Blocking Delaunay Triangulations

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Abstract

Given a set B of n black points in general position, we say that a set of white points W blocks B if in the Delaunay triangulation of $B \cup W$ there is no edge connecting two black points. We give the following bounds for the size of the smallest set W blocking B: (i) $3n/2$ white points are always sufficient to block a set of n black points, (ii) if B is in convex position, $5n/4$ white points are always sufficient to block it, and (iii) at least $n-1$ white points are always necessary to block a set of n black points.

Keywords: Proximity graphs, Delaunay graph, Graph drawing, Witness graphs

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1. Introduction

Proximity graphs were originally defined to capture different notions of proximity in a set of points [8]. A particular proximity graph in a set of points S is defined by assigning to every pair of points in S a region (or family of regions). Then, the graph contains the edge pq if and only if at least one of the regions corresponding to the pair is empty of points of S. Examples of such graphs are the Gabriel graph, the relative neighborhood graph, and the Delaunay triangulation.

Recently, Dulieu [7] and Aronov et al. [3, 4] extended the notion of proximity graphs with the concept of *witness proximity graphs*. In this generalized setting, we have a second point set W. The points of W are the *witnesses* which account for the existence of an edge between two points of S. The authors consider two different possibilities. In the first one, an edge between two points $p, q \in S$ exists if some of the regions corresponding to the pair p, q do not contain any witness point. In the second version, an edge between two points of S exists if there exists a region containing a witness point. In this paper we deal with the first version of the witness Delaunay graph: Given a set B of *black* points and a set W of *white* points (the witnesses), we follow the notation in [3] and consider the graph $DG^{-}(B, W)$. In this graph, two points $p, q \in B$ are adjacent if and only if there exists an open disk which is empty of points of W and whose bounding circle passes through p and q.

In the same paper, the following combinatorial problem is proposed: If B has size n, find the smallest c such that we can always guarantee the existence of a set W, with size cn, and such that $DG^{-}(B, W)$ does not contain any edges. This problem can also be formulated as a very natural stabbing problem: Given a set B of n points, let $\mathcal{C}$ be the set of open disks each having at least two points of B on its bounding circle. We say that a point p *stabs* a disk $D \in \mathcal{C}$ if $p \in D$. Give a bound for the size of a set of points stabbing all disks in $\mathcal{C}$.

Let $\mathcal{D}$ be the set of Delaunay disks of B, i.e., open disks which are empty of points of B and whose bounding circles pass through at least two points of B. In [3] it is shown that in order to stab all disks in $\mathcal{C}$ it is sufficient to stab all disks in $\mathcal{D}$. For the combinatorial problem mentioned previously, they show that in order to stab the set of Delaunay disks generated by a set of n points, $2n - 2$ points are always sufficient, and n points are sometimes necessary. If points of B are in convex position, they improve the upper bound to $4n/3$. Similar problems have been studied from an algorithmic point of view in [1].

In the present work we focus on this combinatorial problem, with a slightly different language that we find more intuitive: For $DG^{-}(B,W)$ having no edges it is necessary and sufficient that points in W stab all disks in $\mathcal{D}$. This in turn is equivalent to the fact that there is no edge connecting two black points in the Delaunay graph of $B \cup W$. If this is the case, we say that the set W *blocks* the set B.

We show the following results for a set B of n black points:

- $3n/2$ white points are always sufficient to block a set B in general position.
- If B is in convex position, then $5n/4$ white points are always sufficient to block B.
- At least $n-1$ white points are always needed to block B.

We assume that the set $B \cup W$ is in *general position* (no three points on a line and no four points on a circle). Throughout this paper, we denote the Delaunay triangulation of S by $DT(S)$, the Voronoi diagram of S by $V(S)$, and the Voronoi region of point $p \in S$ in $V(S)$ by $V_p(S)$.

2. An upper bound for arbitrary point sets

We start with a constructive approach for blocking point sets in general position that utilizes the duality between Delaunay triangulations and Voronoi diagrams.

