

# Characterization of extremal antipodal polygons

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May 23, 2014

## Abstract

Let  $S$  be a set of  $2n$  points on a circle such that for each point  $p \in S$  also its antipodal (mirrored with respect to the circle center) point  $p'$  belongs to  $S$ . A polygon  $P$  of size  $n$  is called *antipodal* if it consists of precisely one point of each antipodal pair  $(p, p')$  of  $S$ . We provide a complete characterization of antipodal polygons which maximize (minimize, respectively) the area among all antipodal polygons of  $S$ . Based on this characterization, a simple linear time algorithm is presented for computing extremal antipodal polygons. Moreover, for the generalization of antipodal polygons to higher dimensions we show that a similar characterization does not exist.

**Keywords:** Antipodal points; extremal area polygons; discrete and computational geometry.

## 1 Introduction

For a point  $p = (x_1, x_2) \in \mathbb{R}^2$ , let  $p' := (-x_1, -x_2)$  be the antipodal point of  $p$ . Consider a set  $S$  of points on a circle centered at the origin such that for each point  $p \in S$  also its antipodal point  $p'$  belongs to  $S$ . We choose one point from each antipodal pair of  $S$  such that their convex hull is as large or as small (w.r.t. its area) as possible. It is easy to see that, with this selection, the largest polygon will have to contain the center of the circle, but the smallest one does not. In Figure 1 an example of a thin (the smallest) and a thick (the largest) polygon is shown. An interesting question, which immediately suggests itself, is whether any polygon of  $S$  containing the center has larger area than any polygon that does not. In this paper, we will formalize the mentioned concepts of thin and thick polygons and answer this question for sets in the plane as well as for higher dimensions.

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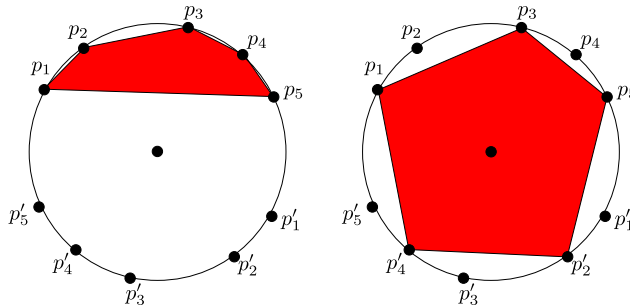


Figure 1: A thin (left) and a thick (right) antipodal polygon.

We start by introducing the problem formally in the plane. A set of  $2n$  points on the unit circle centered at the origin is called an *antipodal point set* if for every point  $p$  it also contains its antipodal point  $p'$ . Let  $S := \{p_1, p'_1, p_2, p'_2, \dots, p_n, p'_n\}$  be such a set. An *antipodal polygon* on  $S$  is a convex polygon having as vertices precisely one point from each antipodal pair  $(p_i, p'_i)$  of  $S$ . A *thin* antipodal polygon  $P$  is an antipodal polygon whose vertices are all consecutive points on the circle. For  $n$  odd, a *thick* antipodal polygon  $P$  is an antipodal polygon such that all its vertices alternate on the circle. For  $n$  even, we add the exception that exactly one pair of points in  $P$  are consecutive on the circle. See Figure 1. Note that an antipodal polygon  $P$  could be thin, thick, or neither, but  $P$  cannot both be thin and thick at the same time. Moreover, a thin antipodal polygon does not contain the center of the circle and a non-thin antipodal polygon always contains it.

In this paper we investigate the following questions:

- Does a thick antipodal polygon always have larger area than a thin antipodal polygon?
- How efficiently can one compute an antipodal polygon with minimal (maximal) area?
- What can be said about antipodal polygons in higher dimensions?

## 1.1 Related work

The questions studied here are related to several other geometric problems, some of which we mention below.

