On the Triangle Vector*

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Abstract

Let S be a set of n points in the plane in general position. In this note we study the so-called triangle vector τ of S. For each cardinality i, $0 \leq i \leq n-3$, $\tau(i)$ is the number of triangles spanned by points of S which contain exactly i points of S in their interior. We show relations of this vector to other combinatorial structures and derive tight upper bounds for several entries of τ, including $\tau(n-6)$ to $\tau(n-3)$.

1 Introduction

Throughout this paper let S be a set of n points in the plane in general position, that is, no three points of S are on a line. We define $\tau(i) \geq 0$ to be the number of i-triangles, that is, triangles spanned by three points in S with exactly i points of S in their interior. The triangle vector of S is defined as $\tau = (\tau(0), \tau(1), \ldots, \tau(n-3))$. For example, if and only if S is in convex position then $\tau(0) = \binom{n}{3}$ and all other entries of τ are zero. If S has a triangular convex hull then $\tau(n-3) = 1$, otherwise $\tau(n-3) = 0$. This trivially implies $\tau(n-3) \leq 1$. Obviously also $\sum_{i=0}^{n-3} \tau(i) = \binom{n}{3}$ and thus $\tau(0) \leq \binom{n}{3}$.

Bounding the rectilinear crossing number $\overline{\text{cr}}(\text{S})$ of the complete geometric graph K_n on S is a central topic in discrete geometry; see [3] for a nice survey. The following relation can be obtained by double counting 4-tuples of points: $\overline{\text{cr}}(\text{S}) = 3\binom{n}{4} - \sum_{i=0}^{\lfloor n/2 \rfloor -1} i(n-i-2)E_i$. Here E_i denotes the number of i-edges in S, that is, the number of edges connecting two points of S with exactly i points of S on one side of the line supporting this edge. Bounding the number of i-edges is therefore used to obtain bounds on the crossing number [3].

Similar, by double-counting the number of 4-tuples of S in non-convex position, we get $\sum_{i=0}^{n-3} i\tau(i) = \binom{n}{4} - \overline{\text{cr}}(\text{S})$. Combining the two relations leads to $\sum_{i=0}^{n-3} i\tau(i) = \sum_{i=0}^{\lfloor n/2 \rfloor -1} i(n-i-2)E_i - 2\binom{n}{4}$. In other words, there is a direct relation between the triangle vector and the vector of the number of i-edges. Therefore, and to lower bound $\overline{\text{cr}}(\text{S}) = \binom{n}{4} - \sum_{i=0}^{n-3} i\tau(i)$, we are interested in upper bounds of the entries of τ. In this note we show that for sufficiently large n we have $\tau(n-4) \leq 3$, $\tau(n-5) \leq 6$, and $\tau(n-6) \leq 10$.

2 Basics

Let $\text{CH}(S)$ be the convex hull of S and $|\text{CH}(S)|$ the number of points from S on the boundary of $\text{CH}(S)$. We call a point of S on the boundary of $\text{CH}(S)$ *extreme* point of S and a line segment connecting two adjacent extreme points an *extreme edge*. If $S = \{p_1, \ldots, p_n\}$, then for simplicity we write $\text{CH}(p_1, \ldots, p_n)$ instead of $\text{CH}(\{p_1, \ldots, p_n\})$.

For $n = 4, \ldots, 11$, Table 1 gives tight upper bounds for the entries of the triangle vector τ. We obtained this by exhaustive computations using the order type data base [1] which contains all combinatorially different point sets of size up to 11.

$n \mid i$	0	1	2	3	4	5	6	7	8
4	4	1							
5	10	2	1						
6	20	6	3	1					
7	35	11	5	3	1				
8	56	19	9	5	3	1			
9	84	30	16	9	6	3	1		
10	120	48	25	14	10	6	3	1	
11	165	66	35	22	16	10	6	3	1

Table 1: Upper bounds for $\tau(i)$ for $0 \leq i \leq n-3$, $n = 4, \ldots, 11$.

In the remaining sections we will provide upper bounds for several entries of τ. To this end the following definition will be useful.

Definition 1 *We denote by*
Δ *the set of all triangles spanned by (solely) inner (i.e., non-extreme) points of S,*
Δ° *the set of all triangles spanned by one extreme point and two inner points of S,*

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$\Delta^{\circ\circ}$ the set of all triangles spanned by two extreme points and one inner point of S,

$\Delta^{\circ\circ\circ}$ the set of all triangles spanned by (solely) extreme points of S.

