Extremal antipodal polygons and polytopes

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Abstract

Let S be a set of $2n$ points on a circle such that for each point $p \in S$ also its antipodal (mirrored with respect to the circle center) point p' belongs to S. A polygon P of size n is called *antipodal* if it consists of precisely one point of each antipodal pair (p, p') of S.

We provide a complete characterization of antipodal polygons which maximize (minimize, respectively) the area among all antipodal polygons of S. Based on this characterization, a simple linear time algorithm is presented for computing extremal antipodal polygons. Moreover, for the generalization of antipodal polygons to higher dimensions we show that a similar characterization does not exist.

Keywords: Antipodal points; extremal area polygons; discrete and computational geometry.

1 Introduction

For a point $p = (x_1, x_2) \in \mathbb{R}^2$, let $p' := (-x_1, -x_2)$ be the antipodal point of p. Consider a set S of points on a circle centered at the origin such that for each point $p \in S$ also its antipodal point p' belongs to S. We choose one point from each antipodal pair of S such that their convex hull is as large or as small (w.r.t. its area) as possible. Intuitively speaking, the largest polygon will have to contain the center of the circle, but the smallest one does not. In Figure 1 an example of a thin (not containing the center) and a thick (containing the center) polygon is shown. An interesting question, which immediately suggests itself, is whether any thick polygon of S has larger area than any thin polygon of S? In this paper, we will formalize the mentioned concepts of thin and thick polygons and answer this question for sets in the plane as well as for higher dimensions.

We start by introducing the problem formally in the plane. The generalization for higher dimensions is straightforward. A set of $2n$ points on the unit circle centered at the origin is called an *antipodal*

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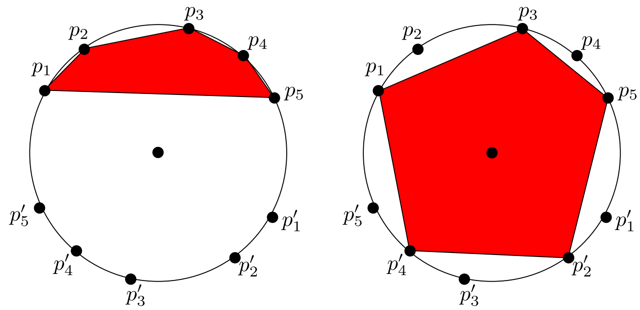

Figure 1: A thin (left) and a thick (right) antipodal polygon.

point set if for every point p it also contains its antipodal point p'. Let $S := \{p_1, p'_1, p_2, p'_2, \ldots, p_n, p'_n\}$ be such a set. An *antipodal* polygon on S is a convex polygon having as vertices precisely one point from each antipodal pair (p_i, p'_i) of S. A *thin* antipodal polygon is an antipodal polygon whose vertices all lie in a half-plane defined by some line through the origin. A *thick* antipodal polygon is an antipodal polygon such that at least $\lceil \frac{n-2}{2} \rceil$ of its vertices lie in both closed half-planes defined by any given line through the origin. See Figure 1. Note that a non-thin antipodal polygon does not need to be thick, but a thick antipodal polygon can never be thin. Moreover, a thin antipodal polygon does not contain the center of the circle and a non-thin antipodal polygon always contains it.

In this paper we investigate the following questions:

- Does a thick antipodal polygon always have larger area than a thin antipodal polygon?
- How efficiently can one compute an antipodal polygon with minimal (maximal) area?
- What can be said about antipodal polygons in higher dimensions?

1.1 Related work

The questions studied here are related to several other geometric problems, some of which we mention below.

Extremal problems: Plane geometry is rich of extremal problems, often dating back till the ancient Greeks. During the centuries many of these problems have been solved by geometrical reasoning. Specifically, extremal problems on convex polygons have attracted the attention of both fields, geometry and optimization. In computational geometry, efficient algorithms have been proposed for computing extremal polygons w.r.t. several different properties [5]. In operations research, global optimization techniques have been extensively studied to find convex polygons maximizing a given parameter [2]. A geometric extremal problem similar to the one studied in this paper was solved by Fejes Tóth [11] almost fifty years ago. He showed that the sum of pairwise distances determined by n points contained in a circle is maximized when the points are the vertices of a regular n-gon inscribed in the circle. Recently, the discrete version of this problem has been reviewed in [12] and problems considering maximal area instead of the sum of inter-point distances have been solved in [9].

