On the Number of Plane Geometric Graphs

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Abstract

We investigate the number of plane geometric, i.e., straight-line, graphs, a set S of n points in the plane admits. We show that the number of plane geometric graphs and connected plane geometric graphs as well as the number of cycle-free plane geometric graphs is minimized when S is in convex position. Moreover, these results hold for all these graphs with an arbitrary but fixed number of edges. Consequently, we provide a unified proof that the cardinality of any family of acyclic graphs (for example spanning trees, forests, perfect matchings, spanning paths, and more) is minimized for point sets in convex position.

In addition we construct a new extremal configuration, the so-called double zig-zag chain. Most noteworthy this example bears $\Theta^*(\sqrt{72}^n) = \Theta^*(8.4853^n)$ triangulations and $\Theta^*(41.1889^n)$ plane graphs (omitting polynomial factors in both cases), improving the previously known best maximizing examples.

1 Introduction

Let us denote by $\mathcal{S}_n$ the set of sets of n points in the plane in general position, that is, no three points of a set $S \in \mathcal{S}_n$ lie on a common line. With $\Gamma_n \in \mathcal{S}_n$ we denote any set of n points in convex position. Throughout this paper we consider *plane geometric graphs* G on top of $S \in \mathcal{S}_n$. That means that the set of vertices of G is S, edges of G are straight-line segments spanned by vertices of S and two edges of G do not intersect in their interior but might have endpoints in common. From now on we use the term graph to denote plane geometric graphs, unless otherwise noted.

In other words, we consider the rectilinear drawing of the complete graph K_n with vertex set $S \in \mathcal{S}_n$ and study its crossing-free subgraphs. The problem of how large the number of such subgraphs may be has been attracting a lot of attention; many references can be found in the handbook [15] and in the lately published book [10]. It has also been proved recently that the set of crossing-free subgraphs can be realized as a polytope [17].

A fundamental contribution was given by Ajtaj et al. [9]: the number of plane graphs on top of any $S \in \mathcal{S}_n$ is bounded from above by some fixed exponential c^n; the bound $c \leq 10^{13}$ was given there and has been successively improved up to $c \leq 344$ [22]. It is worth mentioning that a main tool developed in [9] is the nowadays famous "Crossing Lemma": every plane drawing of a graph with n vertices and $m > 4n$ edges contains at least cm^3/n^2 crossings, for some constant c. This result, independently proved by Leighton [16], has later found many applications.

In fact, the motivation in [9] was to provide an upper bound *for the number of polygonizations* (crossing-free spanning cycles) on top of any $S \in \mathcal{S}_n$. Obviously the bound for generic plane graphs applies, yet better specific bounds have been obtained for polygonizations as well as for plane triangulations, perfect matchings, spanning trees and many other classes of plane graphs; precise references are given later in this paper.

To describe the asymptotic growth of the number of graphs we use the $\mathcal{O}^*()$-, $\Omega^*()$-, and $\Theta^*()$-notation. In these notations we neglect polynomial factors and just give the dominating exponential term. Moreover when the base of the exponent is explicitly given as a numerical value, this has to be seen as an approximation up to the given precision.

Maximal plane graphs, i.e., triangulations, are a case of special interest, because any plane graph can be completed to a triangulation and hence any upper bound $\mathcal{O}^*(\alpha^n)$ on the number of triangulations implies a corresponding upper bound $\mathcal{O}^*(2^{3n}\alpha^n) = \mathcal{O}^*((8\alpha)^n)$ on the number of generic plane graphs, because every triangulation has at most $3n-6$ edges and therefore contains at most 2^{3n} subgraphs, see also Table 2. The current best upper bound for the number of triangulations of 43^n was very recently obtained by Sharir and Welzl [22], improving the $\mathcal{O}^*(59^n)$ bound of Santos and Seidel [20]. The aforementioned bound of $\mathcal{O}^*(344^n)$ for plane graphs is derived from that[1].

On the opposite direction, it is also known that every $S \in \mathcal{S}_n$ admits at least $\Omega^*(2.33^n)$ triangulations, and it has been conjectured that the number of triangulations is minimized when S is the point set called *double circle*, that has $\Theta^*(\sqrt{12}^{\,n})$ triangulations [5].

In this paper we obtain new lower and upper bounds for the maximum and minimum, respectively, number of plane geometric graphs of different types. All given bounds will be exponential bounds of the form α^n where the goal is to optimize the base α.

More precisely, in Section 2 we prove that the number of plane graphs of several classes is minimized by point sets in convex position, a fact that was known for perfect matchings, spanning trees and spanning paths [14, 21]. Here we provide a unified approach that encompasses those results and extends to many more classes.

In Sections 3 and 4 we turn our attention to upper bounds and, in particular, we prove the existence of a certain point set that has $\Theta^*(\sqrt{72}^{\,n}) = \Theta^*(8.4853^n)$ triangulations and $\Theta^*(41.1889^n)$ plane graphs, improving the previously known best maximizing examples and disproving the common belief that the tight upper bound for the number of triangulations would be $\Theta^*(8^n)$.

We use the following notation for the indicated classes of graphs on top of $S \in \mathcal{S}_n$: $\mathsf{sc}(S)$: spanning cycles (Hamiltonian cycles, polygonizations); $\mathsf{pm}(S)$: perfect matchings; $\mathsf{sp}(S)$: spanning paths (Hamiltonian paths); $\mathsf{tr}(S)$: triangulations; $\mathsf{ppt}(S)$: pointed pseudo triangulations; $\mathsf{pt}(S)$: pseudo triangulations; $\mathsf{st}(S)$: spanning trees; $\mathsf{cf}(S)$: cycle-free graphs (forests); $\mathsf{cg}(S)$: connected graphs; $\mathsf{pg}(S)$: all plane graphs. We will use the notation $\mathsf{sc}(n)$ (similar for the other classes) if a given property holds for any point set of cardinality n. The following hierarchy shows these classes of plane graphs:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\left.\begin{array}{l}\text{triangulations }\mathsf{tr}(S)\\ \text{pointed pseudo triangulations }\mathsf{ppt}(S)\end{array}\right\}\begin{array}{l}\text{pseudo}\\ \text{triangulations }\mathsf{pt}(S)\end{array}\\
\text{spanning cycles }\mathsf{sc}(S)\\
\text{spanning trees }\mathsf{st}(S)\\
\text{spanning paths }\mathsf{sp}(S)
\end{array}\right\}\begin{array}{l}\text{connected}\\ \text{graphs}\\ \mathsf{cg}(S)\end{array}\\
\text{cycle-free graphs (forests) }\mathsf{cf}(S)\\
\text{perfect matchings }\mathsf{pm}(S)
\end{array}\right\}\begin{array}{l}\text{all}\\ \text{plane}\\ \text{graphs}\\ \mathsf{pg}(S)\end{array}$$

As mentioned we will show that with the exception of triangulations the cardinality of all these classes is minimized for point sets in convex position. The accurate statement for triangulations would be that the number of plane graphs with $k = 2n-3$ edges is minimized for sets in convex position.

2 Convexity Minimizes

In the following subsections we provide injective mappings of all plane graphs of Γ_n to any set of $\mathcal{S}_n$ such that the number of edges is retained.

[1]In [22] the authors mention further possible improvements for the base constants for the number of triangulations below 43. This will directly implicate further improvements for all related bounds.

2.1 Injective Mappings

Consider the set of all plane graphs $\mathsf{pg}(\Gamma_n)$ on top of Γ_n, an arbitrary set $S \in \mathcal{S}_n$ together with its set of plane graphs, $\mathsf{pg}(S)$. We will show that we can map $G \in \mathsf{pg}(\Gamma_n)$ to a graph $G' \in \mathsf{pg}(S)$ in an injective way such that the number of edges of G and G' is the same. We will provide different mappings to utilize the special properties of connected or cycle-free graphs.

