Edge-Removal and Non-Crossing Configurations in Geometric Graphs

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Abstract

We study the following extremal problem for geometric graphs: How many arbitrary edges can be removed from a complete geometric graph with n vertices such that the remaining graph still contains a certain non-crossing subgraph. In particular we consider perfect matchings and subtrees of a given size. For both classes of geometric graphs we obtain tight bounds on the maximum number of removable edges. We further present several conjectures and bounds on the number of removable edges for other classes of non-crossing geometric graphs.

1 Introduction

A geometric graph is a graph $G = (V, E)$ drawn in the plane, such that V is a point set in general position (meaning that no three points of V lie on a common line) and E is a set of straight-line segments whose endpoints belong to V. A geometric graph is called *non-crossing* if no two edges intersect in their interior, but two edges might have an endpoint in common. Two edges are *disjoint* if they have no point in common.

Extremal problems for geometric graphs have received considerable attention. One problem considered in this area, studied by Erdős, Perles, Kupitz, and Avital and Hanani [1, 11], is to determine the smallest number $e_k(n)$ such that every geometric graph with n vertices and $m > e_k(n)$ edges contains $k+1$ pairwise disjoint edges. Erdős [5] proved that $e_1(n) = n$. For three pairwise disjoint edges, bounds on $e_2(n)$ were given in [1, 7], culminating in $e_2(n) = 2.5n$ (plus a constant), as shown recently by Černý [3]. Bounds for $e_3(n)$ have been obtained in [7, 14].

For general values of k, Goddard et al. [7] showed that $e_k(n) \leq cn(\log n)^{k-4}$ for some constant c. This was improved by Pach and Törőcsik [12] to $e_k(n) \leq k^4 n$, the first upper bound linear in n. Tóth and Valtr [14] further improved this bound to $e_k(n) \leq k^3(n+1)$, and finally Tóth [13] showed that $e_k(n) \leq 2^9 k^2 n$, where the constant 2^9 has since been improved by Felsner [6] to 256. Kupitz proved a lower bound of $e_k(n) > kn$, which was improved to $e_k(n) \geq \frac{3}{2}(k-1)n - 2k^2$ by Tóth and Valtr [14]. It is conjectured that $e_k(n) \leq ckn$ for some constant c.

Research on $e_k(n)$ has focused on small values of k. But k can be as large as $\frac{n}{2} - 1$, in which case we obtain a non-crossing perfect matching. Looking at the problem from this angle, we ask for $e_{n-1}(2n)$, or in other words, we investigate how many (arbitrary) edges can be removed from a complete geometric graph, such that it still contains a non-crossing perfect matching. We show that every complete geometric graph on $2n$ vertices still contains a non-crossing perfect matching after removing any set of $n-1$ edges; that is $e_{n-1}(2n) = \binom{2n}{2} - n$. This bound is achieved for complete geometric graphs on point sets in convex position, meaning that there exists a set of n edges whose removal disallows a non-crossing perfect matching in the remaining graph. For point sets in convex position this question was completely settled by Kupitz and Perles for each k. They showed that if a geometric graph on n vertices in convex position has at least $(k-1)n+1$ edges then the graph contains k disjoint edges, and this bound is tight; see [7].

Our research was motivated by a closely related problem posed by Micha Perles in 2002 and studied by Černý, Dvořák, Jelínek and Kára [4]: How many arbitrary edges can be removed from a complete geometric graph on n vertices such that the remaining graph still contains a non-crossing Hamiltonian path. It is of interest to study this problem for other classes of non-crossing geometric graphs. We consider subtrees of a given size. For the case of spanning trees, removing $n-2$ arbitrary edges from any complete geometric graph on n vertices leaves a graph that still

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contains a non-crossing spanning tree [9]. Removal of more edges is possible if the set of removed edges has certain properties. Benediktovich [2] recently showed that each complete geometric graph on $n \geq 5$ vertices still contains a non-crossing spanning tree after removing any self-crossing 2-factor, i.e., a 2-regular spanning subgraph with two edges sharing an interior point. We show that every complete geometric graph on n vertices still contains a non-crossing subtree that spans $n-k$ vertices after removing $\lceil \frac{kn}{2} \rceil$ arbitrary edges, for $k \geq 2$, and this bound is tight.