*Extremal problems:* Plane geometry is rich of extremal problems, often dating back till the ancient Greeks. During the centuries many of these problems have been solved by geometrical reasoning. Specifically, extremal problems on convex polygons have attracted the attention of both fields, geometry and optimization. In computational geometry, efficient algorithms have been proposed for computing extremal polygons w.r.t. several different properties [6]. In operations research, global optimization techniques have been extensively studied to find convex polygons maximizing a given parameter [3]. A geometric extremal problem similar to the one studied in this paper was solved by Fejes Tóth [14] almost fifty years ago. He showed that the sum of pairwise distances determined by  $n$  points contained in a circle is maximized when the points are the vertices of a regular  $n$ -gon inscribed in the circle. Recently, the discrete version of this problem has been reviewed in [16] and problems considering maximal area instead of the sum of inter-point distances have been solved in [12].

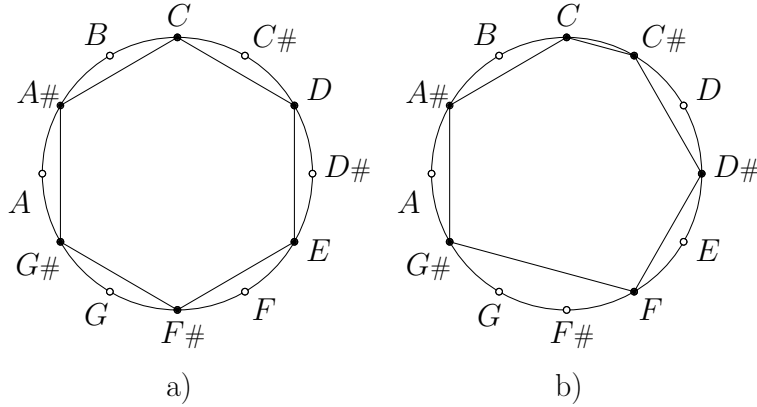


Figure 2: The subsets in a) and b) represent maximally even scales with and without tritones, respectively.

*Stabbing problems:* The problem of stabbing a set of objects by a polygon (transversal problems in the mathematics literature) has been widely studied. For example, in computational geometry, Arkin et al. [1] considered the following problem: a set  $S$  of segments is *stabable* if there exists a convex polygon whose boundary  $C$  intersects every segment in  $S$ ; the closed convex chain  $C$  is then called a (convex) transversal or *stabber* of  $S$ . Arkin et al. [1] proved that deciding whether  $S$  is stabable is an NP-hard problem. In a recent paper [9], the problem of stabbing the set  $S$  of line segments by a simple polygon but with a different criterion has been considered. A segment  $s$  is stabbed by a simple polygon  $P$  if at least one of the two endpoints of  $s$  is contained in  $P$ . Then the problem is: Find a simple polygon  $P$  that stabs  $S$  and has minimum(maximum) area among those that stab  $S$ . In [9], it is shown that if  $S$  is a set of  $n$  pairwise disjoint segments, the problem of computing the minimum and maximum area (perimeter) polygon stabbing  $S$  can be solved in polynomial time. However, for general (crossing) segments the problem is NP-hard. Notice that our problem is a constrained version of the problem studied in [9] in which each segment joins two antipodal points on a circle. As we will show later, our antipodal version (in which all segments intersect at the origin) can be computed in linear time.

*Music Theory:* There exists a surprisingly high number of applications of mathematics to music theory. Questions about variation, similarity, enumeration, and classification of musical structures have long intrigued both musicians and mathematicians. In some cases, these problems inspired mathematical discoveries. The research in music theory has illuminated problems that are appealing, nontrivial, and, in some cases, connected to deep mathematical questions. See for example [4, 5] for introductions to the interplay between mathematics and music.

In our case, an antipodal polygon is related with the *tritone* concept in music theory. Typically, the notes of a scale are represented by a polygon in a clock diagram. In a chromatic scale, each whole tone can be further divided into two semitones. Thus, we can think in a clock diagram with twelve points representing the twelve equally spaced pitches that represent the chromatic universe (using an equal tempered tuning). The pitch class diagram is illustrated in Figure 2. A tritone is traditionally defined as a musical interval composed of three whole tones. Thus, it is any interval spanning six semitones. In Figure 2 a), the polygon represents a scale containing the tritones  $CF\#$ ,  $DG\#$ ,  $EA\#$ . The tritone is defined as a restless interval or dissonance in Western music from the early Middle Ages. This interval was frequently avoided in medieval ecclesiastical singing because of its dissonant quality. The name *diabolus in musica* (the Devil in music) has been applied