Stabbing problems: The problem of stabbing a set of objects by a polygon (transversal problems in the mathematics literature) has been widely studied. For example, in computational geometry,

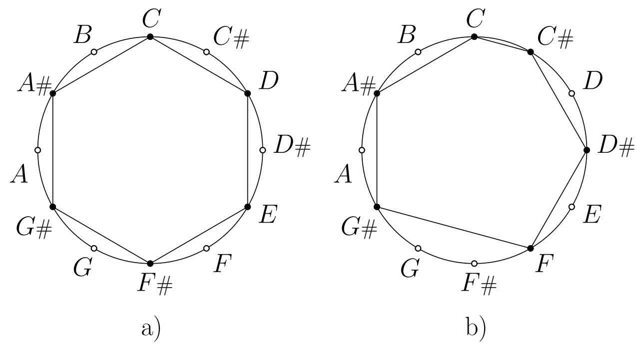

Figure 2: The subsets in a) and b) represent maximally even scales with and without tritones, respectively.

Arkin et al. [1] considered the following problem: a set S of segments is *stabbable* if there exists a convex polygon whose boundary C intersects every segment in S; the closed convex chain C is then called a (convex) transversal or *stabber* of S. Arkin et al. proved that deciding whether S is stabbable is an NP-hard problem. In a recent paper [6], the problem of stabbing the set S of line segments by a simple polygon but with a different criterion has been considered. A segment s is stabbed by a simple polygon P if at least one of the two endpoints of s is contained in P. Then the problem is: Find a simple polygon P that stabs S and has minimum(maximum) area among those that stab S. In [6], it is shown that if S is a set of n pairwise disjoint segments, the problem of computing the minimum and maximum area (perimeter) polygon stabbing S can be solved in polynomial time. However, for general (crossing) segments the problem is APX-hard. Notice that our problem is a constrained version of the problem studied in [6] in which each segment joins two antipodal points on a circle. As we will show later, our antipodal version (in which all segments intersect at the origin) can be computed in linear time.

Music Theory: There exists a surprisingly high number of applications of mathematics to music theory. Questions about variation, similarity, enumeration, and classification of musical structures have long intrigued both musicians and mathematicians. In some cases, these problems inspired mathematical discoveries. The research in music theory has illuminated problems that are appealing, nontrivial, and, in some cases, connected to deep mathematical questions. See for example [3, 4] for introductions to the interplay between mathematics and music.

In our case, an antipodal polygon is related with the *tritone* concept in music theory. Typically, the notes of a scale are represented by a polygon in a clock diagram. In a chromatic scale, each whole tone can be further divided into two semitones. Thus, we can think in a clock diagram with twelve points representing the twelve equally spaced pitches that represent the chromatic universe (using an equal tempered tuning). The pitch class diagram is illustrated in Figure 2 . A tritone is traditionally defined as a musical interval composed of three whole tones. Thus, it is any interval spanning six semitones. In Figure 2 a), the polygon represents a scale containing the tritones $CF\#, DG\#, EA\#$. The tritone is defined as a restless interval or dissonance in Western music from the early Middle Ages. This interval was frequently avoided in medieval ecclesiastical singing because of its dissonant quality. The name *diabolus in musica* (the Devil in music) has been applied to the interval from at least the early 18th century [10].

In this context, an antipodal polygon corresponds to a subset of notes or harmonic scale avoiding the tritone and, according to [9, 12], a maximal antipodal polygon represents a maximally even set that avoids the tritone.

1.2 Our results

In this paper we show that:

Claim 1.1 *For a given antipodal point set $S \in I\!R^2$ every thin antipodal polygon on S has less area than any non-thin antipodal polygon on S.*

In addition we show that the 2-dimensional case is special in the sense that the above result can not be generalized to higher dimensions.