For each of those sets we denote by $|D|_i$ the number of i-triangles in D, $D \in \{\Delta, \Delta^{\circ}, \Delta^{\circ\circ}, \Delta^{\circ\circ\circ}\}$.

3 Upper bound for $\tau(n-4)$

Next we provide a tight upper bound for $\tau(n-4)$. The proofs of the following two lemmas can be found in the full version of this paper.

Lemma 2 *For all $i < |\operatorname{CH}(S)|$, it holds that $\tau(n-i) = 0$.*

Lemma 3 *Let $p_1, p_2, p_3, p_4 \in S$ and $i \in \{0, \ldots, n-3\}$. If both p_1, p_2, p_3 and p_1, p_2, p_4 span i-triangles and $p_3 \neq p_4$, then neither triangle contains the other.*

Theorem 4 $\tau(n-4) \leq 3$ *for $n \geq 5$.*

Proof. We distinguish three cases by different cardinalities of $|\operatorname{CH}(S)|$.

Case 1 $|\operatorname{CH}(S)| \geq 5$
By Lemma 2 it follows that $\tau(n-4) = 0$.

Case 2 $|\operatorname{CH}(S)| = 4$
Exactly $n-4$ points lie in the interior of the polygon spanned by the extreme points of S. Thus every existing $(n-4)$-triangle belongs to $\Delta^{\circ\circ\circ}$.
As $|\Delta^{\circ\circ\circ}| = 4$ and the fact that there are two disjoint pairs of triangles in $\Delta^{\circ\circ\circ}$ it follows that for $n \geq 5$, we get $|\Delta^{\circ\circ\circ}|_{n-4} \leq 2$, thus $\tau(n-4) \leq 2$.

Case 3 $|\operatorname{CH}(S)| = 3$
Exactly $n-3$ points lie in the interior of the polygon spanned by the extreme points of S, thus every existing $(n-4)$-triangle belongs to $\Delta^{\circ\circ}$.
For a fixed line segment assume that it spans an $(n-4)$-triangle with an inner point. Then all other inner points lie within this triangle, thus every further $(n-4)$-triangle spanned by this line segment (and another inner point) is contained in the first one which is a contradiction to Lemma 3. Therefore we know that for every extreme edge of the triangle we have at most one $(n-4)$-triangle, thus $\tau(n-4) \leq 3$.

Overall it follows that $\tau(n-4) \leq 3$. $\square$

Figure 1 provides an example showing that the upper bound $\tau(n-4) \leq 3$ is tight for all $n \geq 6$.

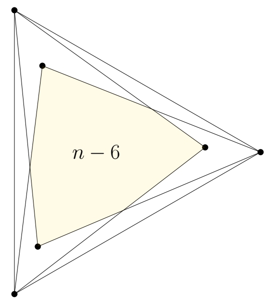

Figure 1: A point set with $n-6$ points in the central cell, showing $\tau(n-4) = 3$ for $n \geq 6$.

4 Upper bound for $\tau(n-5)$

Our next goal is to derive an upper bound for $\tau(n-5)$.

Theorem 5 $\tau(n-5) \leq 6$ *for $n \geq 6$.*

Proof. (Sketch, see full version for a detailed proof) For $n = 6$ the result follows from Table 1, therefore we can assume $n \geq 7$.
Similar to the proof of Theorem 4 we consider different cardinalities of $|\operatorname{CH}(S)|$ in separate cases.

Case 1 $|\operatorname{CH}(S)| \geq 6$
Again Lemma 2 implies $\tau(n-5) = 0$.

Case 2 $|\operatorname{CH}(S)| = 5$
Observe that all $(n-5)$-triangles are in $\Delta^{\circ\circ\circ}$, i.e., are spanned by three extreme points. No cell in the 5-gon is covered by more than five of those triangles, therefore $\tau(n-5) \leq 5$.

Case 3 $|\operatorname{CH}(S)| = 4$
It can be shown that there are no $(n-5)$-triangles in Δ and Δ° and at most two $(n-5)$-triangles in $\Delta^{\circ\circ\circ}$. The last statement follows from the case distinction depicted in Figure 2. For the triangles in $\Delta^{\circ\circ}$, each

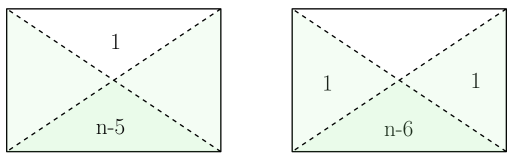

Figure 2: The possible cases for two $(n-5)$-triangles in $\Delta^{\circ\circ\circ}$ for $|\operatorname{CH}(S)| = 4$.

extreme edge of the 4-gon can span at most one $(n-5)$-triangle (cf. Lemma 3), therefore in total we get $\tau(n-5) \leq 6$.