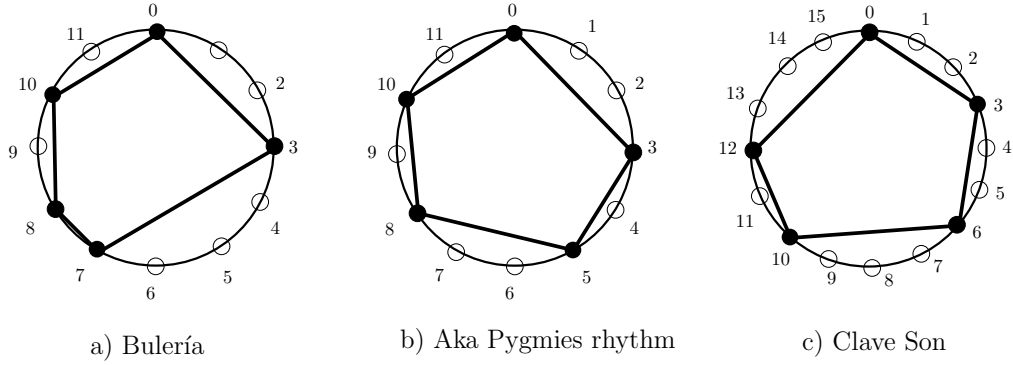


Figure 3: a) The bulería rhythm used in Spain, b) a rhythm used by the Aka Pygmies of Central Africa, c) the clave son of Cuba.

to the interval from at least the early 18th century [13]. In this context, an antipodal polygon corresponds to a subset of notes or harmonic scale avoiding the tritone. On the other hand, one of the properties that musicologists and mathematicians have observed in music in various oral traditions is what is called regularity of the rhythm (or of the musical scale). Regular rhythms have been defined as those which maximize a particular geometric measure [16]. Thus, a maximal antipodal polygon represents a maximally even set that avoids the tritone. A relationship between extremal polygons and musical scales has been showed in [12].

Moreover, inscribed polygons with no antipodal vertices have also been considered in the analysis of musical rhythms [2, 7]. A rhythm has the *rhythmic oddity* property if, when represented on a circle, it does contain two onsets (the black points in the Figure 3) that lie diametrically opposite each other. Notice that the property asserts that one cannot break the circle into two parts of equal length whatever the chosen breaking onsets and then the pattern possesses a particular type of asymmetry. Many musical traditions all over the world have asymmetric rhythmic patterns. Indeed, rhythms with the oddity property (antipodal polygons have such a property) play a fundamental role in traditional music as, for instance, flamenco music, african music or cuban music. The *bulería pattern* used in the flamenco music of Spain (Figure 3 a)), the Aka Pygmies pattern of Central Africa (Figure 3 b)) and the *clave son* in Cuba (Figure 3 c)) can all be considered as antipodal polygons with  $k < n$  vertices. See [8], [2] and [15], respectively, for a detailed study on the preference of theses rhythms in their cultural contexts.

## 1.2 Our results

In this paper we prove the following universal result:

**Claim 1.1** *For a given antipodal point set  $S \in \mathbb{R}^2$  every thin antipodal polygon on  $S$  has less area than any non-thin antipodal polygon on  $S$ .*

In addition we show that the 2-dimensional case is special in the sense that the above result can not be generalized to higher dimensions.

The analogue result holds for thick antipodal polygons when  $n$  is odd but surprisingly turns out to be wrong when  $n$  is even; for  $n$  even we provide an example of an antipodal non-thick polygon having larger area than a thick antipodal polygon. However we are able to prove the following

existencial result:

**Claim 1.2** *For a given antipodal point set  $S \in \mathbb{R}^2$  and every non-thick antipodal polygon on  $S$ , there exists a thick antipodal polygon on  $S$  with larger area.*

Note that above claims imply that an antipodal polygon with minimum (resp. maximum) area is thin (resp. thick).

