On k-Gons and k-Holes in Point Sets

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Abstract

We consider a variation of the classical Erdős-Szekeres problems on the existence and number of convex k-gons and k-holes (empty k-gons) in a set of n points in the plane. Allowing the k-gons to be non-convex, we show bounds and structural results on maximizing and minimizing their numbers. Most noteworthy, for any k and sufficiently large n, we give a quadratic lower bound for the number of k-holes, and show that this number is maximized by sets in convex position. We also provide an improved lower bound for the number of convex 6-holes.

1 Introduction

Let S be a set of n points in general position in the plane. A k-gon is a simple polygon spanned by k points of S. A k-hole is an empty k-gon; that is, a k-gon which contains no points of S in its interior.

Around 1933 Esther Klein raised the following question which was (partially) answered in the classical paper by Erdős and Szekeres [12] in 1935: "Is it true that for any k there is a smallest integer $g(k)$ such that any set of $g(k)$ points contains at least one convex k-gon?" As observed by Klein, $g(4) = 5$, and Kalbfleisch et al. [18] solved the more involved case of $g(5) = 9$. The case $k = 6$ was only solved as recently as 2006 by Szekeres and Peters [23]. They showed that $g(6) = 17$ by an exhaustive computer search. The well known Erdős–Szekeres Theorem [12] states that $g(k)$ is finite for any k. The current best bounds are $2^{k-2} + 1 \leq g(k) \leq \binom{2k-5}{k-2} + 1$; see [13, 24].

Erdős and Guy [11] posed the following generalization: "What is the least number of convex k-gons determined by any set S of n points in the plane?" The trivial solution for the case $k = 3$ is $\binom{n}{3}$. But for convex 4-gons this question is related to the search for the rectilinear crossing number $\bar{cr}(S)$ of S; see the next section for details.

In 1978 Erdős [9] raised the following question for convex k-holes: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?" As had been observed by Esther Klein, every set of 5 points determines a convex 4-hole, and 10 points always contain a convex 5-hole, a fact proved by Harborth [16]. However, in 1983 Horton showed that there exist arbitrarily large sets of points containing no convex 7-hole [17]. It took almost a quarter of a century after Horton's construction to answer the existence question for 6-holes. In 2007/08 Nicolás [20] and independently Gerken [15] proved that every sufficiently large point set contains a convex 6-hole; see also [26].

Erdős also proposed the following variation of the problem [10]. "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?" We know by Horton's construction that $h_k(n) = 0$ for $k \geq 7$. Table 1 shows the current best lower and upper bounds for $k = 3 \ldots 6$; see [3, 4, 5, 7, 25] and Section 6.

$$n^2 - O(n \log n) \leq h_3(n) \leq 1.6196n^2 + o(n^2)$$
$$\frac{n^2}{2} - O(n) \leq h_4(n) \leq 1.9397n^2 + o(n^2)$$
$$3\lfloor \tfrac{n-4}{8} \rfloor \leq h_5(n) \leq 1.0207n^2 + o(n^2)$$
$$\lfloor \tfrac{n-1}{858} \rfloor - 2 \leq h_6(n) \leq 0.2006n^2 + o(n^2)$$

Table 1: Bounds on the number $h_k(n)$ of convex k-holes.

In this paper we generalize the above questions by allowing k-gons and k-holes to be non-convex. Thus whenever we refer to a (general) k-gon or k-hole, unless it is specifically stated to be convex or non-convex, it could be either.

A set of k points in convex position obviously spans precisely one convex k-hole. In contrast, a point set might admit exponentially many different polygonizations (spanning cycles) [8, 14, 22], which implies that the number of k-gons and k-holes can be larger than $\binom{n}{k}$. This makes questions like minimizing or maximiz-