Case 4 $|\operatorname{CH}(S)| = 3$
Obviously we have $|\Delta|_{n-5} = |\Delta^{\circ\circ\circ}|_{n-5} = 0$.
The upper bounds $|\Delta^{\circ\circ}|_{n-5} \leq 6$ and $|\Delta^{\circ}|_{n-5} \leq 3$

are rather easy to derive: each extreme edge spans at most two $(n-5)$-triangles (Lemma 3) and each extreme point spans at most one $(n-5)$-triangle, respectively. The main effort is required to show that in total those two sets contain at most six $(n-5)$-triangles.
If one edge spans two $(n-5)$-triangles, its two adjacent extreme points do not span any $(n-5)$-triangle with two inner points. Therefore we can conclude that for $|\Delta^{\circ\circ}|_{n-5} \geq 4$ (in this case at least one edge spans two (n-5)-triangles) it follows $|\Delta^{\circ}|_{n-5} \leq 1$ and for $|\Delta^{\circ\circ}|_{n-5} \geq 5$ (i.e., at least two edges span two $(n-5)$-triangles, respectively) it holds $|\Delta^{\circ}|_{n-5} = 0$.

This proves the assumption $|\Delta^{\circ} \cup \Delta^{\circ\circ}|_{n-5} \leq 6$, and thus $\tau(n-5) \leq 6$. □

Figure 3 shows two different examples reaching the upper bound $\tau(n-5) = 6$.

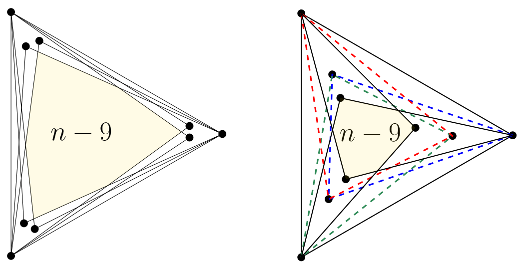

Figure 3: Two point sets, each with $n-9$ points in the central cell, with $\tau(n-5) = 6$.

5 Upper bound for $\tau(n-6)$

For the proof of our next statement we need an additional definition concerning the triangles in $\Delta^{\circ\circ}$.

Definition 6 *By $\Delta_e^{\circ\circ}$ we denote the set of all triangles in $\Delta^{\circ\circ}$ spanned by an extreme edge, i.e., two adjacent extreme points of S and one inner point, and by $\Delta_d^{\circ\circ}$ the set of all triangles in $\Delta^{\circ\circ}$ spanned by a diagonal, i.e., two nonadjacent extreme points of S and one inner point.*

Note that $\Delta_e^{\circ\circ}$ and $\Delta_d^{\circ\circ}$ are disjoint and that $\Delta_e^{\circ\circ} \cup \Delta_d^{\circ\circ} = \Delta^{\circ\circ}$, implying $|\Delta_e^{\circ\circ}|_i + |\Delta_d^{\circ\circ}|_i = |\Delta^{\circ\circ}|_i$.
The proofs of the following statements can be found in the full version.

Lemma 7 *Let $|\operatorname{CH}(S)| = 4$. Then the following implications hold:*

(a) If there are six $(n-6)$-triangles in $\Delta_e^{\circ\circ}$, then $|\Delta^{\circ}|_{n-6} = 0$.

(b) If there are five $(n-6)$-triangles in $\Delta_e^{\circ\circ}$, then $|\Delta^{\circ}|_{n-6} \leq 2$ and $|\Delta^{\circ\circ\circ}|_{n-6} = 0$.

*(c) If there are at least three $(n-6)$-triangles in $\Delta_d^{\circ\circ}$, then $|\Delta^{\circ}|_{n-6} \leq 2$ and $|\Delta^{\circ\circ\circ}|_{n-6} = 0$.
If there are exactly four $(n-6)$-triangles in $\Delta_d^{\circ\circ}$, then $|\Delta^{\circ}|_{n-6} = 0$.*

(d) If there are two $(n-6)$-triangles in $\Delta_d^{\circ\circ}$, then $|\Delta^{\circ} \cup \Delta^{\circ\circ\circ}|_{n-6} \leq 4$.