The analogue result holds for thick antipodal polygons when n is odd but surprisingly turns out to be wrong when n is even; for n even we provide an example of an antipodal non-thick polygon having larger area than a thick antipodal polygon. However we are able to show that:

Claim 1.2 *For a given antipodal point set $S \in I\!R^2$ and every non-thick antipodal polygon on S, there exists a thick antipodal polygon on S with larger area.*

Note that above claims imply that an antipodal polygon with minimum (resp. maximum) area is thin (resp. thick).

2 Thin antipodal polygons

Assume that the clockwise circularly order of S around the origin is $p_1, p_2, \ldots, p_n$, $p'_1, p'_2, \ldots, p'_n$. For every point q in S, let S_q be the thin antipodal polygon that contains q as a vertex and all $n-1$ next consecutive points clockwise from q. Note that all thin antipodal polygons are of this form and that S_q and S'_q are congruent.

First, we prove a lemma regarding the triangles containing a given point of S.

Lemma 2.1 *For a point $p \in S$ let ℓ be the line containing p and p'. Let τ be the triangle determined by p, and its two neighbors in S. Among all triangles that have as vertices p and one point of S in each of the two half-planes defined by ℓ, τ has strictly the smallest area.*

Proof. Let τ' be a triangle with vertices in S, containing p as a vertex and with a vertex in each of the two half-planes defined by ℓ. Assume that τ' is different from τ. Let b be the side opposite to p in τ and b' be the side opposite to p in τ'. Note that b' is at least as large as b, because S is an antipodal point set and ℓ contains the origin. The height of τ' with respect to p is greater than the height of τ with respect to p, as otherwise b' would have to intersect b, which is not possible by construction. Thus the area of τ' is larger than the area of τ. □

We split the proof of Claim 1.1 into the three cases $n = 3$, $n = 4$, and $n \geq 5$.

Lemma 2.2 *For $n = 3$, every thin antipodal polygon on S has an area strictly less than that of any non-thin antipodal polygon on S.*

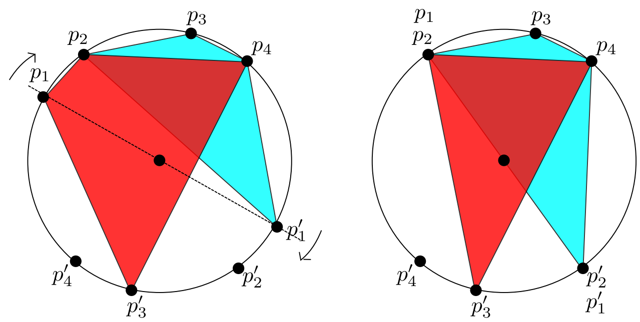

Figure 3: The rotation in the proof of Lemma 2.3 and its limit case.

Proof. In this case the only non-thin polygons are the two triangles τ and τ' with vertex sets $\{p_1, p_2', p_3\}$ and $\{p_1', p_2, p_3'\}$, respectively. Note that τ has the same area as τ'. In addition, by Lemma 2.1, τ has greater area than S_{p_2} and τ' has greater area than S_{p_1} and S_{p_3}. □

Lemma 2.3 *For $n = 4$, every thin antipodal polygon on S has an area strictly less than that of any non-thin antipodal polygon on S.*

Proof. In this case a non-thin antipodal polygon P has exactly two consecutive points; without loss of generality assume that they are p_1 and p_2. Thus P is the convex quadrilateral p_1, p_2, p_4, p_3'. We show that P has greater area than $S_{p_1}, S_{p_2}, S_{p_3'}$ and $S_{p_4'}$.

By Lemma 2.1 the triangle $p_4'p_1p_2$ has less area than the triangle $p_3'p_1p_2$. By Lemma 2.2 the triangle $p_3'p_2p_4$ has an area greater than the triangle $p_3'p_4'p_2$ and also greater than the triangle $p_4'p_2p_3$. Thus P has an area greater than $S_{p_3'}$ and also greater than $S_{p_4'}$. By Lemma 2.1 the triangle $p_1p_2p_3$ has less area than the triangle $p_1p_2p_4$. By Lemma 2.2 the triangle $p_3'p_1p_4$ has an area greater than the triangle $p_1p_3p_4$. Thus P has an area greater than S_{p_1}.