Examples bounding the number of removable edges often are defined on point sets in convex position. We conjecture that for point sets with many points in the interior of the convex hull many more edges can be removed from the complete geometric graph to still guarantee the considered subgraph. We finally briefly consider this problem for other classes of geometric graphs.

In the following, removal of a set E' of edges of a complete geometric graph G is expressed by $G - H$, where E' is the edge set of a subgraph H of G. The edges of E' are called *removed* or *forbidden* edges. We omit several proofs in this abstract.

2 Perfect matchings

In this section we investigate the maximum number of removable edges in a complete geometric graph such that the remaining graph contains a non-crossing perfect matching. We first show a result for abstract graphs.

Theorem 1 *For all $p \geq 2$, for every spanning subgraph $H = (V, E')$ of the complete graph K_{kp} with $|E'| \leq k-1$, the graph $K_{kp} - H$ contains the complete p-partite graph $K_{k,\dots,k}$.*

Proof. For each p we prove the theorem by induction on k. For $k = 1$ the statement is trivial. Assume the statement is true for every number $k' < k$. Now, we are given the complete graph K_{kp} and we are given a spanning subgraph $H = (V, E')$ with $|E'| \leq k-1$. Assume that $|E'| > 0$, as otherwise nothing has to be proved. Observe that there exists a set Q of at least $p-1$ isolated vertices in H and there exists a vertex $v \notin Q$ whose degree is at least 1 in H. Let $N(v)$ denote the set of neighbors of v in H. Define a graph $H' = (V \backslash (Q \cup \{v\}), E^*)$ where E^* is obtained by first taking the set of edges of the induced subgraph of $(V \backslash (Q \cup \{v\}), E')$ and then adding a minimum number of edges to the resulting set, such that $N(v)$ is connected. We have $|E^*| \leq |E'| - 1 \leq k-2$, because we removed $\deg_H(v)$ edges and added at most $\deg_H(v) - 1$ edges to restore the connectedness. By induction, $K_{(k-1)p} - H'$ contains the complete p-partite graph $K_{k-1,\dots,k-1}$. Since $N(v)$ is connected in H', all the vertices of $N(v)$ belong to the same vertex class of $K_{k-1,\dots,k-1}$. Add v to the vertex class containing $N(v)$, and add one vertex in Q to each of the other vertex classes so that $K_{k,\dots,k} \subseteq K_{kp} - H$. □

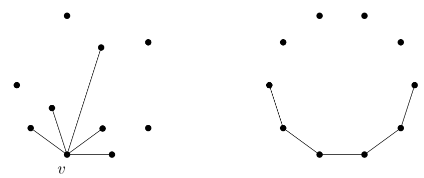

Figure 1: Two examples where removing n edges from the complete geometric graph on a set of $2n$ points disallows a non-crossing perfect matching.

Corollary 2 *For every complete geometric graph G on $2n$ vertices and for every subgraph H of G with at most $n-1$ edges, the geometric graph $G - H$ contains a non-crossing perfect matching. This bound is tight with respect to the cardinality of the set of forbidden edges.*

Proof. Apply the case $p = 2$ of Theorem 1, which states that $G - H$ contains a complete bipartite graph $K_{n,n}$. Color the point set according to this bipartition, say red and blue. This 2-colored point set has a non-crossing red-blue matching; that is, each edge of the matching connects a red and a blue point. Thus, this matching does not use edges of H.

Removing n edges from G does not always leave a non-crossing perfect matching, as can be seen in Figure 1 (left). There, if vertex v is matched to another point not using the drawn 'forbidden' edges, then this segment splits the point set into two sets of odd size, which disallows a non-crossing perfect matching. Thus, the bound of $n-1$ edges is tight. □

Another example that prohibits a non-crossing perfect matching without forbidden edges is shown in Figure 1 (right). In both examples the graph defined by the forbidden edges has one component that contains $n+1$ vertices. The size of the largest component in this graph turns out to be crucial for the existence of a non-crossing perfect matching without forbidden edges. To show this, we first show a result for colored point sets (which extends a known proof for 2-colored point sets).