Lemma 8 *Let $|\operatorname{CH}(S)| = 3$. If an extreme edge $\overline{pq}$ of S spans three $(n-6)$-triangles in $\Delta^{\circ\circ}$, then neither p nor q is incident to any $(n-6)$-triangle in Δ°.*

Theorem 9 *$\tau(n-6) \leq 10$ for $n \geq 8$.*

Proof. (Sketch, see full version for a detailed proof) For $8 \leq n \leq 11$ the result follows from Table 1. Thus we can assume $n \geq 12$.

Case 1 $|\operatorname{CH}(S)| \geq 7$
By Lemma 2 we have $\tau(n-6) = 0$.

Case 2 $|\operatorname{CH}(S)| = 6$
As all possible $(n-6)$-triangles lie in $\Delta^{\circ\circ\circ}$, i.e., are spanned by extreme points of S, the idea is, similar as in the Proof of Case 2 of Theorem 5, to count the number of covering triangles for each cell in the 6-gon. It follows that $|\Delta^{\circ\circ\circ}|_{n-6} \leq 8$, i.e., $\tau(n-6) \leq 8$.

Case 3 $|\operatorname{CH}(S)| = 5$
We consider the triangles in Δ, Δ°, $\Delta^{\circ\circ\circ}$, $\Delta_e^{\circ\circ}$ and $\Delta_d^{\circ\circ}$ separately.
For the first three sets, we get upper bounds $|\Delta|_{n-6} = |\Delta^{\circ}|_{n-6} = 0$ and $|\Delta^{\circ\circ\circ}|_{n-6} \leq 4$.
For the triangles in $\Delta_e^{\circ\circ}$ and $\Delta_d^{\circ\circ}$ we consider two possible cases each. We have either $|\Delta_e^{\circ\circ}|_{n-6} = 3$ and $|\Delta^{\circ\circ\circ}|_{n-6} = 0$ or $|\Delta_e^{\circ\circ}|_{n-6} \leq 2$ and $|\Delta^{\circ\circ\circ}|_{n-6} \leq 4$. Therefore it follows that $|\Delta_e^{\circ\circ} \cup \Delta^{\circ\circ\circ}|_{n-6} \leq 6$.
On the other hand we have $|\Delta_d^{\circ\circ}|_{n-6} \leq 5$, but for $|\Delta_d^{\circ\circ}|_{n-6} = 5$ we can conclude $|\Delta^{\circ\circ\circ}|_{n-6} = 0$.
Overall it follows that $\tau(n-6) \leq 10$.

Case 4 $|\operatorname{CH}(S)| = 4$
In this case we again prove upper bounds for different set individually. However, it is much more tedious to show that in total the number of $(n-6)$-triangles does not exceed the claimed upper bound.
For the separate upper bounds we get

- $|\Delta|_{n-6} = 0$,
- $|\Delta^{\circ}|_{n-6} \leq 4$,
- $|\Delta^{\circ\circ\circ}|_{n-6} \leq 2$,
- $|\Delta^{\circ} \cup \Delta^{\circ\circ\circ}|_{n-6} \leq 5$,
- $|\Delta_d^{\circ\circ}|_{n-6} \leq 4$,
- $|\Delta_e^{\circ\circ}|_{n-6} \leq 6$.

Figure 4 indicates how these upper bounds affect each other and which implications (provided by Lemma 7) are needed to approach the overall upper bound of $\tau(n-6) = |\Delta \cup \Delta^\circ \cup \Delta_e^{\circ\circ} \cup \Delta_d^{\circ\circ} \cup \Delta^{\circ\circ\circ}|_{n-6} \leq 10$.