It remains to show that P has area greater than S_{p_2}. Let ℓ be the line passing through p_1 and p_1'. Rotate ℓ clockwise continuously around the origin, until p_1 meets p_2 and p_1' meets p_2'. See Figure 3. Note that throughout the motion the area of S_{p_2} is strictly increasing. To see that, notice that the height of the triangle with vertices p_2, p_4 and p_1 is strictly increasing, as otherwise, at some point p_1' must intersect the perpendicular bisector of the segment p_2p_4. However, this cannot happen since p_1' reaches p_2' before it reaches this line.

On the other hand, the area of P might at first be strictly increasing, then at some point be strictly decreasing. Moreover, if this is the case, there is a point in time, at which P has the same area as in the beginning of the motion (and will strictly decrease afterwards) and the area of S_{p_2} has increased. Assume then that the motion is such that the area of P is strictly decreasing and the area of S_{p_2} is strictly increasing.

We show that at the end of the motion P and S_{p_2} have equal area, this implies that at the beginning of the motion the area of P is greater than the area of S_{p_2}.

At the end of the motion P coincides with the triangle $p_2p_4p_3'$ and S_{p_2} with the quadrilateral $p_2p_3p_4p_2'$. We split the the quadrilateral $p_2p_3p_4p_2'$ into the triangles $p_2p_3p_4$ and $p_2'p_2p_4$, sharing the side $\overline{p_2p_4}$. The height of the triangle $p_2p_4p_3'$ with respect to $\overline{p_2p_4}$ has the same lenght that the sum of the heights of the triangles $p_2p_3p_4$ and $p_2'p_2p_4$ with respect to $\overline{p_2p_4}$ (It is easy to see by using the triangle $p_4'p_3'p_2'$). Hence Area$(p_2p_4p_3')$ equals Area$(p_2p_3p_4p_2')$. □

We are ready now to prove our first claim.

Theorem 2.4 *Every thin antipodal polygon on S has less area than any non-thin antipodal polygon on S.*

Proof. We proceed by induction on n. By Lemmas 2.2 and 2.3, we assume that $n \geq 5$. Let P be a non-thin antipodal polygon on S. Let T be any triangulation of P. Let p be a vertex of degree two in T and let p' be its antipodal point. Let τ be the only triangle of T having p as a vertex. Let q and r be the two neighbors of p in S. Let τ' be the triangle with vertices p, q and r. By Lemma 2.1 the area of τ' is equal or less than the area of τ.

Now, suppose that τ does not contain the origin in its interior, then the polygon P' with vertices $V(P) \setminus \{p\}$ is a non-thin antipodal polygon for $S \setminus \{p, p'\}$. By induction P' has area greater area than any thin antipodal polygon on $S \setminus \{p, p'\}$. Some of these thin polygons together with τ' form antipodal polygons on S. Using this observation and the fact that the area of S_{p_i} is the same as the area of $S_{p'_i}$, we can show that except for S_p and S_q all antipodal thin polygons on S have area strictly less than P. However, for $n \geq 5$, P can be triangulated so that p is not the middle nor the last vertex (clockwise) of an ear. As any triangulation has two ears. There is an ear that does not contain the origin. The previous arguments (for this ear) show that the area of P is strictly greater than the area of S_p, similarly for S_q. □

3 Thick antipodal polygons

In this section we present two area increasing operations on antipodal polygons. Using a sequence of these operations a non-thick antipodal polygon can be transformed into a thick antipodal polygon, this sequence proves Theorem 3.3.

We begin with an antipodal polygon P. Let q be a point in S. By *flipping* q, we mean the following operation: if q is a vertex of P, then choose q' instead; if q is not a vertex of P then choose q instead of q'. The two operations described in Lemmas 3.1 and 3.2 are sequences of such flips.

Lemma 3.1 *If P has three consecutive points q_1, q_2 and q_3 of S as vertices, then flipping q_2, provides a polygon P of greater area.*

Proof. Let q'_4 be the point after q'_3 in P and q'_0 be the point before q'_1 in P. Let τ_1 be the triangle with vertex set $\{q_1, q_2, q_3\}$ and τ_2 the triangle with vertex set $\{q'_0, q'_2, q'_4\}$. The difference of the areas of P and P' is equal to the difference in the areas of τ_1 and τ_2. However, τ_1 has the same area as the triangle with vertex set $\{q'_1, q'_2, q'_3\}$; by Lemma 2.1 the area of this triangle is less than that of τ_2. □

From now on, we assume that P does not contain three consecutive points of S as vertices. Otherwise we apply the operation described in Lemma 3.1.