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	numbers of k-gons				numbers of k-holes			
	convex	non-convex	general		convex	non-convex	general	
	min	max	min	max	min	max	min	max
$k=4$	$\bar{cr}(n)$ $\Theta(n^4)$	$3\binom{n}{4}-3\bar{cr}(n)$ $\Theta(n^4)$	$\binom{n}{4}$ $\Theta(n^4)$	$3\binom{n}{4}$ $-2\bar{cr}(n)$ $\Theta(n^4)$	$\geq \frac{n^2}{2}-O(n)$ $\leq 1.9397n^2+o(n^2)$ $\Theta(n^2)$ [5, 7]	$\leq \frac{n^3}{2}-O(n^2)$ $\geq \frac{n^3}{2}-O(n^2 \log n)$ $\Theta(n^3)$ [2]	$\geq \frac{5}{2}n^2-O(n)$ $\leq \frac{n^3}{2}+O(n^2)$ $\Omega(n^2)$ [2], $O(n^3)$ [Sec. 5]	$\binom{n}{4}$ $\Theta(n^4)$ [2]
$k=5$	$\Theta(n^5)$ [6]	$10\binom{n}{5}$ $-2(n-4)\bar{cr}(n)$ $\Theta(n^5)$ [3]	$\binom{n}{5}$ $\Theta(n^5)$ [3]	$\Theta(n^5)$ [Sec. 2]	$\geq 3\lfloor\frac{n-4}{8}\rfloor$ $\leq 1.0207n^2+o(n^2)$ $\Omega(n)$ [3], $O(n^2)$ [5]	$\leq n!/(n-4)!$ $\Theta(n^4)$ [Sec. 4]	$\geq 17n^2-O(n)$ $\leq O(n^{\frac{7}{2}})$ $\Omega(n^2)$ [3], $O(n^{\frac{7}{2}})$ [Sec. 5]	$\binom{n}{5}$ $\Theta(n^5)$ [3]
$k\geq 6$	$\Theta(n^k)$ [6]	$\Theta(n^k)$ [Sec. 2]	$\binom{n}{k}$ $\Theta(n^k)$ [Sec. 2]	$\Theta(n^k)$ [Sec. 2]	$k=6: \geq \lfloor\frac{n-1}{858}\rfloor - 2$ $O(n^2)$ [5] $\Omega(n)$ [Sec. 6] $k\geq 7: \emptyset$ [17]	$\leq n!/(n-k+1)!$ $\Theta(n^{k-1})$ [Sec. 4]	$\geq n^2-O(n)$ $\leq O(n^{\frac{k+2}{2}})$ $\Omega(n^2), O(n^{\frac{k+2}{2}})$ [Sec. 5]	$\binom{n}{k}$ $\Theta(n^k)$ [Sec. 3]

Table 2: Bounds on the numbers of convex, non-convex and general k-gons and k-holes for n points and constant k.

ing the number of non-convex and general k-holes more challenging than they might appear at first glance.

Table 2 summarizes known bounds on the numbers of k-gons and k-holes, including the results of this paper. Every entry in the table shows lower and upper bounds, also in explicit form if available. Among other results, we generalize properties concerning 4-holes [2] and 5-holes [3] to $k \geq 6$. In Section 2 we give asymptotic bounds on the number of non-convex and general k-gons. In Section 3 we consider (general) k-holes. We show that for sufficiently small k their number is maximized by sets in convex position, which is not the case for large k. Section 4 provides a tight bound for the maximum number of non-convex k-holes, and Section 5 contains bounds for the minimum number of general k-holes. In Section 6 we improve the lower bound for convex 6-holes, and we conclude with open problems in Section 7.

2 General k-gons

For non-convex k-gons of small cardinalty their number can be related to the rectilinear crossing number $\bar{cr}(S)$ of a set S of n points in the plane. This is the number of proper intersections in the drawing of the complete straight line graph on S. By $\bar{cr}(n)$ we denote the minimum possible rectilinear crossing number over all point sets of cardinality n. Determining $\bar{cr}(n)$ is a well known problem in discrete geometry; see [6, 11] as general references and [1] for bounds on small sets. Asymptotically we have $\bar{cr}(n) \approx 0.38\binom{n}{4} = \Theta(n^4)$.

It is easy to see that the number of convex 4-gons is equal to $\bar{cr}(S)$ and is thus minimized by sets realizing $\bar{cr}(n)$. Moreover, as four points in non-convex position span three non-convex 4-gons, we have at most $3\binom{n}{4} - 3\bar{cr}(n) \approx 1.86\binom{n}{4}$ non-convex and at most $3\binom{n}{4}-2\bar{cr}(n) \approx 2.24\binom{n}{4}$ general 4-gons. All these bounds are tight for point sets which minimize the rectilinear crossing number.

A similar relation has been obtained for the number of non-convex 5-gons in [3]: Any set of n points has at most $10\binom{n}{5} - 2(n-4)\bar{cr}(n) \approx 6.2\binom{n}{5}$ non-convex 5-gons, and again this bound is obtained for sets minimizing the rectilinear crossing number. Note that this number exceeds the maximum number of convex 5-gons. For the number of general 5-gons, and for non-convex and general k-gons with $k \geq 6$, no such direct relations to $\bar{cr}(n)$ are possible.