2.1.1 Mapping for Plane Geometric Graphs

Let G, Γ_n, and S be given as defined above. We first fix root vertices $r \in \Gamma_n$ and $r' \in S$. Each root vertex has to be chosen as an arbitrary, but unique extreme vertex[2]. We then label the remaining vertices of Γ_n in clockwise order around r and the remaining vertices of S around r', respectively, see Figure 1. Consider the (possibly empty) fan of all vertices of Γ_n connected to r in G and connect the vertices with corresponding labels in S to r' in G'. By extending the inserted edges to rays as indicated in the right part of Figure 1, we are left with subsets of equal number of vertices for both, Γ_n and S. We proceed on these subsets in a recursive manner, using the just connected vertices as new, 'local' root vertices. Note that the extension of edges to rays is limited to the interior of each subset. Each local root conquers the set of all vertices in the wedge to its left, including the left-neighbored local root if it exists. The last vertex added to r (in clockwise order) plays a double-role, as it is the local root for its left and right wedge, respectively.

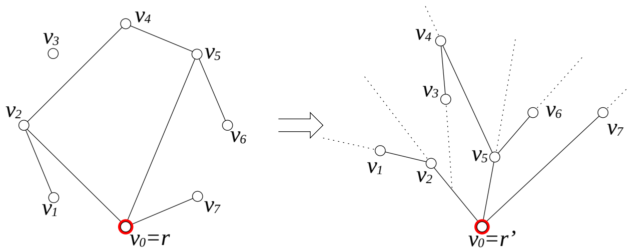

Figure 1: Injective mapping of a plane geometric graph G of Γ_n.

Note that in each recursive step the vertices of a subset are locally relabeled in both sets according to their clockwise order around the local root vertex. As all vertices of a subset lie in a half-plane defined by a line through the local root vertex, there exists a canonical vertex of the subset to start the labeling from. For example, in a second recursive step in Figure 1, v_4 is the local root of the set $\{v_2, v_3, v_4\}$. As the vertices are locally resorted around v_4 the edge v_2v_4 of G maps to the edge v_3v_4 of G'. This shows why the given mapping does not guarantee isomorphic graphs. Another, simpler example is given in Figure 3. See Subsection 2.1.3 for isomorphic mappings for cycle-free plane graphs.

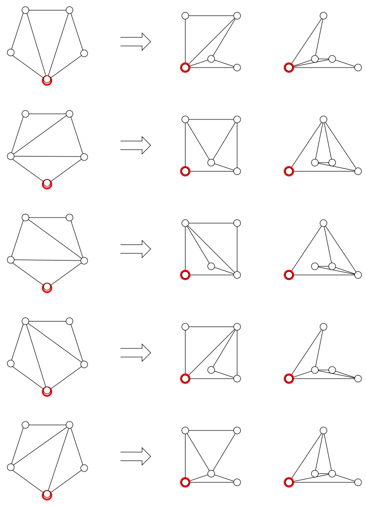

Figure 2: Mapping of all triangulations of Γ_5 to plane graphs with 7 edges on non-convex sets with 5 points.

If a (local) root vertex is not connected to any interior vertex of the subset, in the next recursive step a new root vertex is chosen similar to the first step. For example, in Figure 1, v_7 is neither connected to v_5 nor v_6, such that in the next step v_6 in Γ_n and v_5 in S are the corresponding local root vertices of the subset $\{v_5, v_6\}$ and the edge v_5v_6 is inserted in G'.

The recursion stops if the local root vertex is the only vertex of a subset. As each subset has a strictly smaller cardinality than the previous set (the previous root vertex can never be part of a subset), the process terminates.

[2] For example, we can choose the vertex with the smallest y-coordinate, and in case there are ties, the one with the minimum x-coordinate among them. Note that we use this method in all given figures.

It is intuitive to see how the mapping works for all triangulations of Γ_5 to different point sets, see Figure 2. The interested reader might want to check why some plane graphs with 7 edges on top of the non-convex sets are not generated.

Theorem 1 *For any fixed number k, $0 \leq k \leq 2n-3$, the number of plane geometric graphs with k edges on top of a set of n points is minimized for sets in convex position.*

Proof. To prove the theorem it is sufficient to show that the above mapping is injective. First observe that all recursive steps are independent in the sense that no edges constructed in G' cross the rays separating the subsets of S. Thus the image of G is in fact a plane graph G'. Intuitively the injectivity of our mapping follows then from this independency and the fact that each root vertex under consideration is connected to a uniquely determined subset of vertices.

More formal we prove the statement that there exists an injective mapping by induction over the number of points. The root vertex is chosen in a unique way and splits the problem into smaller subproblems. Note that this is still true if the chosen root of Γ_n is not connected to other vertices, as the cardinality of the remaining problem is reduced by at least one. Thus we can apply the induction hypothesis to get injective mappings for each subset. These sub-mappings are combined in a unique way via the root vertex, and the theorem follows. $\square$

Corollary 2 *The number of plane geometric graphs on top of a set of n points is minimized for sets in convex position.*

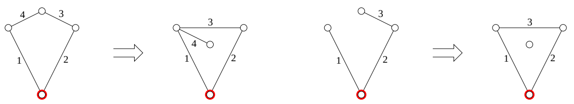

Figure 3: Transformations resulting in non-isomorphic graphs. Corresponding edges are labeled in order of their consideration.

The left side of Figure 3 shows a connected graph with a cycle which is transformed into a non-isomorphic graph. Thus our transformation is not suited for degree preservation, bipartite graphs etc. Also connectivity is not preserved by our mapping as can be seen from the right part of Figure 3. In the next section, we will give a variation of the mapping which preserves connectivity and in Subsection 2.1.3 we will extend this to an isomorphic mapping for cycle-free graphs.

2.1.2 Mapping for Connected Plane Geometric Graphs

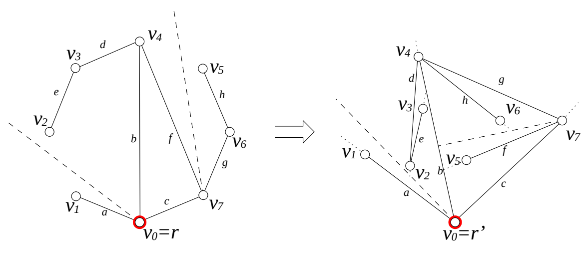

Figure 4: Using linear separators (dashed) to preserve connectedness. Corresponding edges are labeled with the same letter.

In this section, we consider $\mathsf{cg}(\Gamma_n)$, the set of connected graphs on top of Γ_n. Let $G \in \mathsf{cg}(\Gamma_n)$.

From the right part of Figure 3 it can be seen that troubles with connectivity occur when, within the same fan, two neighbored local root vertices r_1 and r_2 of S get connected in G'. This is caused by the resorting of the subset V of vertices between r_1 and r_2 (including r_1) around r_2.

Observe that if there would have already been a connection between r_1 and r_2 in V (not necessarily a direct edge) then connectivity would not have changed. To solve this problem we allow an edge between r_1 and r_2 in G' only if they are connected in the subgraph induced by V. Note that this only preserves connectedness for connected graphs G, but does not guarantee isomorphism.

For a subset $\tilde{S}$ of Γ_n and a graph $\tilde{G}$ of $\tilde{S}$ we call a straight line through a vertex of $\tilde{S}$ which does not intersect any edge of $\tilde{G}$ a linear separator for $\tilde{S}$. For our new mapping we insert a linear separator into a wedge of G formed by two neighbored root vertices r_1 and r_2 of a fan whenever r_1

and r_2 are not connected within the wedge, see Figure 4. In this case the subset of vertices is split into two independent parts and we have two separated recursive steps for the wedge, one with local root r_1, and one with local root r_2, respectively. Thus no edge inserted in G' crosses the linear separator.

Observe that in the example of Figure 4, vertices v_3 and v_6 would be singletons in G' using the mapping of Section 2.1.1, that is, without using linear separators.

That our new mapping indeed respects connectedness can be seen from the fact that each (local) root vertex is properly connected to its subset, and not to a neighbored root vertex of the same fan. Thus the claim follows from the recursive approach by induction.

Note that connectivity plays a crucial role when using linear separators. We can guarantee injectivity of the mapping only if, in the case that there is no linear separator, all vertices of the subset V are connected within the subgraph induced by V. The right part of Figure 5 shows a non-connected graph, where no linear separator exists. As the subset V does not induce a connected graph, the suggested approach does not work. Thus the example shows that we can use this method only for connected graphs.