Lemma 3.2 *Let $q_1, q_2, \ldots, q_m$ $(4 \leq m < n)$ be consecutive points of S. Suppose that:*

- *P contains q_1 and q_2.*
- *P contains either both q_{m-1} and q_m, or neither of them.*

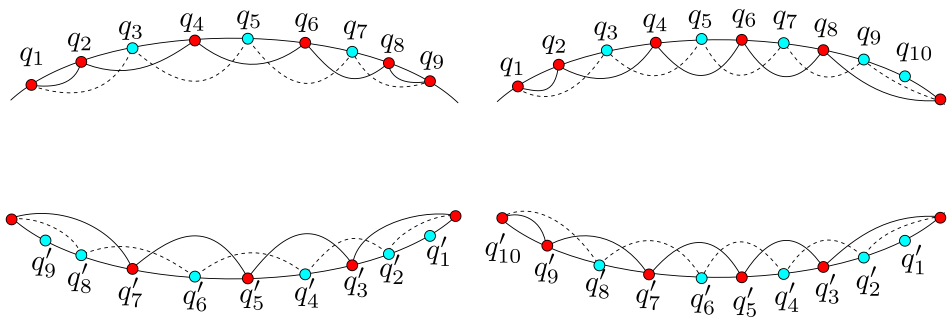

Figure 4: Schematic diagram of the two flip operations described in Lemma 3.2. P is drawn solid and P' is dashed.

- *The points from q_2 to q_{m-1} alternatingly belong to P or not.*

Let P' be the antipodal polygon obtained from P, by flipping each point q_i $(2 \leq i \leq m-1)$. Then P' has greater area than P.

Proof. For each p in $\{q_2, q_2', \ldots, q_{m-1}, q_{m-1}'\}$, let $\tau(p)$ be the triangle such that: has p as a vertex; if p is a vertex of P then the two sides of $\tau(p)$ that contain p, are contained in the boundary of P, while its opposite side is contained in P'; if p is a vertex of P' then the two sides of $\tau(p)$ that contain p, are contained in the boundary of P', while its opposite side is contained in P. The difference in the area of P and the area of P' equals the difference in the areas of those triangles contained in P and those contained in P'. For $4 \leq i \leq m-3$, the area of $\tau(q_i)$ equals the area of $\tau(q_i')$ and one of them is contained in P while the other is contained in P'. Thus the difference in the areas of P and P' depends only on the areas of $\tau(q_2)$, $\tau(q_2')$, $\tau(q_3)$, $\tau(q_3')$, $\tau(q_{m-2})$, $\tau(q_{m-2}')$, $\tau(q_{m-1})$, and $\tau(q_{m-1}')$ Note that the area of $\tau(q_2)$ is smaller than the area of $\tau(q_2')$ and that P contains $\tau(q_2)$ while P' contains $\tau(q_2')$. Similarly for $\tau(q_3)$ and $\tau(q_3')$). See Figure 4.

If P contains both q_{m-1} and q_m, then $\tau(q_{m-1})$ is contained in P and $\tau(q_{m-1}')$ is contained in P'. In this case the area of $\tau(q_{m-1})$ is smaller than the area of $\tau(q_{m-1}')$.

If P does not contain q_{m-1} and q_m, then $\tau(q_{m-1}')$ is contained in P and $\tau(q_{m-1})$ is contained in P'. In this case the area of $\tau(q_{m-1}')$ is smaller than the area of $\tau(q_{m-1})$. The same argument can by apply to $\tau(q_{m-2})$ and $\tau(q_{m-2}')$). Thus, in all cases the area of P is smaller than the area of P'. $\square$

Note that in the operation described in Lemma 3.2 the number of pairs of consecutive points that are either both on P or not in P decreases. Moreover, no three consecutive points all in P or all not in P are created at the same time .

We are now ready to prove the second claim.