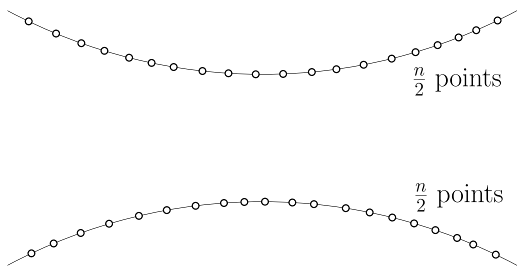

Figure 1: The so-called double chain.

Polygonizations, also called spanning cycles, can be considered as k-gons of maximal size (i.e., $k = n$). García et al. [14] construct a point set with $\Omega(4.64^n)$ spanning cycles, the so-called double chain $DC(n)$, which is currently the best known example; see Figure 1. The upper bound on the number of spanning cycles of any n-point set was improved several times during the last years, most recently to $O(70.21^n)$ [22] and $O(68.664^n)$ [8], neglecting polynomial factors in the asymptotic expressions. The minimum is obtained by point sets in convex position, which have exactly one spanning cycle.

For the number of general k-gons this implies a lower bound of $\binom{n}{k}$, as well as an upper bound of $O\left(68.664^k\binom{n}{k}\right)$. For constant k, we obtain $\Theta(n^k)$. On the other hand, the double chain provides $\Omega(n^k)$ non-convex k-gons, where $k \geq 4$ is again a constant. To see this, choose one vertex from the upper chain of

$DC(n)$ and $k-1 \geq 3$ vertices from the lower chain of $DC(n)$, and connect them to a simple, non-convex polygon. This gives at least $\frac{n}{2}\binom{n/2}{k-1} = \Omega(n^k)$ non-convex k-gons. As the lower bound on the maximal number of non-convex k-gons asymptotically matches the upper bound on the maximal number of general k-gons, we get our first result.

Lemma 1 *Let S be a set of n points in the plane in general position and $k \geq 3$ a constant. Then the maximum number of non-convex k-gons in S is $\Theta(n^k)$ and the maximum number of general k-gons in S is also $\Theta(n^k)$.*

3 Maximizing the number of (general) k-holes

In [2] it is shown that the number of 4-holes is maximized for point sets in convex position if n is sufficiently large. It was conjectured that this is true for any constant $k \geq 4$. The following theorem settles this conjecture in the affirmative.

Theorem 2 *For every $k \geq 4$ and $n \geq 2(k-1)!\binom{k}{4}+k-1$, the number of k-holes is maximized by a set of n points in convex position.*

Proof. Every non-convex k-hole has as its vertex set a non-convex k-tuple, and every non-convex k-tuple has at least one triangle formed by three extreme points (i.e., points on the convex hull of the k-tuple) that contains points of the k-tuple in its interior. So consider such a non-empty triangle Δ. We count the number of non-convex k-holes having the three vertices of Δ as extreme points. Note that any such k-hole can be reduced to a (not necessarily simple) non-empty $(k-1)$-gon by removing a reflex vertex from its boundary.

Denote by $\mathcal{K}$ the set of (not necessarily simple) non-empty $(k-1)$-gons having the vertices of Δ on their convex hull. First, $|\mathcal{K}|$ can be bounded from above by the number of (not necessarily simple) possibly empty $(k-1)$-gons having the three vertices of Δ on their boundary, which is $\frac{(k-2)!}{2}\binom{n-3}{k-4}$.

Further, every $(k-1)$-gon in $\mathcal{K}$ can be completed to a (simple) non-convex k-hole in at most $k-1$ ways by adding a reflex vertex. Thus the number of non-convex k-holes having all vertices of Δ on their convex hull is bounded from above by

$$(k-1)\frac{(k-2)!}{2}\binom{n-3}{k-4} = \frac{(k-1)!}{2}\binom{n-3}{k-4}.$$

Considering convex k-holes, observe that every k-tuple gives at most one convex k-hole. Denote by N the number of k-tuples that do *not* form a convex k-hole, and by T the number of non-empty triangles. Then we get (1) as a first upper bound on the number of (general) k-holes of a point set.

$$\binom{n}{k} - N + \left(\frac{(k-1)!}{2}\binom{n-3}{k-4}\right) \cdot T \qquad (1)$$