## 2 Thin antipodal polygons

Assume that the clockwise circularly order of  $S$  around the origin is  $p_1, p_2, \dots, p_n, p'_1, p'_2, \dots, p'_n$ . For every point  $q$  in  $S$ , let  $P_q$  be the thin antipodal polygon that contains  $q$  as a vertex and all  $n-1$  next consecutive points clockwise from  $q$ . Note that all thin antipodal polygons are of this form and that  $P_q$  and  $P'_q$  are congruent, that is,  $P_q$  and  $P'_q$  are the same polygon in different positions.

First, we prove a lemma regarding the triangles containing a given point of  $S$ .

**Lemma 2.1** *For a point  $p \in S$  let  $\ell$  be the line containing  $p$  and  $p'$ . Let  $\tau$  be the triangle determined by  $p$ , and its two neighbors in  $S$ . Among all triangles that have as vertices  $p$  and one point of  $S$  in each of the two half-planes defined by  $\ell$ ,  $\tau$  has strictly the smallest area.*

**Proof.** Let  $\tau'$  be a triangle with vertices in  $S$ , containing  $p$  as a vertex and with a vertex in each of the two half-planes defined by  $\ell$ . Assume that  $\tau'$  is different from  $\tau$ . Let  $b$  be the side opposite to  $p$  in  $\tau$  and  $b'$  be the side opposite to  $p$  in  $\tau'$ . Note that  $b'$  is at least as large as  $b$ , because  $S$  is an antipodal point set and  $\ell$  contain the origin. The height of  $\tau'$  with respect to  $p$  is greater than the height of  $\tau$  with respect to  $p$ , as otherwise  $b'$  would have to intersect  $b$ , which is not possible by construction. Thus the area of  $\tau'$  is larger than the area of  $\tau$ .  $\square$

We split the proof of Claim 1.1 into the three cases  $n = 3$ ,  $n = 4$ , and  $n \geq 5$ .

**Lemma 2.2** *For  $n = 3$ , every thin antipodal polygon on  $S$  has an area strictly less than that of any non-thin antipodal polygon on  $S$ .*

**Proof.** In this case the only non-thin polygons are the two triangles  $\tau$  and  $\tau'$  with vertex sets  $\{p_1, p'_2, p_3\}$  and  $\{p'_1, p_2, p'_3\}$ , respectively. Note that  $\tau$  has the same area as  $\tau'$ . In addition, by Lemma 2.1,  $\tau$  has greater area than  $P_{p_2}$  and  $\tau'$  has greater area than  $P_{p_1}$  and  $P_{p_3}$ .  $\square$

**Lemma 2.3** *For  $n = 4$ , every thin antipodal polygon on  $S$  has an area strictly less than that of any non-thin antipodal polygon on  $S$ .*

**Proof.** In this case a non-thin antipodal polygon  $P$  has exactly two consecutive points; without loss of generality assume that they are  $p_1$  and  $p_2$ . Thus  $P$  is the convex quadrilateral  $p_1, p_2, p_4, p'_3$ . We show that  $P$  has greater area than  $P_{p_1}, P_{p_2}, P_{p'_3}$  and  $P_{p'_4}$ .

By Lemma 2.1 the triangle  $p'_4 p_1 p_2$  has less area than the triangle  $p'_3 p_1 p_2$ . By Lemma 2.2 the triangle  $p'_3 p_2 p_4$  has an area greater than the triangle  $p'_3 p'_4 p_2$  and also greater than the triangle  $p'_4 p_2 p_3$ . Thus  $P$  has an area greater than  $P_{p'_3}$  and also greater than  $P_{p'_4}$ . By Lemma 2.1 the triangle  $p_1 p_2 p_3$  has