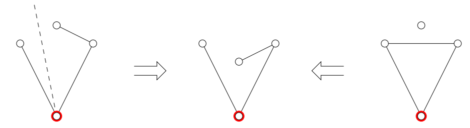

Figure 5: Using linear separators in combination with non-connected graphs causes loss of injectivity.

Theorem 3 *For any fixed number k, $0 \leq k \leq 2n-3$, the number of connected plane geometric graphs with k edges on top of a set of n points is minimized for sets in convex position.*

Corollary 4 *The number of connected plane geometric graphs on top of a set of n points is minimized for sets in convex position.*

2.1.3 Isomorphic Mapping for Cycle-Free Plane Geometric Graphs

Recall that with $\mathsf{cf}(\Gamma_n)$ we denote the set of all cycle-free graphs on top of Γ_n. Let $G \in \mathsf{cf}(\Gamma_n)$ and let us point out that G is not necessarily connected.

From Figure 5 it can be seen that the mapping of the last section is non-isomorphic if cycles of G are broken or cycles in G' are closed. Therefore we now extend the mapping of the previous section. As G has been a connected graph in the last section, we had at most one linear separator for each wedge. Now, as G is cycle-free but possibly not connected, we are going to use multiple linear separators per wedge. We insert a linear separator between two vertices of the wedge whenever these two vertices are not connected within the wedge, see Figure 6.

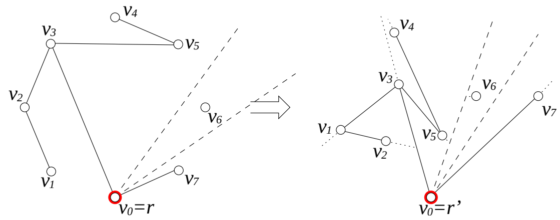

Figure 6: Cycle-free graphs: Using multiple linear separators guarantees isomorphic mapping.

As G is cycle-free there is always at least one linear separator per inner wedge. In addition we get a linear separator for each connected component which lies within a wedge but is not connected to the root vertex. Thus, situations like in Figure 5 cannot occur. Moreover, any recursive subset is independent from other subsets not only in the sense that no other edges cross, but also that there are no other incident edges from outside V except the ones from the (local) root. Thus, the mapping indeed generates isomorphic graphs.

The local labeling in recursive steps might still change. Although we get an isomorphic graph G', the final labeling of S needs not to be the one obtained by sorting the vertices around r'. See, for example, v_1 and v_2 in Figure 6.

n	all plane graphs		connected graphs	
	minimum	maximum	minimum	maximum
3	8	8	4	4
4	48	64	23	38
5	352	768	156	494
6	2 880	13 824	1 162	9 482
7	25 216	270 336	9 192	187 318
8	231 168	6 443 008	75 819	4 478 792
9	2 190 848	164 429 824	644 908	113 290 236
10	21 292 032	4 612 423 680	5 616 182	$\geq$ 3 145 329 136
11	211 044 352	$\geq$ 131 922 001 920	49 826 712	$\geq$ 88 754 921 232
	A054726		A007297	

Table 1: Minimum and maximum numbers of plane graphs and connected graphs, respectively, for small sets. The last row gives the sequence identification codes in Sloans Integer Sequence encyclopedia [24]. See Table 3 for asymptotic bounds.

Theorem 5 *There exists a mapping of all cycle-free plane geometric graphs on top of a set of n points in convex position to isomorphic plane geometric graphs on top of any set of n points.*

Corollary 6 *The number of (plane geometric) spanning trees, spanning paths, perfect matchings, and cycle-free graphs (forests) is minimized for point sets in convex position.*

Note that for graphs with at least $k \geq 3$ edges the minimum stated in the presented theorems and corollaries is unique. That is, for any set S of n points, $S \neq \Gamma_n$, there exist strictly more such graphs than on top of Γ_n.

That the number of crossing-free perfect matchings and spanning trees is minimized for convex sets has been shown in [14]. A similar result has been obtained for the number of spanning paths [21].

To obtain the numbers in Table 1 we made use of the exhaustive data base of all point sets in the plane [2, 6, 7]. It is interesting to observe that there seems to be a close relation between maximizing the number of plane graphs and connected graphs on the one hand and minimizing the rectilinear crossing number (see e.g. [3, 7]) on the other hand. For $n \leq 6$ there is always one unique example which minimizes the rectilinear crossing number and maximizes the number of plane graphs and connected graphs at the same time. For $n = 7, ..., 11$ there are $(2, 2, 1, 2, 3)$ sets which maximize the number of plane graphs and among them there is for each n one set which is the only one maximizing the number of connected graphs[3]. All these sets stem from the $(3, 2, 10, 2, 374)$, $n = 7, ..., 11$, sets minimizing the rectilinear crossing number.

The mentioned observations indicate that sets maximizing the number of plane graphs or the number of connected graphs, respectively, have to minimize the rectilinear crossing number. Nevertheless, there is no (inverse) monotonous relation between these two properties: there exist sets which, when compared to other sets, have a larger rectilinear crossing number and a higher number of plane graphs at the same time.

Moreover, there is no monotonous relation between the number of plane graphs and the number of connected graphs. For $n \geq 7$ there exist point sets S_1 and S_2, such that there exist more plane graphs on top of S_1 than on top of S_2, but for connected graphs the situation is converse.

The second column of Table 4 in Section 4 shows the asymptotic growth of several types of graphs for the convex set Γ_n. Except for triangulations the given bounds are lower bounds for these types of graphs.

To our knowledge the number of spanning paths for Γ_n has not been explicitly stated before. For completeness we include the following simple argumentation.

[3]As indicated in Table 1, for $n = 10$ and $n = 11$, respectively, we still have to check all remaining sets, i.e., sets not minimizing the rectilinear crossing number, to ensure extremality of the given values.

Type	per triangulation	Type	per triangulation
$\mathsf{sc}(n)$	$\mathcal{O}^*(\sqrt{6}^n)$ [21]	$\mathsf{pt}(n)$	$\mathcal{O}^*(3^n)$ [19]
$\mathsf{pm}(n)$	$\mathcal{O}^*(\sqrt{3}^n)$ [*]	$\mathsf{st}(n)$	$\mathcal{O}^*(5.\dot{3}^n)$ [18]
$\mathsf{sp}(n)$	$\mathcal{O}^*(3^n)$ [21]	$\mathsf{cf}(n)$	$\mathcal{O}^*(6.75^n)$ [*]
$\mathsf{tr}(n)$	1	$\mathsf{cg}(n)$	$\mathcal{O}^*(8^n)$
$\mathsf{ppt}(n)$	$\mathcal{O}^*(3^n)$ [19]	$\mathsf{pg}(n)$	$\mathcal{O}^*(8^n)$ [14]

Table 2: Number of different types of graphs per triangulation. For easier comparison all expressions are given by their numerical values, see the related references for exact formulas ([*] stands for the paper at hand).

To count the number of spanning paths for Γ_n start with an arbitrary vertex and extend the path edge-wise. For each edge, except the last one, there are precisely two ways to continue as otherwise the plane path cannot span the whole set. These two choices are the first unused vertices in clockwise and counterclockwise direction, respectively, as seen from the starting vertex. The last edge is uniquely determined. Thus there are 2^{n-2} such paths. As we have n potential vertices to start with and each path has two 'beginnings' this gives the total of $n \cdot 2^{n-3}$ spanning paths for Γ_n.

3 On Upper Bounds

In order to show the bound of $\mathcal{O}^*(344^n)$ for the number of plane graphs, the upper bound on the number of triangulations has been used. As any triangulation of S has at most $3n-6$ edges, it contains at most 2^{3n-6} plane subgraphs. Therefore we get the bound $|\mathsf{pg}(n)| \leq 2^{3n-6}|\mathsf{tr}(n)| = 344^n$. The last equality comes from the currently best upper bound for $|\mathsf{tr}(n)| \leq 43^n$ given in [22].

For the maximum number of cycle-free graphs in a triangulation we get an upper bound of $\mathcal{O}(\binom{3n-6}{n-1}) = \mathcal{O}^*(6.75^n)$. For the special case of spanning trees this has recently been improved to $\mathcal{O}^*(5.\dot{3}^n)$ by observing that the dual tree has bounded degree 3 [18].

