

Convexifying monotone polygons while maintaining internal visibility

O. Aichholzer¹, M. Cetina², R. Fabila-Monroy³, J. Leaños⁴, G. Salazar², J. Urrutia⁵

¹ Institute for Software Technology, University of Technology, Graz, Austria
oaich@ist.tugraz.at

² Instituto de Física, Universidad Autónoma de San Luis Potosí, México
gsalazar@ifisica.uaslp.mx

³ Departamento de Matemáticas, Cinvestav, México
ruyfabila@math.cinvestav.edu.mx

⁴ Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas, México
jleanos@mate.reduaz.mx

⁵ Instituto de Matemáticas, Universidad Nacional Autónoma de México
urrutia@matem.unam.mx

Abstract. Let P be a simple polygon on the plane. Two vertices of P are visible if the open line segment joining them is contained in the interior of P . In this paper we study the following questions posed in [5, 6]: (1) Is it true that every non-convex simple polygon has a vertex that can be continuously moved such that during the process no vertex-vertex visibility is lost and some vertex-vertex visibility is gained? (2) Can every simple polygon be convexified by continuously moving only one vertex at a time without losing any internal vertex-vertex visibility during the process?

We provide a counterexample to (1). We note that our counterexample uses a monotone polygon. We also show that question (2) has a positive answer for monotone polygons.

Introduction

Let P be a simple polygon with vertices $\{p_1, \dots, p_n\}$. We say that two vertices of P are P -visible if the relative interior of the line segment joining them is contained in the interior of P . The *visibility graph* $VG(P)$ of P is the graph with vertex set $\{p_1, \dots, p_n\}$ in which two vertices of P are adjacent if they are P -visible. A classical problem in computational geometry is that of convexifying simple polygons; that is, using a given fixed set of transformations that can be applied to the vertices and edges of P , try to transform P into a convex polygon in such a way that some properties of P are preserved. The first formulation of a problem of this kind was proposed by Erdős [3], who proposed a strategy to convexify a non-convex polygon by using *flipturns*; see also [1, 2, 7, 8].

Our starting point is the following question posed by Satyan L. Devadoss in the Open Problem Session at CCCG 2008 [5, 6]:

Question: Given a simple polygon P and its visibility graph $VG(P)$, can the vertices of P be moved continuously (one at a time or simultaneously) along paths so that:

- the simplicity of the polygon P is maintained all the time, and
- the visibility graph of P never loses edges, only gains them.

In discussions after the workshop, the following specific questions were raised [4]:

- (1) Is it true that every non-convex simple polygon has a vertex p that can be continuously moved so that while p moves, $VG(P)$ gains at least one extra edge, and never loses any?
- (2) Can every simple polygon be convexified by continuously moving several vertices in sequence, but only one at a time, such that $VG(P)$ never loses any edge?

Our aim is to prove that Question (2) has a positive answer for monotone polygons. On the other hand, we give an example that shows that the answer to Question (1) is negative, even for monotone polygons. For recent results on this topic, see also [6].

1 Polygons and visibility

Let $\{p_0, \dots, p_{n-1}\}$ be a set of points in the plane. A simple polygon P with vertex set $\{p_0, \dots, p_{n-1}\}$ is a non-crossing piecewise linear curve formed by the union of the closed line segments $\overline{p_i p_{i+1}}$ joining p_i to p_{i+1} , $i = 0, \dots, n-1$, addition taken mod n . The interior of P is the area bounded by P . Note that we consider the interior of P as an open set, i.e., vertices and edges of P do not belong to the interior.

A polygon P is x -monotone if any vertical line intersects the interior of P in at most one connected component. For simplicity we will use the term monotone when referring to an x -monotone polygon. We will suppose without loss of generality that no vertical line passes through two vertices of P . Let u and v be the leftmost and rightmost vertices of P . Clearly there are two edge-disjoint paths contained in P joining u to v , one above the other. The first will be called the *upper chain* of P , the second the *lower chain* of P .

A basic operation that we will use in this paper is that of moving the elements of $\{p_0, \dots, p_{n-1}\}$ around the plane. As the vertices of P move, strictly speaking the polygon P defined by its vertices changes, nevertheless, abusing our terminology a bit, we will always refer to it as P . In other words, as the elements of $\{p_0, \dots, p_{n-1}\}$ move around the plane, they and P *keep their respective identities*. We will restrict our point moves to those that do not destroy the simplicity of P .

The convex hull of P is denoted by $CH(P)$. A vertex of P that lies on the boundary of $CH(P)$ is called a *convex hull vertex*, otherwise it is called an *interior vertex*. A simple polygon P divides $\mathbb{R}^2 \setminus P$ into two regions, a bounded region called the *interior of P* , and an unbounded region called the *exterior of P* . We say the two vertices u and v of P are P -visible if the relative interior of the line segment \overline{uv} joining them is contained in the interior of P . We call $\{u, v\}$ a *visibility pair*. Note that according to our definition, consecutive vertices of P are not visible. Let $\mathcal{N}(P)$ be the set of pairs of vertices of P that are not P -visible. As consecutive vertices of P are not P -visible, $|\mathcal{N}(P)| \geq n$.

The visibility graph $VG(P)$ is the graph with vertex set $\{p_0, \dots, p_{n-1}\}$ in which two vertices are adjacent if they form a visibility pair. Note that if the vertices of P move, the set of visible pairs of P may change, and in turn $VG(P)$ may also change.

A *point-move* operation on P is the translation of a vertex of P , say p_i , (the moving point) from an initial position on the plane (x_i, y_i) to a final position (x'_i, y'_i) along the line segment joining (x_i, y_i) to (x'_i, y'_i) .