Theorem 1. *Let B be a set of n black points in general position. There always exists a set W of at most $3n/2$ white points that blocks B.*

Proof. Let I be the largest independent set in $DT(B)$, and $C = B \setminus I$ its complement. Because every triangulation is 4-colorable, we know that $|C| \leq 3n/4$. We are going to show that B can be blocked by adding two white points in a close neighborhood of each point in C.

First, for each $p \in C$ we choose a point $\eta(p) \in C$ among the neighbors of p in $DT(B)$. This is always possible, because if pqr is a triangle in $DT(B)$, then it cannot happen that q and r are both in I.

Then, for each $p \in C$ we choose a point x_p (not in B) in the interior of the Voronoi cell $V_p(B)$, and satisfying the following conditions: (i) The ray

$x_p p$ intersects the Voronoi edge of $V(B)$ that separates $V_p(B)$ from $V_{\eta(p)}(B)$. Let y_p be this point of intersection. (ii) In the case in which $q = \eta(p)$ and $p = \eta(q)$, x_p and x_q have to be chosen in such a way that $y_p \neq y_q$ (see Figure 1).

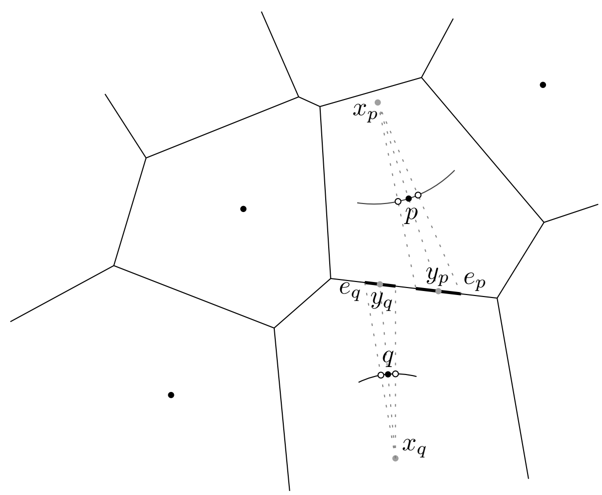


Figure 1: Blocking a black point by placing two white points in its Voronoi cell.

Now we assign a segment e_p to each point $p \in C$ such that e_p is centered at y_p and contained in the edge of $V(B)$ separating the Voronoi regions of p and $\eta(p)$. If $q = \eta(p)$ and $p = \eta(q)$, we choose the intervals e_p and e_q small enough to be disjoint.

Next, we add two white points in $V_p(B)$ in the following way. Consider the circle that is centered at x_p and passes through p, and place the white points at the intersections of this circle with the line segments defined by x_p and the endpoints of e_p. Note that neither x_p nor y_p belongs to our set of white points.

Once we have done this for every point $p \in C$, we claim that in the Voronoi diagram of the resulting set no pair of black points have adjacent regions. The only area where p could be closer to some black point than one of the two shielding white points constructed for it is inside the wedge defined by the bisectors of p and these two white points. The apex of the wedge is $x_p \in V_p(B)$, and only point $q = \eta(p)$ has the possibility to be a Voronoi neighbor for p. But by construction, the intervals e_p and e_q are disjoint, so this does not happen. $\square$

3. An upper bound for convex sets

For the special case of point sets in convex position we improve the $4n/3$ bound in [3] by translating the problem into a combinatorial setting.

We call a triangle of a triangulation an *ear* if two of its edges are boundary edges of the convex hull. The vertex adjacent to both of them is the *tip* of the ear. A triangle without edges on the boundary of the convex hull is an *inner triangle*.

Considering the properties of the Delaunay triangulation, we propose the following two simple operations to block Delaunay edges. *Blocking a single edge* is done by placing a white point arbitrarily close to the center of the edge. For inner edges this can be done on any of its two sides, and for edges of the convex hull the white point has to be placed slightly outside the convex hull. *Blocking a vertex* p with two white points is achieved in the following way: Consider a line ℓ passing through p and leaving the rest of the point set on one side, say the left. Let D be an empty open disk on the left side of ℓ, bounded by a circle tangent to ℓ at p. The part of D outside the convex hull of the point set is divided into two connected regions. Placing one white point into each of these two regions *blocks* p, as the two white points become neighbors in an updated combined Delaunay triangulation.