Theorem 3 *Let S be a set of colored points in general position in the plane with $|S|$ even. Then S admits a non-crossing perfect matching such that every edge connects two points of distinct colors if and only if at most half the points in S have the same color.*

A related problem considering long alternating paths for multicoloured point sets was studied in [10].

Corollary 4 *For every complete geometric graph G on $2n$ vertices, and for every subgraph H of G with at most n vertices in each component, the geometric graph $G-H$ contains a non-crossing perfect matching.*

Note that Corollary 4 also implies Corollary 2.

Conjecture 1 *For every complete geometric graph G on a set of $2n$ points with $k \geq n-2$ of them in the interior of the convex hull and for every subgraph H of G which has at most $k+1$ edges, the geometric graph $G-H$ contains a non-crossing perfect matching.*

3 Non-crossing subtrees

In this section we investigate how many arbitrary edges can be removed from any complete geometric graph such that the remaining graph still contains a non-crossing tree of a given size. It turns out that the connectivity of the subgraph H defined by the removed edges is crucial for the size of the largest non-crossing subtree. We recall that the *connectivity* of a graph G is the size of a smallest vertex cut. A *vertex cut* of a connected graph G is a set of vertices whose removal disconnects G. A graph is called *k-connected* if its connectivity is k or greater.

Lemma 5 *For every complete geometric graph G on n vertices and for every subgraph H of G with connectivity k, the geometric graph $G-H$ contains a non-crossing subtree on $n-k$ vertices.*

In particular, Lemma 5 implies that for every subgraph H with $n-1$ edges of a complete geometric graph G on n vertices, the geometric graph $G-H$ contains a non-crossing subtree that spans $n-1$ vertices. Also, for every disconnected subgraph H of a complete geometric graph G, the geometric graph $G-H$ contains a non-crossing spanning tree.

Theorem 6 *For every $2 \leq k \leq n-1$, for every complete geometric graph G on n vertices, and for every subgraph H of G with at most $\lceil kn/2 \rceil$ edges, the geometric graph $G-H$ contains a non-crossing subtree that spans $n-k$ vertices. Moreover, the complete geometric graph G on n points in convex position has a subgraph H with $\lceil kn/2 \rceil$ edges such that $G-H$ has no non-crossing tree on $n-k+1$ vertices.*

Again we omit the proof and remark that for the convex complete geometric graph G considering as subgraph H the *Harary graph $H_{k,n}$* [8], see Figure 2, yields the desired result.

We remark that also for point sets with many interior points we can not remove more edges than in the convex case to guarantee a non-crossing subtree of a given size in the remaining graph.

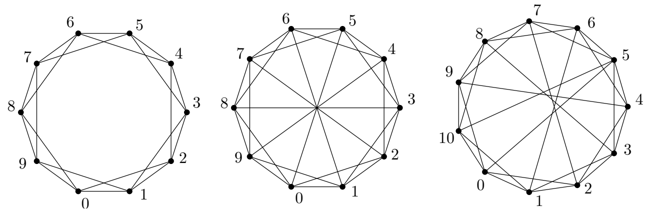

Figure 2: The Harary graphs $H_{4,10}$, $H_{5,10}$ and $H_{5,11}$.

4 More classes of non-crossing geometric graphs

4.1 Spanning paths

Černý et al. [4] showed that for any subgraph $H = (V, E')$ of the convex complete geometric graph G on n vertices with $|E'| \leq \lceil \frac{n}{2} \rceil - 1$, the geometric graph $G-H$ contains a non-crossing spanning path.