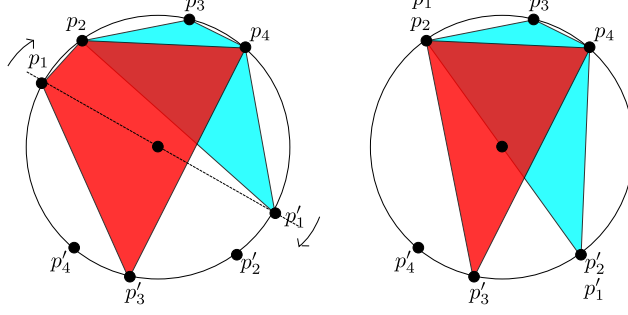


Figure 4: The rotation in the proof of Lemma 2.3 and its limit case.

less area than the triangle  $p_1p_2p_4$ . By Lemma 2.2 the triangle  $p'_3p_1p_4$  has an area greater than the triangle  $p_1p_3p_4$ . Thus  $P$  has an area greater than  $P_{p_1}$ .

It remains to show that  $P$  has area greater than  $P_{p_2}$ . Let  $\ell$  be the line passing through  $p_1$  and  $p'_1$ . Rotate  $\ell$  clockwise continuously around the origin, until  $p_1$  meets  $p_2$  and  $p'_1$  meets  $p'_2$ . See Figure 4. Note that throughout the motion the area of  $P_{p_2}$  is strictly increasing. To see that, notice that the height of the triangle with vertices  $p_2, p_4$  and  $p_1$  is strictly increasing, as otherwise, at some point  $p'_1$  must intersect the perpendicular bisector of the segment  $p_2p_4$ . However, this cannot happen since  $p'_1$  reaches  $p'_2$  before it reaches this line.

On the other hand, the area of  $P$  might at first be strictly increasing, then at some point be strictly decreasing. Moreover, if this is the case, then there is a point in time, at which  $P$  has the same area as in the beginning of the motion (and will strictly decrease afterwards) and the area of  $P_{p_2}$  has increased. Thus assume then that the motion is such that the area of  $P$  is strictly decreasing and the area of  $P_{p_2}$  is strictly increasing.

We show that at the end of the motion  $P$  and  $P_{p_2}$  have equal area. This implies that at the beginning of the motion the area of  $P$  is greater than the area of  $P_{p_2}$ .

At the end of the motion  $P$  coincides with the triangle  $p_2p_4p'_3$  and  $P_{p_2}$  with the quadrilateral  $p_2p_3p_4p'_2$ . We split the quadrilateral  $p_2p_3p_4p'_2$  into the triangles  $p_2p_3p_4$  and  $p'_2p_2p_4$ , sharing the side  $\overline{p_2p_4}$ . The height of the triangle  $p_2p_4p'_3$  with respect to  $\overline{p_2p_4}$  has the same length as the sum of the heights of the triangles  $p_2p_3p_4$  and  $p'_2p_2p_4$  with respect to  $\overline{p_2p_4}$  (It is easy to see by using the triangle  $p'_4p'_3p'_2$ ). Hence,  $\text{Area}(p_2p_4p'_3)$  equals  $\text{Area}(p_2p_3p_4p'_2)$ .  $\square$

We are now ready to prove our first claim.

**Theorem 2.4** *Every thin antipodal polygon on  $S$  has less area than any non-thin antipodal polygon on  $S$ .*

**Proof.** We proceed by induction on  $n$ . By Lemmas 2.2 and 2.3, we assume that  $n \geq 5$ . Let  $P$  be a non-thin antipodal polygon on  $S$ . Let  $T$  be any triangulation of  $P$ . Let  $p$  be a vertex of degree two in  $T$  such that the triangle of  $T$  having  $p$  as a vertex does not contain the origin in its interior. Let  $\tau$  be the only triangle of  $T$  having  $p$  as a vertex. Let  $q$  and  $r$  be the two neighbors of  $p$  in  $S$ . Let  $\tau'$  be the triangle with vertices  $p, q$  and  $r$ . By Lemma 2.1 the area of  $\tau'$  is equal or less than the area of  $\tau$ .