To obtain an improved upper bound from (1), we need to derive a good lower bound for N. To this end, consider again a non-empty triangle Δ. As Δ is not empty, none of the $\binom{n-3}{k-3}$ k-tuples that contain all three vertices of Δ forms a convex k-hole. On the other hand, for such a k-tuple, all of its $\binom{k}{3}$ contained triangles might be non-empty. We obtain $T \cdot \binom{n-3}{k-3}/\binom{k}{3}$ as a lower bound for N, and thus (2) as an upper bound for the number of k-holes.

$$\binom{n}{k} + \left(\frac{(k-1)!}{2}\binom{n-3}{k-4} - \frac{\binom{n-3}{k-3}}{\binom{k}{3}}\right) \cdot T \qquad (2)$$

For $n \geq 2(k-1)!\binom{k}{4} + k - 1$ this is at most $\binom{n}{k}$, the number of k-holes of a set of n points in convex position, which proves the theorem. □

The above theorem states that convexity maximizes the number of k-holes for $k = O(\frac{\log n}{\log\log n})$ and sufficiently large n. Moreover, the proof implies that any non-empty triangle in fact reduces the number of empty k-holes. Thus it follows that, for $k = O(\frac{\log n}{\log\log n})$ and n sufficiently large, the maximum number of convex k-holes is strictly larger than the maximum number of non-convex k-holes; see also the next section.

At the other extreme, for $k \approx n$ the statement does not hold: As already mentioned in the introduction, a set of k points spans at most one convex k-gon, but might admit exponentially many different non-convex k-gons.

Theorem 3 *The number of k-holes in the double chain $DC(n)$ on n points is at least*

$$\binom{\frac{n-4}{2}}{\frac{n-k}{2}} \cdot \frac{n-k+2}{2} \cdot \Omega(4.64^k).$$

Proof. Recall that $DC(n)$ admits $\Omega(4.64^n)$ polygonizations. Thus, for a double chain on k points ($k/2$ points on each chain), we have $\Omega(4.64^k)$ different k-polygonizations. We distribute the remaining $n-k$ points among all possible positions, meaning that for each k-polygonization, we obtain the double chain on n points with a k-hole drawn, as shown in Figure 2.

In their proof, García et al. count paths that start at the first vertex of the upper chain and end at the last vertex of the lower chain. Before the first vertex on the lower chain, they add an additional point q to complete these paths to polygonizations. We slightly extend this principle, by also adding an additional point p on the upper chain after the last vertex. Then we complete each path C to a polygonization in one of the following ways: Either we add p to C directly next to $p_{\frac{k}{2}-1}$ and then complete C via q, obtaining P_q, or we add q to C directly next to q_1, and close the polygonization via p, obtaining P_p.

Note that this changes the number of polygonizations only by a constant factor and thus does not influence

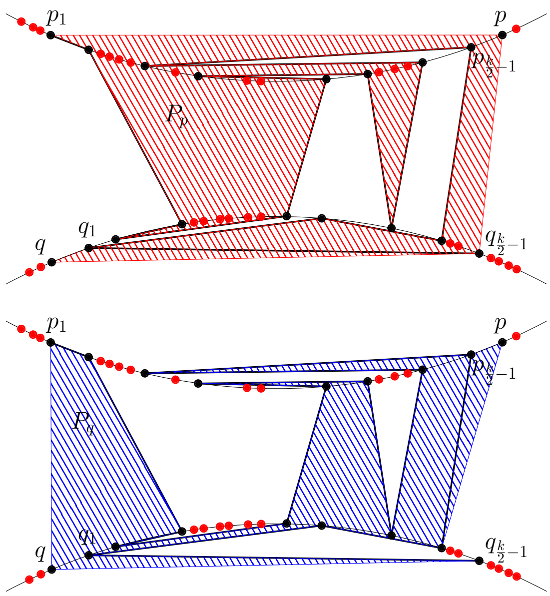

Figure 2: Two ways to complete a path to a polygonization.

the asymptotic bound. However, the interior of P_q is the exterior of P_p, meaning that if we place a point somewhere on the double chain and it lies inside P_q, then it lies outside P_p, and vice versa. It follows that, in one of the two polygonizations, at least half of the $k+2$ positions to insert points are outside the polygonization. Hence we can distribute the $\frac{n-k}{2}$ points on each chain to at least $\frac{k}{2}+1$ possible positions in total. Now, on one of the two chains we have at least $\frac{k}{4}+1$ positions; see again Figure 2. More precisely, there are $\frac{k}{4}+j+1$ positions on this chain (where $0 \leq j < \frac{k}{4}$), and on the other chain there are (at least) $\max\{2, \frac{k}{4}-j\}$ positions. Using this, we obtain

$$\binom{\frac{n-k}{2}+\frac{k}{4}+j}{\frac{n-k}{2}} \cdot \max\left\{\binom{\frac{n-k}{2}+1}{\frac{n-k}{2}}, \binom{\frac{n-k}{2}+\frac{k}{4}-j-1}{\frac{n-k}{2}}\right\}$$

possibilities to place the remaining points on the two chains. This factor is minimized for $j = \frac{k}{4}-2$, which yields the claimed lower bound for the number of k-holes of $DC(n)$. □

