k}) \setminus \{p_k, p_{n-k}\}$. Moreover, as s_k is non-expanding in direction "−" we have $\angle r p_k p_{n-k} < \frac{\pi}{2}$. Application of Claim 1 to (p_k, p_{n-k}) and $\tau = +$ yields a $\frac{3\pi}{2}$-open spanning path for $S^+(p_k, p_{n-k})$ starting with $\langle p_k, p_{n-k} \rangle$. Similarly, applying Claim 1 to (p_{n-k}, p_k) and $\tau = +$ we obtain a $\frac{3\pi}{2}$-open spanning path for $S^+(p_{n-k}, p_k)$ starting with $\langle p_{n-k}, p_k \rangle$. Combining both paths provides the desired $\frac{3\pi}{2}$-open spanning path for S. This path has p as one of its endpoints by construction.

In a symmetric way, we can handle the case that there is a point $s \in S^+(p_k, p_{n-k}) \setminus \{p_k, p_{n-k}\}$ for which $\angle p_{n-k} p_k s > \frac{\pi}{2}$. Finally, if neither of the points q and s exist, we can apply Claim 1 to (p_k, p_{n-k}) and $\tau = -$ as well as to (p_{n-k}, p_k) and $\tau = -$ and in this way obtain a $\frac{3\pi}{2}$-open spanning path for S which has p as one of its endpoints. □