Reconsidering the presented blocking operations we transform the whole setting into a combinatorial framework. We call blocking a single edge *coloring an edge* with cost 1, and blocking a vertex *coloring a vertex* with cost 2, where the latter also colors all incident edges. Thus, our task can be rephrased as coloring all edges of a given triangulation with minimal cost. Let $C(n)$ denote the maximum minimal cost over all sets of n points in convex position. Clearly, an upper bound on $C(n)$ is also an upper bound on the number of white points needed in the geometric setting, while the converse may not be true: It is conceivable that additional operations exist that allow us to block edges in the geometric setting in a more efficient way.

An (n, a, k)-*cut* of a triangulation T of a set of n points is a separation of the n points into two disjoint groups A and B with $|A| = a$ and $|B| = n - a$, plus a coloring of A with cost k such that any edge of T incident to a point in A is colored, see Figure 2.

Lemma 2. *If for a triangulation T of a convex n-gon there exists an (n, a, k)-cut, then the cost of coloring T is at most $C(n - a) + k$.*

Proof. Let A and B be the two sets as defined for the (n, a, k)-cut. We use the coloring of A given by the cut and remove all colored vertices and edges.

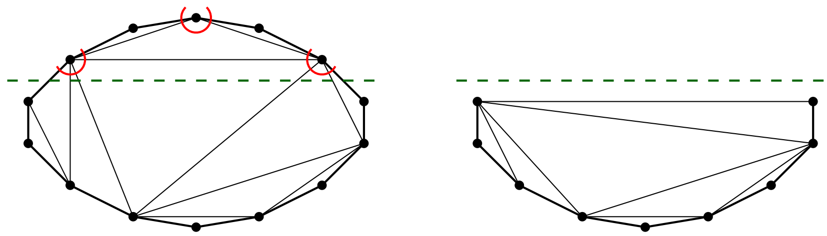
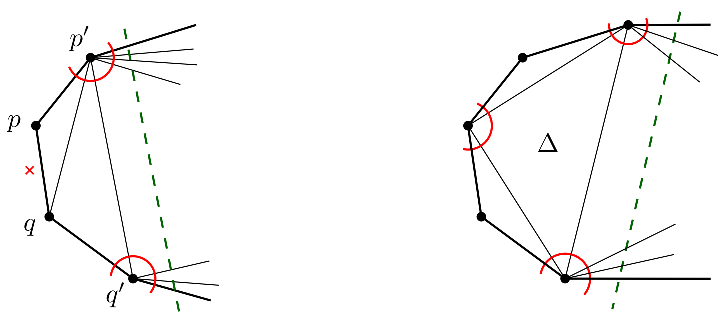


Figure 2: An $(14,5,6)$-cut and the retriangulated subset.

We complete the remaining graph of B to a full triangulation of the convex set B by (arbitrarily) retriangulating the holes induced by removing A (cf. Figure 2, right), and color this triangulation of B with cost at most $C(n-a)$. Combining the two colorings of A and B (where we can ignore colored edges of B which are not part of T), we obtain a coloring of T with cost at most $C(n-a)+k$. □



(a) (b)

Figure 3: The two cases for a convex set: removing an ear (left), and removing an inner triangle with two incident ears (right).

Theorem 3. $C(n) \leq \frac{5n}{4}$.

Proof. We prove the statement by induction on the number n of vertices. For the induction base it is straightforward that for $n \leq 3$ we have $C(n) \leq n$. So assume the statement is true for any set of size $n' < n$, and consider a triangulation T of n points. We distinguish two cases.

Case 1. Assume that there exists an ear of T with tip p such that a neighbor q of p (neighborhood is with respect to the convex hull) has precisely one incident inner edge, see Figure 3(a). We color the two other neighbors p' and q' of p and q, respectively, as well as the edge pq. With $A = \{p, q, p', q'\}$ this induces an $(n, 4, 5)$-cut of T. By Lemma 2 we have $C(n) \leq 5 + C(n-4) \leq 5 + \frac{5(n-4)}{4} = \frac{5n}{4}$, where the last inequality follows from the induction hypothesis.