4 An upper bound for non-convex k-holes

The following theorem shows that, asymptotically, the maximum number of non-convex k-holes is smaller than the maximum number of convex k-holes.

Theorem 4 *For any constant $k \geq 3$, the number of non-convex k-holes in a set of n points is bounded by $O(n^{k-1})$ and there exist sets with $\Theta(n^{k-1})$ non-convex k-holes.*

Proof. We first show that there are at most $O(n^{k-1})$ non-convex k-holes by giving an algorithmic approach to generate all non-convex k-holes. We represent a non-convex k-hole by the counter-clockwise sequence of its vertices, where we require that the last vertex is reflex. Note that any non-convex k-hole has $r \geq 1$ such representations, where r is the number of its reflex vertices. Thus the number of different representations is an upper bound on the number of non-convex k-holes.

We have n possibilities to choose the first vertex v_1, $n-1$ for the second vertex v_2, and so on. Several of the sequences obtained might lead to non-simple polygons, but we are only interested in an upper bound. For the second-last vertex v_{k-1} we have $n-k+2$ possibilities, but the last vertex v_k is uniquely defined. As v_k is required to be reflex and the polygon has to be empty, we have to use the inner geodesic connecting v_{k-1} back to v_1. Only if this geodesic contains exactly one point, namely v_k, we do obtain one non-convex k-hole (again ignoring possible non-simplicity). Thus we obtain at most $n!/(n-k+1)! = O(n^{k-1})$ non-convex k-holes.

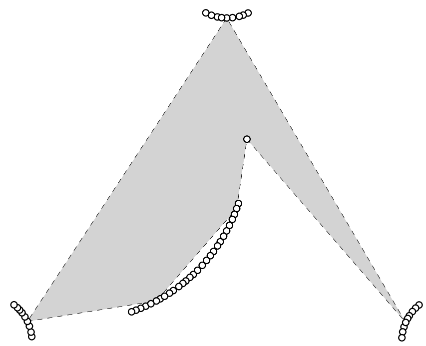

Figure 3: A set with $\Theta(n^{k-1})$ non-convex k-holes.

For an example which achieves this bound see Figure 3. Each of the four indicated groups of points contains a linear fraction of the point set; e.g. $\frac{n}{4}$ points. It is sufficient to only consider the k-holes with triangular convex hull of the type indicated in the figure, which sums to $\Omega(n^3 \cdot \binom{n}{k-4}) = \Omega(n^{k-1})$ non-convex k-holes. □

5 On the minimum number of (general) k-holes

Every set of k points admits at least one polygonization. Using this obvious fact, we obtain the following result.

Theorem 5 *Let S be a set of n points in the plane in general position. For every $c < 1$ and every $k \leq c \cdot n$, S contains $\Omega(n^2)$ k-holes.*

Proof. We follow the lines of the proof of Theorem 5 in [2]. Consider the point set S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and sets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ of cardinality at least k is

$$\sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^{n} 1 = \frac{(n-k+1)(n-k+2)}{2} = O(n^2).$$

For each $S_{i,j}$ use the $k-2$ points of $S_{i,j} \setminus \{p_i, p_j\}$ which are closest to the segment $p_i p_j$ to obtain a subset of k points including p_i and p_j. Each such set contains at least one k-hole which has p_i and p_j among its vertices. Moreover, as p_i and p_j are the left and rightmost points of $S_{i,j}$, they are also the left and rightmost points of this k-hole. This implies that any k-hole of S can count for at most one set $S_{i,j}$, which gives a lower bound of $\Omega(n^2)$ for the number of k-holes in S. □

Theorem 6 *For every constant $k \geq 4$ and every $n = m^2 \geq k$, there exist sets with n points in general position that admit at most $O(n^2(\sqrt{n}\log n)^{k-3})$ k-holes.*

Proof. The point set S we consider is the squared Horton set of size $\sqrt{n} \times \sqrt{n}$; see [25]. Roughly speaking, S is a grid which is perturbed such that every set of originally collinear points forms a Horton set. Using Pick's Theorem [21], it can be shown that for any two points $p, q \in S$, the number of empty triangles in S that contain the edge pq is $O(\sqrt{n}\log n)$, regardless of the choice of p and q; details will be given in the full version of this paper.