If the set S of n points has $k \leq \frac{n}{2}-2$ interior points, then we can not remove more than $\lceil \frac{n}{2} \rceil - 1$ edges of G; because each spanning path contains a perfect matching, for n even, and Figure 1 (left) shows that after removal of $\lceil \frac{n}{2} \rceil$ edges, the remaining graph does not even contain a non-crossing perfect matching.

Conjecture 2 *For every complete geometric graph G on a set of n points with $k \geq \lceil \frac{n}{2} \rceil - 2$ of them in the interior of the convex hull and for every subgraph H of G which has at most $k+1$ edges, the geometric graph $G-H$ contains a non-crossing spanning path.*

4.2 Spanning cycles

Point sets in convex position only admit one non-crossing spanning cycle. Therefore, removal of only one edge disallows such a cycle.

Conjecture 3 *For every complete geometric graph G on a set of n points with k of them in the interior of the convex hull and for every subgraph H of G which has at most $\lceil \frac{k}{2} \rceil$ edges, the geometric graph $G-H$ contains a non-crossing spanning cycle.*

Figure 3 shows an example where removal of $\lceil \frac{k+4}{2} \rceil$ edges disallows a non-crossing spanning cycle, for $k = n-3$.

4.3 Triangulations and pseudo-triangulations

For each point set S there exist edges which appear in every triangulation of S, for example edges of the convex hull. We call these edges *unavoidable edges*. Edges which do not appear in every triangulation are called *avoidable edges*. Clearly, removal of only one unavoidable edge of the complete geometric graph on S disallows a triangulation for S. Thus, for removal we only consider avoidable edges.

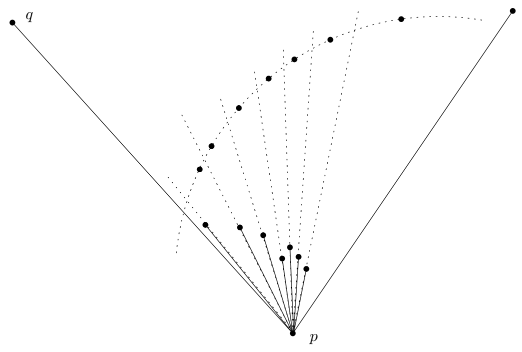

Figure 3: This point set contains no non-crossing spanning cycle if we disallow the $\lceil \frac{k+4}{2} \rceil$ drawn edges.

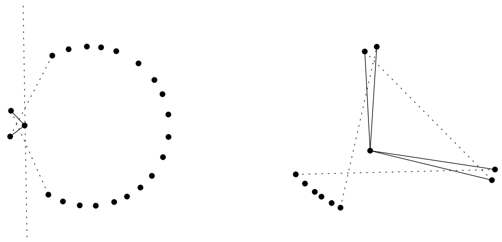

Figure 4: Removal of two avoidable edges disallows a triangulation (left) and removal of four avoidable edges disallows a pseudo-triangulation (right).

Theorem 7 *For each subgraph $H = (V, E')$ of the complete geometric graph G on n vertices in convex position where E' is a set of at most $n - 3$ avoidable edges, the geometric graph $G - H$ contains a triangulation. This bound is tight with respect to the cardinality of E'.*

Interestingly, in the case of triangulations less (avoidable) edges can be removed if we also consider interior points. Figure 4 (left) shows an example.

Lemma 8 *There exist point sets with interior points, such that removal of two avoidable edges disallows a triangulation.*

We finally consider pseudo-triangulations. A *pseudo-triangle* is a simple polygon that has exactly three interior angles less than π. A *pseudo-triangulation* of a point set S is a partition of the convex hull of S into pseudo-triangles whose vertex set is exactly S. For the considered problem, pseudo-triangulations behave similar to triangulations. Note that for point sets in convex position triangulations and pseudo-triangulations coincide.

Lemma 9 *There exist point sets with interior points, such that removal of four avoidable edges disallows a pseudo-triangulation.*

Figure 4 (right) shows an example. The four solid edges are avoidable. To see that their removal disallows a pseudo-triangulation, observe that the face incident to the interior vertex with angle greater than π has to have at least four convex vertices, whereas a pseudo-triangle has exactly three.

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