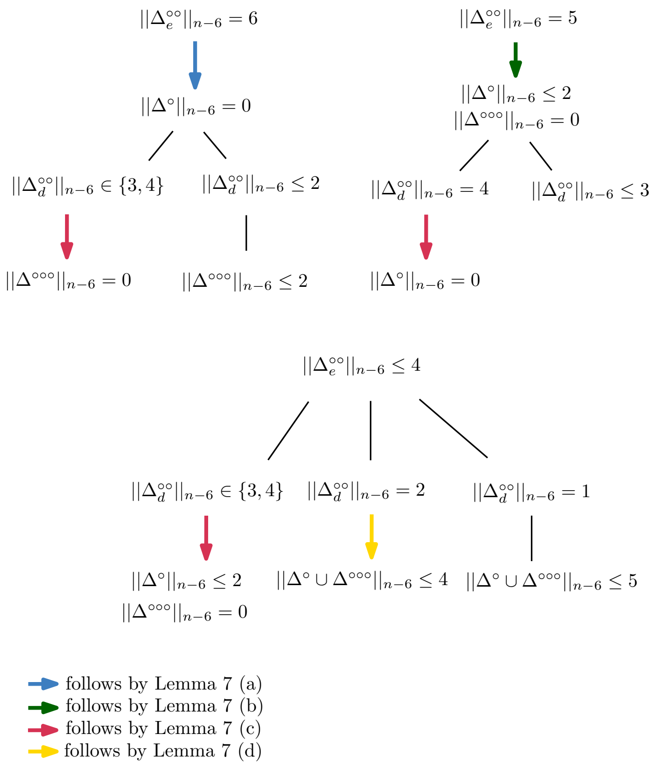

Figure 4: Overview on how to show that $\tau(n-6) \leq 10$. Both statements for the separately considered subsets are used and implications between the subsets provided by Lemma 7 are indicated by colored arrows.

Case 5 $|\operatorname{CH}(S)| = 3$
For the case of $n-3$ inner points we get the following upper bounds:

- $|\Delta|_{n-6} \leq 1$
- $|\Delta^\circ|_{n-6} \leq 6$
- $|\Delta^{\circ\circ}|_{n-6} \leq 9$
- $|\Delta^{\circ\circ\circ}|_{n-6} = 0$

To satisfy the overall upper bound, we distinguish between several cases for the number of $(n-6)$-triangles in $\Delta^{\circ\circ}$.

Case 5a $|\Delta^{\circ\circ}|_{n-6} \geq 7$
We apply Lemma 8 to this case. Thus $|\Delta^{\circ\circ}|_{n-6} \geq 7$ implies $|\Delta^\circ|_{n-6} \leq 2$ and $|\Delta^{\circ\circ}|_{n-6} \in \{8, 9\}$ implies $|\Delta^\circ|_{n-6} = 0$. Combined with the upper bound for $(n-6)$-triangles in Δ we are done.

Case 5b $|\Delta^{\circ\circ}|_{n-6} \leq 3$
In this case the separated upper bounds directly sum up to 10.

Cases 5c and 5d $|\Delta^{\circ\circ}|_{n-6} = 4, 5$
In both cases a more sophisticated case analysis has to be made. For example we show which extreme points span how many (n-6)-triangles and how they are related.

Case 5e $|\Delta^{\circ\circ}|_{n-6} = 6$
In that case each extreme point spans two $(n-6)$-triangles. Using further observations it follows that either $|\Delta^{\circ\circ}|_{n-6} \leq 3$ or else $|\Delta^{\circ\circ}|_{n-6} \leq 4$ and $|\Delta|_{n-6} = 0$.

This concludes the case $|\operatorname{CH}(S)| = 3$.

In summary for all cases we obtained the claimed upper bound $\tau(n-6) \leq 10$. $\square$

Figure 5 shows a point set obtaining the upper bound $\tau(n-6) = 10$, implying that Theorem 9 is tight.

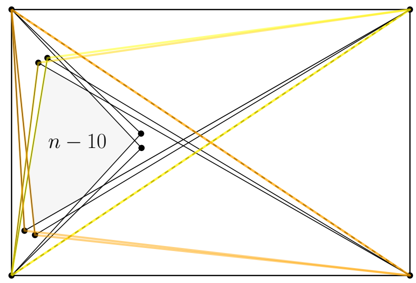

Figure 5: A point set with $\tau(n-6) = 10$. Six $(n-6)$-triangles in $\Delta_e^{\circ\circ}$ are drawn in black; four $(n-6)$-triangles in $\Delta_d^{\circ\circ}$ are drawn in yellow and orange.

6 Conclusion

We have shown tight upper bounds for $\tau(n-6)$ to $\tau(n-4)$. This leads us to the following conjecture, which holds for $k \leq 6$.

Conjecture 10 *For a constant k, $3 \leq k \leq 10$, and n large enough we have $\tau(n-k) \leq \binom{k-1}{2}$.*

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