Taking the average vertex degree in a triangulation into account, a bound of $\mathcal{O}^*(3^n)$ for the number of spanning paths in a given triangulation can be shown [21]: Starting at an arbitrary beginning vertex construct the path edge-wise step by step. At each vertex the number of possible ways to continue, that is, to choose the next edge, is its effective degree. Here effective degree means that we do not count edges of the triangulation which have already been considered before. The reason is that once a vertex v has been visited, no edge incident to v can be used later on to continue the path (even if the edge has not been chosen, as v cannot be visited again). Hence, every edge is considered only once and therefore the sum of the effective degree of all vertices is the number of edges of the triangulation, bounded by $3n-6$. Essentially the number of different spanning paths is the product of the effective degrees. This product is maximized by uniformly distributing the over-all degree, and the given bound follows.

Using the same arguments we get a bound of $\mathcal{O}^*(\sqrt{3}^n)$ for the number of crossing-free perfect matchings in a triangulation. Here a single step consists of choosing the leftmost unused vertex and matching it to one of its (effective) incident neighbors.

The bound on $|\mathsf{sp}(n)|$ also implies an upper bound for the number of spanning cycles: For every spanning cycle $C \in \mathsf{sc}(S)$, $S \in \mathcal{S}_n$, we get n spanning paths by omitting one edge of C. As any two elements of $\mathsf{sc}(S)$ differ by at least two edges this implies $|\mathsf{sp}(S)| \geq n \cdot |\mathsf{sc}(S)|$. An improved bound for $|\mathsf{sc}(n)|$ of $\mathcal{O}^*(\sqrt{6}^n)$ per triangulation was provided by Raimund Seidel and reported in [21].

Most of the upper bound constructions for various classes of graphs are based on the just mentioned relation to triangulations. For example, the bound on the number of cycle-free graphs in a triangulation implies the upper bound of $\mathcal{O}^*(290.25^n)$ for the number of cycle-free graphs given in Table 3. Only recently different approaches have been investigated, an outstanding result being the $\mathcal{O}^*(10.05^n)$ bound for crossing-free perfect matchings [21]. Syntactically their proof follows the proof of [20] bounding the number of triangulations, but uses novel ideas and refined observations.

Type	Lower Bound	Number for Γ_{10}	Upper Bound	
$\mathsf{sc}(n)$	1	1	$\Omega^*(4.64^n)$ [14]	$\mathcal{O}^*(86.81^n)$ [21][4]
$\mathsf{pm}(n)$	$\Theta^*(2^n)$ [13, 14]	42	$\Omega^*(3^n)$ [14]	$\mathcal{O}^*(10.05^n)$ [21][4]
$\mathsf{sp}(n)$	$\Theta^*(2^n)$	1 280	$\Omega^*(4.64^n)$ [14]	$\mathcal{O}^*(100.88^n)$ [21][4]
$\mathsf{tr}(n)$	$\Omega^*(2.33^n)$ $\mathcal{O}^*(3.47^n)$ [5]	250	$\Omega^*(8.48^n)$ [*]	$\mathcal{O}^*(43^n)$ [22]
$\mathsf{ppt}(n)$	$\Theta^*(4^n)$ [4]	1 430	$\Omega^*(12^n)$ [8]	$\mathcal{O}^*(129^n)$ [19]
$\mathsf{pt}(n)$	$\Theta^*(4^n)$ [4]	1 430	$\Omega^*(20^n)$ [8]	$\mathcal{O}^*(129^n)$ [19]
$\mathsf{st}(n)$	$\Theta^*(6.75^n)$ [13]	246 675	$\Omega^*(10.42^n)$ [11]	$\mathcal{O}^*(229.33^n)$ [22, 23]
$\mathsf{cf}(n)$	$\Theta^*(8.22^n)$ [13]	2 117 283	$\Omega^*(11.62^n)$ [*]	$\mathcal{O}^*(290.25^n)$ [*]
$\mathsf{cg}(n)$	$\Theta^*(10.39^n)$ [13]	5 616 182	$\Omega^*(35.49^n)$ [*]	$\mathcal{O}^*(344^n)$ [22]
$\mathsf{pg}(n)$	$\Theta^*(11.65^n)$ [13]	21 292 032	$\Omega^*(41.18^n)$ [*]	$\mathcal{O}^*(344^n)$ [22]

Table 3: Asymptotic bounds for various classes of graphs. All types except triangulations are minimized for sets in convex position. References for upper bounds give either the presented bounds or their relation to the number of triangulations.

Type	Convex Set	Double Circle	Double Chain	Double Zig-Zag Chain
$\mathsf{sc}(n)$	1	$\mathcal{O}^*(4.83^n)$ [*]	$\Omega^*(4.64^n)$ $\mathcal{O}^*(5.61^n)$ [14]	
$\mathsf{pm}(n)$	$\Theta^*(2^n)$ [14]	$\Theta^*(2.20^n)$ [*]	$\Theta^*(3^n)$ [14]	
$\mathsf{sp}(n)$	$\Theta^*(2^n)$	$\mathcal{O}^*(4.83^n)$ [*]	$\Omega^*(4.64^n)$ [14]	
$\mathsf{tr}(n)$	$\Theta^*(4^n)$	$\Theta^*(\sqrt{12}^n)$ [5, 8]	$\Theta^*(8^n)$ [14]	$\Theta^*(8.48^n)$ [*]
$\mathsf{ppt}(n)$	$\Theta^*(4^n)$	$\Theta^*(\sqrt{28}^n)$ [8]	$\Theta^*(12^n)$ [8]	
$\mathsf{pt}(n)$	$\Theta^*(4^n)$	$\Theta^*(\sqrt{40}^n)$ [8]	$\Theta^*(20^n)$ [8]	
$\mathsf{st}(n)$	$\Theta^*(6.75^n)$ [13]	$\Omega^*(7.07^n)$ [*]	$\Omega^*(10.42^n)$ [11]	
$\mathsf{cf}(n)$	$\Theta^*(8.22^n)$ [13]	$\Omega^*(8.55^n)$ [*]	$\Omega^*(11.62^n)$ [*]	
$\mathsf{cg}(n)$	$\Theta^*(10.39^n)$ [13]	$\Omega^*(11.83^n)$ [*]	$\Omega^*(35.49^n)$ [*]	$\Omega^*(32.49^n)$ [*]
$\mathsf{pg}(n)$	$\Theta^*(11.65^n)$ [13]	$\Theta^*(15.0046^n)$ [*]	$\Theta^*(39.80^n)$ [14][*]	$\Theta^*(41.19^n)$ [*]

Table 4: Special configurations and their asymptotic number of graphs.

Interestingly the result for crossing-free perfect matchings can be used to bound the number of spanning cycles and spanning paths [21]: Observe that for even n every spanning cycle contains two crossing-free perfect matchings. Thus $|\mathsf{sc}(n)| \leq |\mathsf{pm}(n)|^2$. Similarly every spanning path contains a crossing-free perfect matching on n points and a crossing free perfect matching on $n-2$ points (omit start and endpoint). We thus get an upper bound of $\mathcal{O}^*(100.88^n)$ for both structures. Recently this bound has been improved to $\mathcal{O}^*(86.81^n)$ for spanning cycles [21][4].

4 Special Configurations

In the following subsections, we consider two configurations, namely the double circle [5] and the double chain [14], as well as a new configuration, the so-called double zig-zag chain (DZZC). We will show that the DZZC is a new extremal example for maximizing the asymptotic number of triangulations and of plane graphs on top of a given point set.

4.1 Double Circle and Double Chain

While the double chain is already known for quite some time [14], let us briefly repeat the definition of the double circle [5]: Let S_e be a set of $2m-k$ points in convex position, and let S_i be a set of k points interior to $CH(S_e)$, $k \leq m$, each one infinitesimally close to a different midpoint of an edge of $CH(S_e)$. When $|S_e| = |S_i| = k = m$ the configuration is the double circle.

[4]Further improvements on the constants have been made very recently, and will be reported in [23]. To our knowledge the currently best bounds are $|\mathsf{sc}(n)| = \mathcal{O}^*(74.60^n)$, $|\mathsf{pm}(n)| = \mathcal{O}^*(9.22^n)$, and $|\mathsf{sp}(n)| = \mathcal{O}^*(85.01^n)$.