To estimate the number of k-holes in S, we will use triangulations and their dual: For a triangulation of a k-hole, the dual is a binary tree where every node represents a triangle. It can be rooted at any triangle that has an edge on the boundary of the k-hole; see [19]. It is well known that there are $C_{k-2} = O(4^k \cdot k^{-\frac{3}{2}})$ such rooted binary trees [19]. Although exponential in k, this bound is constant in the size n of S.

Now pick an empty triangle Δ in S and an arbitrary rooted binary tree B. Consider all k-holes which contain Δ and admit a triangulation that is represented by B rooted at Δ. As the number of empty triangles incident to an edge in S is $O(\sqrt{n}\log n)$, each of the $n-3$ edges in B yields $O(\sqrt{n}\log n)$ possibilities to continue a triangulated k-hole, and we obtain an upper bound of $O((\sqrt{n}\log n)^{k-3})$ for the number of triangulations of k-holes for Δ that represent B.

Multiplying this by the (constant) number of rooted binary trees of size $k-2$ does not change the aysmptotics and thus yields an upper bound of $O((\sqrt{n}\log n)^{k-3})$ for the number of all triangulations of all k-holes containing Δ. As any k-hole can be triangulated, this is also an upper bound for the number of k-holes containing Δ.

Finally, there are $O(n^2)$ empty triangles in S (see again [25]), and thus we obtain $O(n^2(\sqrt{n}\log n)^{k-3})$ as an upper bound for the number of k-holes in S. □

Note that the Horton set has $\Omega(n^3)$ 4-holes. A general super-quadratic lower bound for the number of 4-holes would solve a conjecture of Bárány to the positive, showing that every point set contains an edge that spans a super-constant number of 3-holes; see e.g. [6], Chapter 8.4, Problem 4. This would also imply a quadratic lower bound for the number of convex 5-holes. So far, not even a super-linear bound is known for the latter problem [6].

6 An improved lower bound for convex 6-holes

Gerken [15] showed that each set of at least 1717 points in general position contains a convex 6-hole. This immediately implies that each set of n points contains a linear number of convex 6-holes, namely at least $\lfloor \frac{n}{1717} \rfloor$. In the following we slightly improve on this bound. We start by showing a result for monochromatic convex 6-holes in two-colored point sets.

Lemma 7 *Each set of r red points and b blue points in general position in the plane with $r \geq 1716 \lceil \frac{b}{2} \rceil + 1717$ contains a convex red 6-hole.*

Proof. Consider a non-crossing perfect matching of the blue points; if b is odd, then allow one isolated point p. We extend the segments (in both directions) one by one, until each segment either hits another segment, the line of a previously extended segment or goes to infinity. If b is odd, we take an arbitrary segment through p and extend it as well. Altogether, this results in a decomposition of the plane into $\lceil \frac{b}{2} \rceil + 1$ convex regions. As the red points lie inside these regions, it follows by the pigeon-hole principle that at least one of these regions contains 1717 red points, and thus a red convex 6-hole by [15]. □

Theorem 8 *Each set S of n points in general position in the plane contains at least $\lfloor \frac{n-1}{858} \rfloor - 2$ convex 6-holes.*

Proof. We prove the statement by contradiction. Assume that the point set S contains strictly less than $\lfloor \frac{n-1}{858} \rfloor - 2$ convex 6-holes, and color the points of S red. Now we eliminate all red convex 6-holes by placing an additional blue point inside each of them, such that the resulting two-colored point set is in general position. By this, at most $b \leq \lfloor \frac{n-1}{858} \rfloor - 3$ blue points are added. Therefore the number n of red points is at least

$$n \geq 858(b+3) + 1 \geq 1716 \left\lceil \frac{b}{2} \right\rceil + 1717.$$

By Lemma 7, any such two-colored point set contains a convex red 6-hole, a contradiction. □

7 Conclusion

We have shown various lower and upper bounds on the numbers of convex, non-convex, and general k-holes and

k-gons in point sets. Several questions remain unsettled. For example, some of the presented bounds are not tight, like the classic question for the minimum number of convex k-holes for $k \leq 6$. Maybe the most intriguing open question in this context is whether there exists a super-quadratic lower bound for the number of general k-holes for $k \geq 4$.

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