We say that a vertex move is *visibility-preserving*, if the following holds: If p_j and p_k were P -visible, they remain P -visible while p_i moves. If in addition the number of edges of $VG(P)$ increases, then we call it a *visibility-increasing* vertex move.

Our main results are the following:

Theorem 1.1 *There are polygons that have no visibility-increasing vertex moves.*

Theorem 1.2 *Every monotone polygon can be converted with a sequence of visibility-preserving moves.*

2 A counterexample to question (1)

Consider the monotone polygon P shown in Figure 1. The coordinates of the vertices of P are $a = (-100, 0)$, $b = (-63, 40)$, $c = (-61, 40)$, $d = (-33, 2)$, and $e = (0, 45)$. The points $\{f, g, h, i\}$ are obtained from the points $\{a, b, c, d\}$ by reflecting them along the y -axis. Points b' to h' are obtained from the points b to h by a reflection along the x -axis.

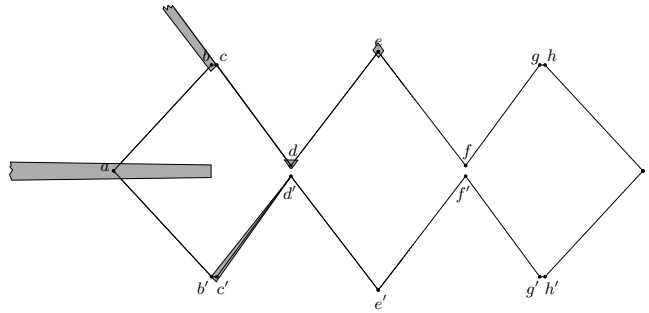


FIGURE 1. A monotone polygon without visibility-increasing vertex moves. Shaded areas indicate visibility-preserving regions.

To show that P does not admit any visibility-increasing vertex move, it is sufficient to consider the vertices of P in the set $\{a, b, c', d, e\}$. The remaining cases follow by symmetry. For each of these vertices, we show in Figure 1 the open shaded region into which any of these points can be translated without losing any visibility pairs in P . It is now easy to see that there is no single vertex move that is visibility increasing.

3 Visibility-preserving vertex moves

For a point $q \in \mathbb{R}^2$ and some $\delta > 0$, we denote by $B_\delta(q)$ the closed disk with radius δ with center at point q . Let $P = \{p_0, \dots, p_{n-1}\}$ be a set of points in the plane in general position. We say that $\delta > 0$ is a *safe threshold* of P if there are no three elements p_i, p_j , and p_k of P such that $B_\delta(p_i)$, $B_\delta(p_j)$, and $B_\delta(p_k)$ are all intersected by a line ℓ . Equivalently, we can say that δ is a safe threshold of P if there are no three points $p_i, p_j, p_k \in P$ such that when we translate each of them to a point within δ distance of them, they become aligned.

It is not hard to see that every point set in the plane in general position has a safe threshold δ and that if a vertex move is not visibility-preserving, then at some point while moving the vertex it becomes collinear with two other vertices of P . However, the following lemma shows that collinearity is no problem for our approach. With $V^\circ(P)$ we denote the set of vertices interior to the convex hull of P .

Lemma 3.1 *Let P be a monotone polygon. Then there is a sequence of visibility-preserving vertex moves of some vertices of P such that at the end of the sequence,*

the vertices of P are in general position, P remains monotone, and $|V^\circ(P)| + |\mathcal{N}(P)|$ does not increase during the vertex movements.

The proof of Lemma 3.1 works by induction on the number of *collinear triples* in $V(P)$. Details are omitted in this extended abstract. We are now ready to give a brief sketch of the proof of Theorem 1.2.

By Lemma 3.1, we can assume that $V(P)$ is in general position. We proceed by induction on the sum of the number of interior vertices plus the number of non-visible pairs. If the vertices of P are in convex position, there is nothing to prove. Observe that P is convex if $|V^\circ(P)| + |\mathcal{N}(P)| = n$. Suppose then that $|V^\circ(P)| + |\mathcal{N}(P)| > n$ and assume that the theorem holds for all polygons Q with $|V^\circ(Q)| + |\mathcal{N}(Q)| < |V^\circ(P)| + |\mathcal{N}(P)|$.

Since P is not convex, suppose w.l.o.g. that there are $k \geq 1$ interior vertices of P on its upper chain. Relabel them as v_1, v_2, \dots, v_k , in increasing order with respect to their x -coordinate. Let $\delta > 0$ be a safe threshold for the *initial position* of $V(P)$. Our algorithm will execute the following basic procedure **BP** once:

BP: *One at a time from left to right, move v_1, v_2, \dots, v_k upwards, by a distance δ .*

Once v_1, v_2, \dots, v_k have all been moved, we execute **BP** repeatedly until one of the following occurs: (1) a vertex in $\{v_1, v_2, \dots, v_k\}$ reaches the convex hull of P , (2) a new visible pair occurs, (3) or the visibility-preserving property is lost.

If we stop because (1) or (2) occurs, then we are done, by our induction hypothesis. Using the monotonicity of P we can show that (3) does not happen before a visibility-increasing event, which proves the theorem. Details are omitted in this extended abstract.

4 Conclusion

Several open questions arise from our work: How many vertex moves do we need to convexify a monotone polygon? Can this number be bounded by a polynomial? If we allow only vertical moves we can construct a polygon where the number of vertex moves is unbounded, but how about general moves? What happens if we allow more than one vertex to move at a time? We conclude with the following conjecture.

Conjecture 4.1 *Every simple polygon can be convexified by a sequence of visibility-preserving 1-vertex moves.*

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