4.1.1 The Number of Plane Geometric Graphs of the Double Circle

Theorem 7 *Let $|\mathsf{pg}(DC_n)|$ be the number of plane geometric graphs of the double circle containing n points. Then $|\mathsf{pg}(DC_n)| = \sum_{i=0}^{\frac{n}{2}} (-1)^i \cdot \binom{\frac{n}{2}}{i} \cdot 2^{i+\frac{n}{2}} \cdot |\mathsf{pg}(\Gamma_{n-i})|$, where $|\mathsf{pg}(\Gamma_m)|$ is the number of plane geometric graphs of the convex m-gon.*

Proof. We move every interior point of the double circle to the outside of its corresponding convex hull edge e, see Figure 7. A plane geometric graph of the double circle might have crossings in the resulting drawing. We count the number of plane graphs of this point set which do not contain any of the former convex hull edges (i.e., an edge e in the right drawing of Figure 7). Since a convex hull edge of the double circle appears in exactly half of all plane graphs, we simply have to multiply this number by $2^{\frac{n}{2}}$.

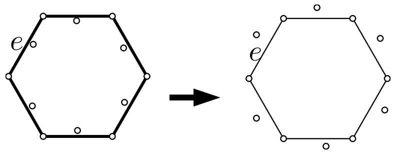

Figure 7: Moving the interior points to the outside of the convex hull.

If edge e belongs to a plane graph, then there are four possibilities to connect (or not) the point which is separated by e to the plane graph. Hence, we assume this separated point does not belong to the graph. But then, e is a convex hull edge and appears in half of all graphs. Since our new point set is convex, the edge e appears in $\frac{|\mathsf{pg}(\Gamma_{n-1}))|}{2} \cdot 4$ plane graphs of $\mathsf{pg}(\Gamma_n)$. Similarly, i edges 'of type e' appear simultaneously in $\frac{|\mathsf{pg}(\Gamma_{n-i})|}{2^i} \cdot 4^i$ plane graphs of $\mathsf{pg}(\Gamma_n)$. Now, in order to count all plane graphs not containing an edge 'of type e' we can use the inclusion-exclusion principle. We obtain $|\mathsf{pg}(DC_n)| = \left(|\mathsf{pg}(\Gamma_n)| - \frac{n}{2} \cdot 2 \cdot |\mathsf{pg}(\Gamma_{n-1})| + \binom{\frac{n}{2}}{2} \cdot 4 \cdot |\mathsf{pg}(\Gamma_{n-2})| - \cdots + (-1)^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} \cdot |\mathsf{pg}(\Gamma_{\frac{n}{2}})|\right) \cdot 2^{\frac{n}{2}}$. □

Corollary 8 $|\mathsf{pg}(DC_n)| = \Theta^*\left(\left(4\sqrt{7+5\sqrt{2}}\right)^n\right) = \Theta^*(15.0045^n)$.

Proof. We know that $|\mathsf{pg}(\Gamma_m)| = \Theta^*\left(\left(6+4\sqrt{2}\right)^m\right)$ [13] and therefore we get $|\mathsf{pg}(DC_n)| = \sum_{i=0}^{\frac{n}{2}} (-1)^i \cdot \binom{\frac{n}{2}}{i} \cdot 2^{i+\frac{n}{2}} \cdot \Theta^*\left(\left(6+4\sqrt{2}\right)^{n-i}\right) = \Theta^*\left(\sum_{i=0}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{i} \cdot \left(\frac{-2}{6+4\sqrt{2}}\right)^i\right) \cdot (6+4\sqrt{2})^n \cdot 2^{\frac{n}{2}}\right)$ Using the binomial theorem $\sum_{i=0}^{t} \binom{t}{i} \cdot x^i = (1+x)^t$ we get $|\mathsf{pg}(DC_n)| =$ $\Theta^*\left(\left(1 - \frac{1}{3+2\sqrt{2}}\right)^{\frac{n}{2}} \cdot (6\sqrt{2}+8)^n\right)$ which simplifies to the given formula. □

4.1.2 The Number of Crossing-Free Perfect Matchings in the Double Circle

Let $\mu(2m,k)$ denote the number of crossing-free perfect matchings in $S_e \cup S_i$ for the special case in which all the points in S_i correspond to k *consecutive* edges of $CH(S_e)$.

Lemma 9 *The following equalities hold: ($2 \le k \le m-1$, $m \ge 3$)*

(a) $\mu(2m,0) = C_m$;

(b) $\mu(2m,1) = \mu(2m,0) + \mu(2m-2,0) = C_m + C_{m-1}$;

(c) $\mu(2m,k) = \mu(2m,k-1) + \mu(2m-2,k-2)$;

(d) $\mu(2m,m) = \mu(2m,m-1) + \mu(2m-2,m-3)$;

where C_m is the m-th Catalan number.

Proof. Equality (a) is well known, see for example [13]. Equality (d) and the first part of (b) can be proved assuming special cases in the proof for equality (c). So we provide the proof for equality (c) in detail.

Let a and b be the endpoints of an edge of $CH(S_e)$ such that there is a point $c \in S_i$ close to the midpoint of ab, and one of the neighboring edges of $CH(S_e)$ has also a point from S_i but the other one does not (refer to Figure 8). Let us call C_1 this configuration of points, and let C_2 be the configuration obtained from C_1 by replacing c with a point d exterior to $CH(S_e)$ and very close to the midpoint of ab. Let C_3 be the configuration obtained from C_1 after the removal of the points a and b.

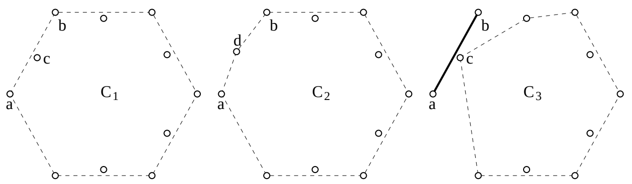

Figure 8: Illustrating the recursion for $\mu(2m,k)$ for $\mathsf{pm}(DC)$.

Now notice that crossing-free perfect matchings in C_1 are in one-to-one correspondence with the perfect matchings in C_2, with d playing the role of c, with the exception of those perfect matchings in which a is matched with b; these are in one-to-one correspondence with the perfect matchings in C_3, which proves the claim. $\square$

Lemma 10 *For $2 \le k \le m-1$, $m \ge 3$ we have $\mu(2m,k) = \sum_{s=0}^{k+1} \binom{k+1-s}{s} C_{m-s}$.*

Proof. First notice that the binomial coefficient is 0 for $k+1-s < s$. The proof is straightforward using Lemma 9 and induction (with base case $\mu(2m,2) = C_m + 2C_{m-1}$). $\square$

Finally, combining Lemma 9 with Lemma 10 we obtain the number of perfect matchings of the double circle.

Corollary 11 *The number of crossing-free perfect matchings of the double circle is*

$$\mu(2m,m) = \sum_{i=0}^{m} \binom{m-i}{i} C_{m-i} + \sum_{j=0}^{m-2} \binom{m-2-j}{j} C_{m-1-j}.$$

The preceding result can be used in order to obtain an asymptotic estimate of $\mu(2m,m)$. The generic term in the expression is roughly $\binom{(1-\alpha)m}{\alpha m} C_{(1-\alpha)m} = \Theta^*(2^{(1-\alpha)\left[H(\frac{\alpha}{1-\alpha})+2\right]m})$, cf. Section 4.1.5. Elementary computations show that the exponent is maximized for $\alpha = (2-\sqrt{2})/4 \approx 0.1464466$, therefore the number of perfect matchings of a double circle with $n = 2m$ points is $\Theta^*(2^{2.271553m}) = \Theta^*(4.828427^m) = \Theta^*(\sqrt{4.828427}^n) = \Theta^*(2.197368^n)$.

Theorem 12 *The double circle of n points has $\Theta^*(2.197368^n)$ crossing-free perfect matchings* [5].