Case 2. Otherwise, all neighbors of the tip of an ear are incident to at least two inner edges. This is equivalent to the fact that all ears are adjacent to inner triangles. Because in any triangulation of a convex set the number of ears is equal to the number of inner triangles plus 2 (this follows from considering the dual tree), there exists at least one inner triangle Δ adjacent to two ears. We color the three vertices of Δ, see Figure 3(b). The tips of the two ears incident to Δ together with the three vertices of Δ form our set A. As A has cardinality 5, this induces an $(n, 5, 6)$-cut of T, and similar as before we have $C(n) \leq 6 + C(n-5) < \frac{5n}{4}$. □

Corollary 4. *Let B be a set of n black points in convex position. There always exists a set W of at most $5n/4$ white points that blocks B.*

4. A lower bound for arbitrary point sets

In this section we provide a lower bound on the number of points needed to block any given set, again using independent vertices. In [3] it is shown that there exist sets of n points that need n points to be blocked. We prove that *any* set of n points requires *at least* $n - 1$ points to be blocked.

Lemma 5. *The size of an independent set in the Delaunay triangulation of a set of n points is at most $\lfloor \frac{n+1}{2} \rfloor$.*

Proof. It is known that every Delaunay triangulation contains a perfect matching of its vertices [6]. Consider such a perfect matching M, and an independent set I. Then for every edge in M, at most one of its endpoints can be in I. If n is odd, then the non-matched point can be in I as well. □

Note that Lemma 5 describes a special property of the Delaunay triangulation, as there exist sets of n points, which can be triangulated in a way that the triangulation has an independent set of size as much as $\frac{2n-2}{3}$. For example, take a set of k white points and triangulate it arbitrarily. Place one black point in the interior of each white triangle. Further, place one black point outside but close to each convex hull edge. Complete the full set of

$n = 3k - 2$ points to a triangulation with k white and $2k - 2$ independent black points.

Theorem 6. *For any set B of n black points, at least $n - 1$ white points are necessary to block it.*

Proof. Assume that the white point set W, $|W| = m$, blocks B. Then the joint Delaunay triangulation $DT(B \cup R)$ does not contain any edge between two black vertices, which implies that B is an independent set in $DT(B \cup R)$. Thus, by Lemma 5, we have $n \leq \lfloor \frac{n+m+1}{2} \rfloor$, and consequently $m \geq n - 1$. □

5. Discussion

We have shown that for blocking a set B of n black points, $3n/2$ white points are sufficient for any set B, and $5n/4$ white points are sufficient if B is in convex position. Our approach is constructive, i.e., an algorithm to compute the set W can be designed by following the lines given above. We do not go into detail here, as the focus of our paper is on combinatorial aspects of the problem. Thus we let algorithmic details for future work.

Further, we know that we always need $n - 1$ white points for blocking any set with n black points. As already mentioned, in [3] the authors show that there exist point sets B of n black points which need n white points to be blocked. Figure 4 shows the intuition behind their proof: The touching points of the coins define the set B, shown as black dots. As the touching coins form cycles (inducing the solid Delaunay edges), $n = |B|$ equals the number of coins. Note that the coins correspond to a subset of interior-disjoint Delaunay disks. Thus the number of required white points is at least the number of coins. The dashed edges complete the Delaunay triangulation.

So far we have not been able to construct a set that needs more than n white points to be blocked, and to the best of our knowledge, no example is known that can in fact be blocked with only $n - 1$ points. Thus we state the following conjecture.

Conjecture 1. *For any set B of n black points in convex position, n white points are necessary and sufficient to block B.*

In fact, from what is currently known, the conjecture might be true even for point sets in general position.