Then the polygon  $P'$  with vertices  $V(P) \setminus \{p\}$  is a non-thin antipodal polygon for  $S \setminus \{p, p'\}$ . By induction,  $P'$  has greater area than any thin antipodal polygon on  $S \setminus \{p, p'\}$ . Some of these thin

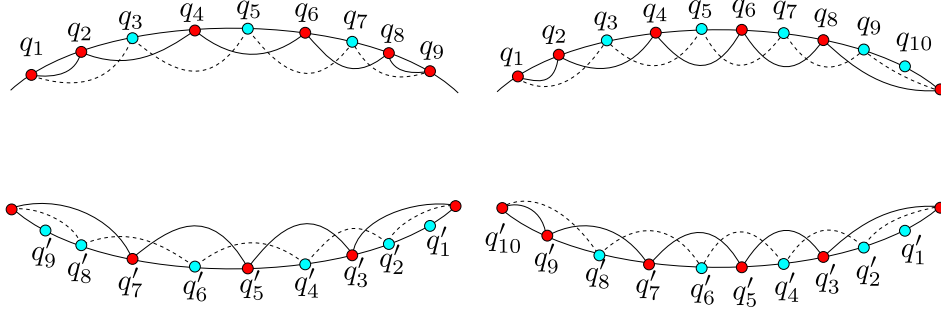


Figure 5: Schematic diagram of the two flip operations described in Lemma 3.2.  $P$  is drawn solid and  $P'$  is drawn dashed.

polygons (specifically, any one that includes both  $r$  and  $q$ ) together with  $\tau'$  form antipodal polygons on  $S$ . Using this observation and the fact that the area of  $P_{p_i}$  is the same as the area of  $P_{p'_i}$ , it follows that, except for  $P_p$  and  $P_q$  ( $P_{p'}$  and  $P_{q'}$ ), all antipodal thin polygons on  $S$  have area strictly less than  $P$ . However, for  $n \geq 5$ ,  $P$  can be triangulated so that  $p$  is not a vertex of degree two in the triangulation. Using the previous arguments for this new triangulation show that the area of  $P$  is strictly greater than the area of  $P_p$  ( $P_{p'}$ ), and similarly for  $P_q$  ( $P_{q'}$ ).  $\square$

### 3 Thick antipodal polygons

In this section we present two area increasing operations on antipodal polygons. Using a sequence of these operations a non-thick antipodal polygon can be transformed into a thick antipodal polygon. This sequence will prove Claim 1.2.

We begin with an antipodal polygon  $P$ . Let  $q$  be a point in  $S$ . By *flipping*  $q$ , we mean the following operation: if  $q$  is a vertex of  $P$ , then choose  $q'$  instead; if  $q$  is not a vertex of  $P$  then choose  $q$  instead of  $q'$ . The two operations described in Lemmas 3.1 and 3.2 are sequences of such flips.

**Lemma 3.1** *If  $P$  has three consecutive points  $q_1, q_2$  and  $q_3$  of  $S$  as vertices, then flipping  $q_2$  provides a polygon  $P'$  of greater area.*

**Proof.** Let  $q'_4$  be the point after  $q'_3$  in  $P$  and  $q'_0$  be the point before  $q'_1$  in  $P$ . Let  $\tau_1$  be the triangle with vertex set  $\{q_1, q_2, q_3\}$  and  $\tau_2$  the triangle with vertex set  $\{q'_0, q'_2, q'_4\}$ . The difference of the areas of  $P$  and  $P'$  is equal to the difference in the areas of  $\tau_1$  and  $\tau_2$ . However,  $\tau_1$  has the same area as the triangle with vertex set  $\{q'_1, q'_2, q'_3\}$ ; by Lemma 2.1 the area of this triangle is less than that of  $\tau_2$ .  $\square$

From now on, we assume that  $P$  does not contain three consecutive points of  $S$  as vertices. Otherwise we apply the operation described in Lemma 3.1.

**Lemma 3.2** *Let  $q_1, q_2, \dots, q_m$  ( $4 \leq m < n$ ) be consecutive points of  $S$ . Suppose that:*

- $P$  contains  $q_1$  and  $q_2$ .
- $P$  contains either both  $q_{m-1}$  and  $q_m$ , or neither of them.

- Every second point from  $q_3$  to  $q_{m-1}$  belongs to  $P$ .