With the same arguments as used at the end of section 3 we obtain

Corollary 13 *The double circle of n points has $\mathcal{O}^*(4.828427^n)$ spanning cycles and spanning paths.*

4.1.3 Lower Bounds on the Number of (Plane Geometric) Connected Graphs, Cycle-Free Graphs and Spanning Trees of the Double Circle

We use a recursion that differs only in configuration C_3 (see Figure 9) from the one used in Section 4.1.2 for the crossing-free perfect matchings. As this approach is identical for connected graphs, cycle-free graphs and spanning trees, we will first do a general analysis and then provide a table (see Table 5) with the different constants and results. Again, let $\mu(2m,k)$ denote the number of the considered graphs in $S_e \cup S_i$ for the special case in which the k points in S_i correspond to consecutive edges of $CH(S_e)$.

[5] The precise base can be computed from the formula $\sqrt{2^{\frac{\log 16+\sqrt{2}(\log 4-\log 8)+(\sqrt{2}-2)\log(2-\sqrt{2})+(2+\sqrt{2})\log(2+\sqrt{2})}{\log 16}}}$.

Similar to Section 4.2 we call edges that have to appear in every triangulation on the double circle unavoidable. Edges that are not unavoidable are called *optional edges*.

Observe that the graphs of configuration C_1 are in one-to-one correspondence to those of configuration C_2 with the exception of those which use optional edges from point c as well as the edge ab. The graphs of configuration C_1 which use both the edge ab and at least one optional edge from c, are in one-to-one correspondence to the graphs of configuration C_3 with the exception of those which use (in addition) optional edges from the points a or b.

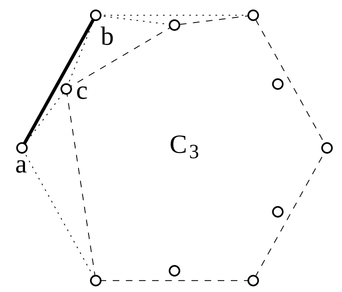

Figure 9: Recursion for $\mu(2m,k)$ for $\mathsf{cg}(DC)$, $\mathsf{cf}(DC)$ and $\mathsf{st}(DC)$. (bold edges must appear)

Note, that there exist some unused edges (dotted) between the points a and b and the rest of the point set (dashed polygon) in configuration C_3. For $\mathsf{cg}(DC)$ any combination except inserting no edge is valid, for $\mathsf{cf}(DC)$ we may insert one of the unused edges or none and for $\mathsf{st}(DC)$ exactly one of these edges has to be inserted. We use the constants η_1, η_k and η_m as we get different numbers of combinations for $k=1$, $2 \le k \le m-1$ and $k=m$. Furthermore, we denote with κ the basis of the exponential terms of the particular graphs in the convex setting, see Table 5.

Lemma 14 *For $\mu(2m,k)$ for $\mathsf{cg}(DC)$, $\mathsf{cf}(DC)$ and $\mathsf{st}(DC)$ we get with $2 \le k \le m-1$:*

1. $\mu(2m,0) = \Theta^*(\kappa^{2m})$
2. $\mu(2m,1) \ge \mu(2m,0) + \eta_1 \cdot \mu(2m-2,0)$
3. $\mu(2m,k) \ge \mu(2m,k-1) + \eta_k \cdot \mu(2m-2,k-2)$
4. $\mu(2m,m) \ge \mu(2m,m-1) + \eta_m \cdot \mu(2m-2,m-3)$

Lemma 15 *For $2 \le k \le m-1$, $m \ge 3$ we have:* $\mu(2m,k) \ge \sum_{i=0}^{k} \left\{ \binom{k-i}{i} \cdot \eta_k^i \cdot \mu(2m-2i,0) \right\}$

Proof. Notice that a binomial coefficient $\binom{r}{s}$ is 0 if $s < 0$ or $r < s$. Using Lemma 14 and induction we prove $\mu(2m,k) \ge \sum_{i=0}^{k} \left\{ \left[\binom{k+1-i}{i} - \binom{k-i}{i-1} \cdot \frac{\eta_k - \eta_1}{\eta_k} \right] \cdot \eta_k^i \cdot \mu(2m-2i,0) \right\}$ with base case $\mu(2m,2) \ge \mu(2m,0) + \left(2 - \frac{\eta_k - \eta_1}{\eta_k}\right) \cdot \eta_k \cdot \mu(2m-2,0)$. Knowing that $\binom{k+1-i}{i} - \binom{k-i}{i-1} = \binom{k-i}{i}$ proves the inequality. □

Corollary 16 *For the number of considered graphs in the double circle we get:*

$$\mu(2m,m) \ge \sum_{i=0}^{m} \left\{ \binom{m-1-i}{i} \cdot \eta_k^i \cdot \mu(2m-2i,0) \right\} + \eta_m \cdot \sum_{i=0}^{m-2} \left\{ \binom{m-3-i}{i} \cdot \eta_k^i \cdot \mu(2m-2i-2,0) \right\}$$

The generic term in the expression is roughly $\binom{(1-\alpha)m}{\alpha m} \cdot \Theta^*\left(\kappa^{2(1-\alpha)m}\right) \cdot \eta_k^{\alpha m}$ and optimizing α we get the asymptotic lower bounds shown in Table 5.

Observe that for connected graphs a fixed edge e of the convex hull appears in at least half of the graphs. This allows us to improve the configuration C_3 from before by adding the point a to the remaining point set (dashed polygon), see Figure 10. By only using half the number of connected graphs of the remaining point set, we ensure that edge e' appears in every counted graph. Point b is connected via the edge ab and therefore we are free to use the dotted edges in Figure 10 in every possible combination, resulting in an additional factor of 8 (respectively 4 if $k=1$).

Lemma 17 *We improve $\mu(2m,k)$ for $\mathsf{cg}(DC)$: $(2 \le k \le m,\ m \ge 3)$*

1. $\mu(2m,0) = \Theta^*(10.39^{2m})$

graph	η_1	η_k	η_m	κ	α	Ω^*
$\mathsf{cg}(DC_n)$	15	31	63	10.39 [13]	0.158896	$\Omega^*(11.5367^n)$
$\mathsf{cf}(DC_n)$	5	6	7	8.22 [13]	0.070494	$\Omega^*(8.5506^n)$
$\mathsf{st}(DC_n)$	4	5	6	6.75 [13]	0.083182	$\Omega^*(7.0787^n)$

Table 5: Constants and results for connected graphs, cycle-free graphs and spanning trees in the double circle.

2. $\mu(2m,1) \geq \mu(2m,0) + \frac{1}{2} \cdot 4 \cdot \mu(2m-1,0)$
3. $\mu(2m,k) \geq \mu(2m,k-1) + \frac{1}{2} \cdot 8 \cdot \mu(2m-1,k-2)$

Corollary 18 *For $2 \leq k \leq m$, $m \geq 3$ we have $\mu(2m,k) \geq \sum_{i=0}^{k} \left\{ \binom{k-i}{i} \cdot 4^i \cdot \mu(2m-i,0) \right\}$ respectively $\mu(2m,m) \geq \sum_{i=0}^{m} \left\{ \binom{m-i}{i} \cdot 4^i \cdot \mu(2m-i,0) \right\}$*

The generic term in the expression is roughly $\binom{(1-\alpha)m}{\alpha m} \cdot \Theta^*\big(10.39^{(2m-\alpha m)}\big) \cdot 4^{\alpha m}$ and with $\alpha = 0.186269$ this gives for the number of connected plane geometric graphs of the double circle of n points: $|\mathsf{cg}(DC_n)| = \Omega^*(11.8321^n)$.