Theorem 3.3 *For every non-thick antipodal polygon on S, there exists a thick antipodal polygon on S of greater area.*

Proof. For n odd, an antipodal polygon Q is thick if and only if its points alternate between being in Q and not being in Q. For n even, an antipodal polygon is thick if and only if its points alternate between being in Q and not in Q, with the exception of exactly one pair of consecutive points which are both in Q (and its antipodal points not in Q).

Assume that all possible operations of Lemmas 3.1 and 3.2 have been applied to a non-thick antipodal polygon P, then P contains at most one pair of consecutive points in S as vertices and P is a thick polygon. □

Corollary 3.4 *For n odd, every thick antipodal polygon on S has greater area than a non-thick antipodal polygon on S.*

Proof. In this case there are only two antipodal thick polygons and they have the same area. □

We now provide an example of a set of points and a non-thick antipodal polygon that has greater area than a thick antipodal polygon on this set.

Theorem 3.5 *For $n \geq 6$ even, there exist point sets with a non-thick antipodal polygon of greater area than a thick antipodal polygon.*

Proof. Place p_1 and p_2 arbitrarily close to $(1, 0)$; thus p'_1 and p'_2 are arbitrarily close to $(-1, 0)$. Place $p_3, \ldots, p_n$ arbitrarily close to $(0, 1)$; thus $p'_3, \ldots, p'_n$ are arbitrarily close to $(0, -1)$. Let P be the thick antipodal polygon that contains both p_1 and p_2 as vertices. Let Q be any non-thick antipodal polygon that contains p_1, p'_2, p_3 and p'_4 as vertices. Note that P is arbitrarily close to the triangle with vertices $(0, 1)$, $(0, -1)$ and $(1, 0)$; Q is arbitrarily close to the quadrilateral with vertices $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(0, -1)$. Thus the area of P is arbitrarily close to 1, while the area of Q is arbitrarily close to 2. □

4 The algorithms

It is worth mentioning that the general algorithmic version of the problem in which the input is a set of line segments, each connecting two points on the circle, has been proved to be NP-hard [6]. Surprisingly, the antipodal version can be easily solved by using above characterizations.

Theorem 4.1 *Antipodal polygons with minimum or maximum the area can be computed in linear time.*

Proof. According to Theorem 2.4, an antipodal polygon with minimum area is a thin antipodal polygon. Thus, since there exist $O(n)$ thin polygons, we can sweep in a linear number of steps around the circle and update in constant time the area of two consecutive thin polygons. On the other hand, according to Theorem 3.3, if n is odd, there are only two thick antipodal polygons (the alternating polygons). For n even, there exists a linear number of thick polygons (having two consecutive points and the rest in alternating position). In the last case, a linear sweep around the circle can also be used to compute in linear time a thick antipodal polygon that maximizes the area. □

5 Higher Dimensions: Antipodal Polytopes

In this section we consider the analogous problem in higher dimensions. Assume therefore that all points are now placed on the unit d-dimensional sphere. Instead of antipodal polygons we thus have antipodal polytopes. For a thin antipodal polytope all its points lie on one side of some hyperplane passing through the origin.

In dimension 3 or greater Theorem 2.4 does not hold—there are antipodal point sets $S \subset \mathbb{R}^d$ such that there exists an antipodal thin polytope with greater d-dimensional volume than a non-thin antipodal polytope on S. We start by providing a three dimensional example and then argue how to generalize it to higher dimensions.

For some small $\varepsilon > 0$, let $\delta = \sqrt{1-2\varepsilon^2}$ and consider the set S_1 of the five points $v_1 := (0,0,1)$, $v_2 := (\delta,\varepsilon,\varepsilon)$, $v_3 := (-\delta,\varepsilon,\varepsilon)$, $v_4 := (\varepsilon,\delta,\varepsilon)$, and $v_5 := (\varepsilon,-\delta,\varepsilon)$. Let S be the antipodal point set consisting of S_1 and all its antipodal points. The convex hull of S_1 is a pyramid with a square base (with corners $v_2,\ldots,v_5$) which lies in the horizontal plane just ε above the origin. The top of the pyramid is at height 1. Thus, this pyramid does not contain the origin in its interior, and for $\varepsilon \to 0$ the volume of the pyramid converges to $2/3$.