Independently, the algorithmic issue of finding a minimal set of blocking white points arises as a natural question for future work. A related problem

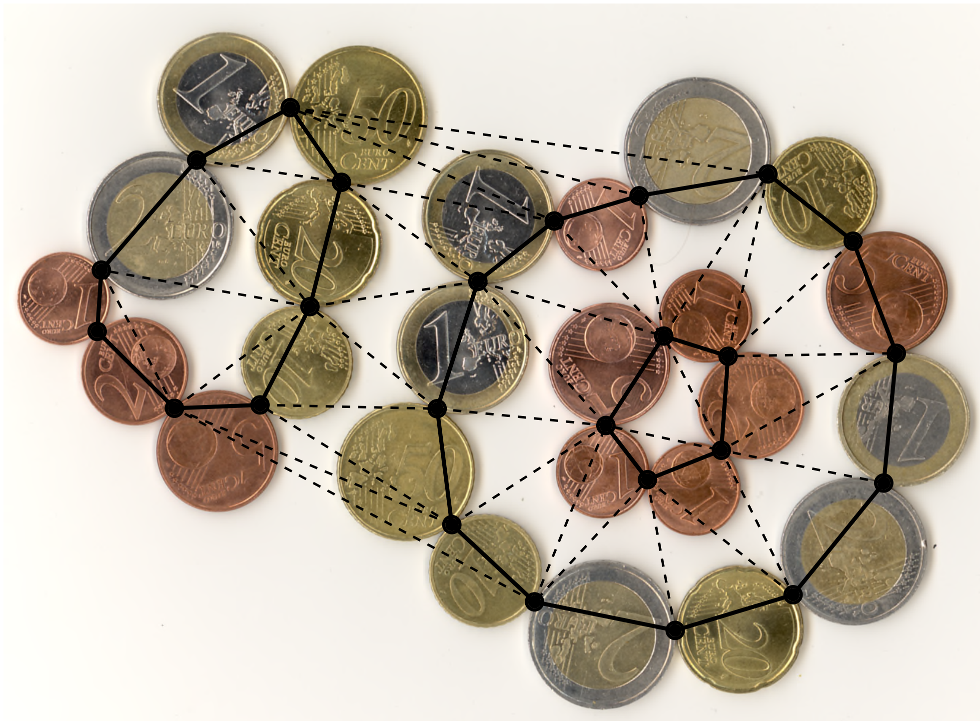


Figure 4: Euros proving a lower bound of n white points.

has been studied recently by de Berg et al. [5], although the objective is not blocking edges, but deleting edges by removing points: Given a set of black and white sites, it is NP-hard to compute the minimum number of white sites that have to be removed so that the union of the Voronoi cells of the black points is a connected region. The problem studied in this paper can also be related to the Voronoi games started in [2]. Suppose that player one has n black points that can be placed in the plane, and player two has m white points that can be placed to block black connections. This setting reveals a whole family of interesting and difficult open problems.

References

[1] S.I. Ahmed, M. Hasan, and A. Sopan. Vindictive Voronoi games and stabbing Delaunay circles. Proc. ISVD 2010, pp. 124-131.

[2] H.-K. Ahn, S.-W. Cheng, O. Cheong, M. Golin, and R. van Oostrum. Competitive facility location: the Voronoi game. Theoretical Computer Science 310 (2004) 457-467.

[3] B. Aronov, M. Dulieu, and F. Hurtado. Witness (Delaunay) Graphs. Computational Geometry: Theory and Applications 44 (2011) 329-344.

[4] B. Aronov, M. Dulieu, and F. Hurtado. Witness Gabriel Graphs. Computational Geometry: Theory and Applications, in press. DOI:10.1016/j.comgeo.2011.06.004.

[5] M. de Berg, D. Gerrits, A. Khosravi, I. Rutter, C. Tsirogiannis, and A. Wolff. How Alexander the Great brought the Greeks together while inflicting minimal damage to the Barbarians. Proc. 26th European Workshop on Computational Geometry 2010, pp. 73-76.

[6] M.B. Dillencourt. Toughness and Delaunay Triangulations. Discrete and Computational Geometry 5 (1990) 575-601.

[7] M. Dulieu. PhD thesis, Polytechnic Institute of NYU. In preparation.

[8] J.W. Jaromczyk and G.T. Toussaint. Relative neighborhood graphs and their relatives. P-IEEE 80 (1992) 1502-1517.