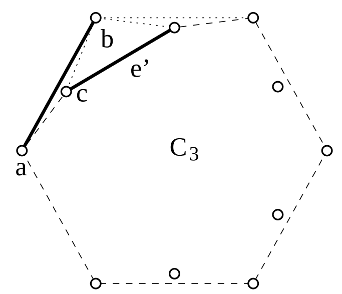

Figure 10: Improved recursion for $\mu(2m,k)$ for $\mathsf{cg}(DC)$. (bold edges must appear)

4.1.4 Lower Bound on the Number of Connected Plane Geometric Graphs of the Double Chain

Theorem 19 *The double chain has $\Omega^*(35.49^n)$ connected plane geometric graphs.*

Proof. Consider all graphs of the double chain such that the subgraphs of both convex $\frac{n}{2}$-gons of the double chain are connected. Adding one vertical edge of the convex hull of the double chain gives a connected graph. From [14] and Section 4.2.1 we know that there are $\Theta^*(39.80^n)$ plane graphs in the double chain. Moreover a convex n-gon has $\Theta^*(11.65^n)$ plane graphs and $\Theta^*(10.39^n)$ connected graphs [13]. We thus have to correct the number 39.80^n by the factor $(\frac{10.39}{11.65})^n$ and get a lower bound of $\Omega^*(35.49^n)$ for the number of connected graphs of the double chain. □

4.1.5 Lower Bound on the Number of Cycle-Free Plane Geometric Graphs of the Double Chain

In this section we show that the double chain contains $\Omega^*(11.6268^n)$ cycle-free graphs. This improves the previous bound of $\Omega^*(11.09^n)$ given in [14].

Theorem 20 *The double chain has $\Omega^*(11.6268^n)$ cycle-free plane geometric graphs.*

Proof. Let $F_{n,k}$ be the number of cycle-free graphs (forests) consisting of k components on a convex n-point set Γ_n. Flajolet and Noy [13] provided a formula for $F_{n,k}$.

$$F_{n,k} = \frac{1}{2n-k} \binom{n}{k-1} \binom{3n-2k-1}{2n-k-1}.$$

We count the number of cycle-free graphs of the double chain (there are $n/2$ points on each chain) by first counting the number of cycle-free graphs $F_{n/2,k}$ on the two convex sets for a suitable value of k. Then we choose one point of each tree, and we count the number of plane graphs in the *interior part* (the area between the two concave chains) of the double chain, thereby only using chosen points. Proceeding along the lines of the proof of Theorem 21, see Section 4.2.1 we get $\Theta^*(3.41421356^{2k})$ plane graphs in the interior part of a double chain with k points on each chain. Observe that any triangulation of the (open) interior of the double chain forms a spanning tree and therefore we do not obtain cycles. Furthermore, the number of cycle-free graphs of Γ_n is at least $F_{n,k}$ for any fixed value of k. Hence, for asymptotic counting we can restrict our attention to $F_{n,\alpha n}$ (respectively $F_{\frac{n}{2},\alpha\frac{n}{2}}$). Then the relevant terms of the equation above are $\binom{n}{\alpha n} \cdot \binom{n(3-2\alpha)}{n(2-\alpha)}$. Combining the three terms we get the lower bound on the number of cycle-free graphs $\left(\binom{\frac{n}{2}}{\alpha\frac{n}{2}} \cdot \binom{\frac{n}{2}(3-2\alpha)}{\frac{n}{2}(2-\alpha)}\right)^2 \cdot 3.41421356^{\alpha n}$.

Using Stirling's formula $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ one derives the well known estimate $\binom{\gamma n}{\delta n} = \Theta(n^{-\frac{1}{2}} 2^{\gamma H(\frac{\delta}{\gamma})n})$ where $H(\alpha) = -(\alpha \log_2 \alpha + (1-\alpha)\log_2(1-\alpha))$ denotes the binary entropy function. This gives $\left(2^{H(\alpha)\frac{n}{2}} \cdot 2^{(3-2\alpha)H(\frac{2-\alpha}{3-2\alpha})\frac{n}{2}}\right)^2 \cdot 3.41421356^{\alpha n} = 2^{\left(H(\alpha)+(3-2\alpha)H(\frac{2-\alpha}{3-2\alpha})+(\log_2(3.41421356)\cdot\alpha)\right)n}$.

For $\alpha = 0.40338$ the expression yields $\Omega^*(11.6268^n)$ cycle-free graphs for the double chain. $\square$

Observe that the above method results in a lower bound of $\Omega^*(9.65^n)$ on the number of spanning trees on the double chain which is worse than the bound $\Omega^*(10.42^n)$ of Dumitrescu [11].

4.2 The Double Zig-Zag Chain - a new Extremal Configuration

In this section we combine the double circle and the double chain in order to obtain a new extremal configuration, the so-called double zig-zag chain (DZZC).

While the double circle has $\Theta^*(\sqrt{12}^n)$ triangulations and is thus conjectured to minimize this number [5], the double chain was up to now the configuration with the asymptotically highest number of triangulations, namely $\Theta^*(8^n)$ [14]. It was widely believed (including most of the authors) that this could be the true upper bound for the number of triangulations.

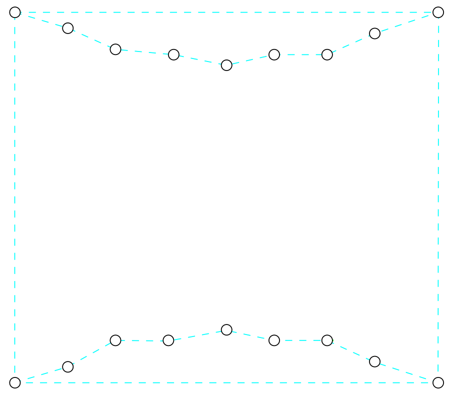

Figure 11: Double zig-zag chain: combining double circle and double chain.

To obtain the double zig-zag chain, take two distorted double circles with $\frac{n}{2}$ points each and combine them within a convex quadrilateral as indicated in Figure 11. The example is similar to the double chain, but instead of two concave chains we now have two zig-zag chains of edges splitting the area of the quadrilateral into three parts. These edges are unavoidable in the sense that they are not crossed by any other edge spanned by the point set. For example, the edges of the zig-zag chains will have to be part of any triangulation. Note that the two zig-zag configurations are at sufficient distance from each other such that any vertex of a chain can 'see' all vertices of the opposite chain, that is, an edge connecting a vertex of the lower chain to a vertex of the upper chain does not cross a zig-zag edge. Both, upper and lower part, are double circles with $\frac{n}{2}$ vertices and thus $\Theta^*(\sqrt{12}^{n/2})$ triangulations each. To be precise, there is one interior vertex near the horizontal convex-hull edges missing in each subset, which does not influence the asymptotic counting arguments for the upper and lower parts.

4.2.1 The Number of Plane Geometric Graphs of the Double Zig-Zag Chain

Theorem 21 *The double zig-zag chain of n points contains $\Theta^*(41.1889^n)$ plane geometric graphs.*

Proof. The double zig-zag chain consists of three parts separated by the two zig-zag chains. The number of graphs can be counted independently for each part.

We first count the number of plane graphs in the interior of the double zig-zag chain (the area between the two zig-zag chains). To this end we are going to choose m points, $0 \leq m \leq \frac{n}{2}$, from each zig-zag chain. The points of each zig-zag chain can be viewed as lying on a smaller and a larger circle. For the upper zig-zag chain we choose i_1 points from the larger and j_1 points from the smaller circle, $i_1 + j_1 = m$. Similarly we choose i_2 and j_2 points from the lower zig-zag chain.

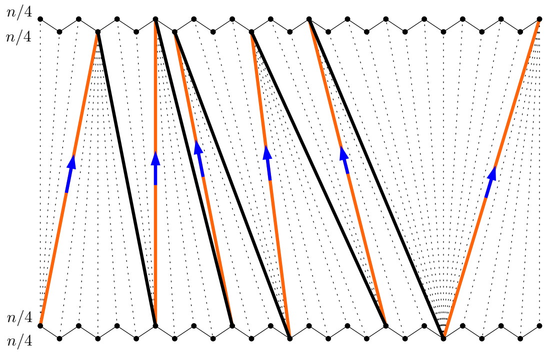

Figure 12: Counting plane graphs in the interior of the double zig-zag chain.

There are $\sum_{0 \leq m \leq \frac{n}{2}} \sum_{i_1, i_2, i_1+j_1=i_2+j_2=m} \binom{\frac{n}{4}}{i_1} \cdot \binom{\frac{n}{4}}{j_1} \cdot \binom{\frac{n}{4}}{i_2} \cdot \binom{\frac{n}{4}}{j_2}$ ways to do so. Then we connect the chosen points by pairing a point from the upper zig-zag chain with a point from the lower zig-zag chain by scanning them from left to right. We say that these pairs are connected by black edges, shown as dark edges in Figure 12. Next we draw red edges (marked by arrows in Figure 12) connecting the black edges in a way that we connect the lower endpoint of the 'left' black edge to the upper endpoint of the 'right' black edge. If the starting (ending) point of the lower (upper) zig-zag chain is not used by a black edge, then the first (last) red edge uses this point instead of an endpoint of a black edge.