To obtain our second polyhedra first flip the vertex v_1 to $v_1' := (0,0,-1)$. This gives a similar upside-down pyramid, which contains the origin in its interior. By also flipping v_2 to $v_2' := (-\delta,-\varepsilon,-\varepsilon)$, we essentially halve the base of the pyramid to be a triangle. We denote the resulting point set by $S_2 = \{v_1', v_2', v_3, v_4, v_5\} \subset S$. Note that v_2' and v_3 are rather close together. As the triangle v_3, v_4, v_5 lies above the origin, the convex hull of S_2 still contains the origin in its interior. Moreover, the volume of the convex hull of S_2 converges to $1/3$ for $\varepsilon \to 0$, and thus towards half of the volume of the convex hull of S_1.

So together these two polyhedra constitute an example which shows that Theorem 2.4 can not be generalized to higher dimensions: S is a set of five antipodal pairs of points on the surface of the 3-dimensional unit sphere such that the convex hull of S_1 does not contain the origin, while the convex hull of S_2 does. But in the limit the volume of the convex hull of S_1 becomes twice as large as the volume of the convex hull of S_2.

It is straight forward to observe that this example can be generalized to any dimension $d \geq 4$. There we have $2d-1$ antipodal pairs of points, where we set $\delta = \sqrt{1-(d-1)\varepsilon^2}$ and every point has one coordinate at $\pm\delta$ and the remaining coordinates at $\pm\varepsilon$, analogous to the 3-dimensional case. For $d-1$ of the coordinate axes two such pairs are 'aligned' as in the 3-dimensional example, and for the last axis there is only one such pair. The resulting polytope does not contain the origin. Flipping the vertex of the singular pair and one vertex for all but one aligned pairs results in a polytope which contains the origin, but has a volume of only $1/2^{d-2}$ of the first polytope.

We call a d-dimensional antipodal polytope *thick* if the number of vertices in any half-space defined by a hyperplane through the origin contains at least $\left\lfloor \frac{n-d}{2} \right\rfloor$ points of the polytope. Note that this definition generalizes the two dimensional case.

It is not clear that for a given antipodal set in $\mathbb{R}^d$ an antipodal thick polytope should exist. However, for every $n \geq d$, there exists antipodal sets in $\mathbb{R}^d$ that admit an antipodal thick polytope. We use the following Lemma.

Lemma 5.1 (Gale's Lemma [7]). *For every $d \geq 0$ and every $k \geq 1$, there exists a set $X \subset S^d$ of $2k+d$ points such that every open hemisphere of S^d contains at least k points of X.*

From the proof of Gale's Lemma in [8] (page 64), it follows that the provided set does not contain an antipodal pair of points. Let X be the set provided by Gale's Lemma for $k = \left\lceil \frac{n-d}{2} \right\rceil$. If necessary remove a point from X so that X consists of exactly n points. Let X' be the set of antipodal points of X. Set $S := X \cup X'$. Let P be the antipodal polytope on S with X as a vertex set. It follows from Gale's Lemma that P is thick.

6 Open problems

Let us assume that we are given a circular lattice with an antipodal set of $2n$ points (evenly spaced) and we would like to compute an extremal antipodal k-polygon with $k < n$ vertices. This problem is significantly different to the considered case $k = n$. Recall that, for $k = n$, the linear algorithms proposed in this paper are strongly based on the simple characterization for the extremal antipodal polygons. Namely, the minimal thin antipodal polygon has consecutive vertices and the thick one has an alternating configuration. It is not difficult to come up with examples for which that characterization does not hold in the general case $k < n$. On the other hand, finding the extremal antipodal $(n-1)$-polygon, called $(2n, n-1)$-problem for short, can be easily reduced to solve $O(n)$ times the $(2(n-1), n-1)$-problem. To see this, observe that in the $(2n, n-1)$-problem an antipodal pair is not selected and can thus be removed from the input. This approach gives a simple $O(n^k)$ time algorithm for solving the general $(2n, k)$-problem. This leaves as open problem to prove if the $(2n, k)$-problem can be solve in subquadratic time.

Instead of area, it is also interesting to consider other extremal measures, like perimeter or the sum of inter-point distances. Finally, for higher dimensions, we leave the existence of thick polytopes for arbitrary antipodal point sets as an open problem.

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