We complete the drawing to a triangulation of the interior by connecting the two zig-zag chains in a fan like manner to the endpoints of the red and black edges. Let us color these edges gray (dotted edges in Figure 12). Note that we can flip some of the gray edges (connected to the smaller circle of a zig-zag chain) to obtain different triangulations, still containing the red and black edges. Here a flip exchanges the two diagonals of the convex quadrilateral formed by the two adjacent triangles.

Next we consider all subgraphs of the obtained triangulations which contain the black segments. For a flippable gray edge we have three possibilities (draw it, delete it, flip it), for a non-flippable gray edge we have two possibilities. We also have two possibilities for the red edges (draw it or not). Note that we do not flip red edges for the following reason. It is important to observe that any obtained subgraph is uniquely assigned to its triangulation, that is, to the given set of black edges. In other words, for a given subgraph its triangulation can be uniquely restored: We can always detect (and insert) the red and black edges: By starting from the leftmost point of the lower zig-zag chain we draw an edge (or simply detect whether it is already there) to the rightmost visible point on the upper zig-zag chain. This gives us the first red edge and the starting point of the next black edge. The black edge is determined as the rightmost edge incident to this vertex and going back to the lower zig-zag chain. Continuing in the same manner this gives us the remaining red and black edges. Note that the above argumentation still goes through when flippable gray edges have been flipped arbitrarily.

We now count the number of the constructed subgraphs. There are $\frac{n}{2} - (j_1 + j_2)$ flippable gray edges (one for each non-chosen point of the two smaller circles). Analogously, there are $\frac{n}{2} - (i_1 + i_2)$ non-flippable gray edges. And there are m black and $m + \{-1, 0, +1\}$ red edges.

Thus, we obtain the number of plane geometric graphs of the interior of the double zig-zag chain by multiplying all factors as $\sum_{0 \leq m \leq \frac{n}{2}} \sum_{i_1, i_2, i_1+j_1=i_2+j_2=m} \binom{\frac{n}{4}}{i_1} \cdot \binom{\frac{n}{4}}{j_1} \cdot \binom{\frac{n}{4}}{i_2} \cdot \binom{\frac{n}{4}}{j_2} \cdot 3^{\frac{n}{2}-(j_1+j_2)} \cdot 2^{\frac{n}{2}-(i_1+i_2)+m+\{-1,0,+1\}}$.

Neglecting polynomial factors, the asymptotic of this sum is determined by its largest element, since we have only a linear number of terms. As $i_1 + j_1$ is independent from $i_2 + j_2$, the maximum is obtained for $i_1 = i_2 = c \cdot \frac{n}{4}$ and $j_1 = j_2 = d \cdot \frac{n}{4}$ by symmetry. We have to maximize the term $\binom{\frac{n}{4}}{c \cdot \frac{n}{4}}^2 \cdot \binom{\frac{n}{4}}{d \cdot \frac{n}{4}}^2 \cdot 3^{(1-d)\frac{n}{2}} \cdot 2^{(1-\frac{c}{2}+\frac{d}{2})\frac{n}{2}}$ for $0 \leq c, d \leq 1$. Using the estimate from Stirling's formula, as described in Section 4.1.5, we get $c = (\sqrt{2}-1)$ and $d = \frac{\sqrt{2}}{3+\sqrt{2}}$ which evaluates to $\Theta^*(3.88215^n)$ plane graphs for the inner part of the double zig-zag chain.

The two outer parts of the double zig-zag chain are double circles without their convex hull edges. Using Corollary 8 we obtain that the total number of graphs is $\Theta^*(3.88215^n \cdot \frac{15.0045^n}{2^{\frac{n}{2}}})$. $\square$

In [14] a lower bound of $\Omega^*(39.80^n)$ plane graphs of the double chain has been shown. Using the argumentation from above we will strengthen this to $\Theta^*(39.80^n)$, which implies that the double zig-zag chain has in fact asymptotically more plane graphs than the double chain. The main difference to the above approach is that for a chain of the double chain we do not need to distinguish between points of the smaller and larger circles, but choose all vertices from the same chain.

In addition, as none of the edges in the interior part is flippable any more, red and gray edges provide now a factor of 2. Thus if we choose $c \cdot \frac{n}{2}$ points of each chain for the black edges, the general term of the sum simplifies to $\binom{\frac{n}{2}}{c\frac{n}{2}}^2 \cdot 2^{n-c\frac{n}{2}}$. This term is maximized for $c = \sqrt{2}-1$ leading to $\Theta^*(39.79898^n)$ plane geometric graphs of the double chain, the exact base being $20+14\sqrt{2}$.

4.2.2 The Number of Triangulations of the Double Zig-Zag Chain

Theorem 22 *The double zig-zag chain of n points contains $\Theta^*(8.48528^n)$ triangulations.*

Proof. To count the number of triangulations for the interior part of the double zig-zag chain we use the same approach as in Section 4.2.1. As every constructed edge has to be in the triangulations, we only get a factor 2 for gray, flippable edges. Non-flippable edges and red edges do not provide a multiplicative factor. We thus get $\sum_{0 \le m \le \frac{n}{2}} \sum_{i_1, i_2, i_1+j_1=i_2+j_2=m} \binom{\frac{n}{4}}{i_1} \cdot \binom{\frac{n}{4}}{j_1} \cdot \binom{\frac{n}{4}}{i_2} \cdot \binom{\frac{n}{4}}{j_2} \cdot 2^{\frac{n}{2}-(j_1+j_2)}$ triangulations. Again for symmetry reasons we assume $i_1 = i_2 = c \cdot \frac{n}{4}$ and $j_1 = j_2 = d \cdot \frac{n}{4}$ for the maximal term of the sum and we have to maximize the term $\binom{\frac{n}{4}}{c \cdot \frac{n}{4}}^2 \cdot \binom{\frac{n}{4}}{d \cdot \frac{n}{4}}^2 \cdot 2^{(1-d)\frac{n}{2}}$ for $0 \le c, d \le 1$.

Like in the previous section we use the estimate from Stirling's formula and conclude that $c = \frac{1}{2}$ gives the maximum for the first factor, namely $\Theta^*(2^{\frac{n}{2}})$. For d we have to maximize $2^{H(d)\frac{n}{2}+(1-d)\frac{n}{2}}$, which is equivalent to maximizing $H(d) - d$. It is well known that the maximum is obtained for $d = \frac{1}{3}$, resulting in $\Theta^*(2^{\frac{\log_2 3}{2} n})$. Combining both factors gives $\Theta^*(2^{H(1/3)n/2+5n/6})$ triangulations, which can be simplified to $\Theta^*(\sqrt{6}^n) = \Theta^*(2.44949^n)$ triangulations for the interior part.

In total we thus get $\Theta^*(\sqrt{12}^{\frac{n}{2}})^2 \cdot \Theta^*(\sqrt{6}^n) = \Theta^*((6\sqrt{2})^n) = \Theta^*(8.48528^n)$ triangulations for the double zig-zag chain. $\square$

5 Further Work and Open Problems

The most challenging question is of course to close the gap between maximizing examples and upper bounds. Here Sharir and Welzl [21] recently have made enormous progress on the upper bounds.

Obviously further work is required to improve (or even fill in) the entries in Tables 2 to 4. In Tables 2 and 3 the goal is to close or at least narrow the gaps between lower and upper bounds, while for Table 4 several entries, especially for the double zig-zag chain, are missing.

Several results given in these tables, like the bound of $\Omega^*(32.4944^n)$ for the number of connected plane geometric graphs for the DZZC, do not improve previous bounds. We thus do not explicitly give the proofs behind these bounds in the paper at hand, but encourage the interested reader to come up with improved bounds.

Concerning our lower bound construction an interesting question is the following: Does there exist an example that shows that an isomorphic mapping from any plane graph on top of a convex point set to any other point set (of the same cardinality) is not possible? If such an example does not exist, can we find a unified isomorphic mapping that works for all graphs?

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