Let  $P'$  be the antipodal polygon obtained from  $P$ , by flipping each point  $q_i$  ( $2 \leq i \leq m-1$ ). Then  $P'$  has greater area than  $P$ .

**Proof.** Note that  $\mathcal{T} := (P \setminus P') \cup (P' \setminus P)$  is a set of interior disjoint triangles. For each  $p$  in  $\{q_2, q'_2, \dots, q_{m-1}, q'_{m-1}\}$  let  $\tau(p)$  be the triangle in  $\mathcal{T}$  that contains  $p$  as a vertex. The difference in the area of  $P$  and the area of  $P'$  equals the difference in the areas of those triangles contained in  $P$  and those contained in  $P'$ . For  $4 \leq i \leq m-3$ , the area of  $\tau(q_i)$  equals the area of  $\tau(q'_i)$  and one of them is contained in  $P$  while the other is contained in  $P'$ . Thus, the difference in the areas of  $P$  and  $P'$  depends only on the areas of  $\tau(q_2)$ ,  $\tau(q'_2)$ ,  $\tau(q_3)$ ,  $\tau(q'_3)$ ,  $\tau(q_{m-2})$ ,  $\tau(q'_{m-2})$ ,  $\tau(q_{m-1})$ , and  $\tau(q'_{m-1})$ . Note that the area of  $\tau(q_2)$  is smaller than the area of  $\tau(q'_2)$  and that  $P$  contains  $\tau(q_2)$  while  $P'$  contains  $\tau(q'_2)$ . Similarly for  $\tau(q_3)$  and  $\tau(q'_3)$ . See Figure 5.

If  $P$  contains both  $q_{m-1}$  and  $q_m$ , then  $\tau(q_{m-1})$  is contained in  $P$  and  $\tau(q'_{m-1})$  is contained in  $P'$ . In this case the area of  $\tau(q_{m-1})$  is smaller than the area of  $\tau(q'_{m-1})$ .

If  $P$  does not contain  $q_{m-1}$  and  $q_m$ , then  $\tau(q'_{m-1})$  is contained in  $P$  and  $\tau(q_{m-1})$  is contained in  $P'$ . In this case the area of  $\tau(q'_{m-1})$  is smaller than the area of  $\tau(q_{m-1})$ . The same argument can be applied to  $\tau(q_{m-2})$  and  $\tau(q'_{m-2})$ . As a consequence, in all cases the area of  $P$  is smaller than the area of  $P'$ .  $\square$

We are now ready to prove the Claim 1.2.

**Theorem 3.3** *For every non-thick antipodal polygon on  $S$ , there exists a thick antipodal polygon on  $S$  of greater area.*

**Proof.** The idea is to transform any non-thick antipodal polygon into a thick antipodal polygon by using flipping transformations. Let  $Q$  be a non-thick antipodal polygon, then  $S$  contains three consecutive points as vertices of  $Q$  or  $S$  contains a sequence  $q_1, q_2, \dots, q_m$  ( $4 \leq m < n$ ) fulfilling  $Q$  contains  $q_1$  and  $q_2$ ,  $Q$  contains either both  $q_{m-1}$  and  $q_m$ , or neither of them and every second point from  $q_3$  to  $q_{m-1}$  belongs to  $Q$ . Using lemmas 3.1 and 3.2 properly we obtain an antipodal polygon with greater area than  $Q$ . Applying these operations successively until they can no longer be applied we obtain a thick polygon. Note that in the operations described in lemmas 3.1 and 3.2 the number of consecutive points in  $S$  as vertices of the non-thick antipodal polygon decreases.  $\square$

**Corollary 3.4** *For  $n$  odd, every thick antipodal polygon on  $S$  has greater area than a non-thick antipodal polygon on  $S$ .*

**Proof.** In this case there are only two antipodal thick polygons and they have the same area.  $\square$

We now provide an example of a set of points and a non-thick antipodal polygon that has greater area than a thick antipodal polygon on this set.

**Theorem 3.5** *For  $n \geq 6$  even, there exist point sets with a non-thick antipodal polygon of greater area than a thick antipodal polygon.*

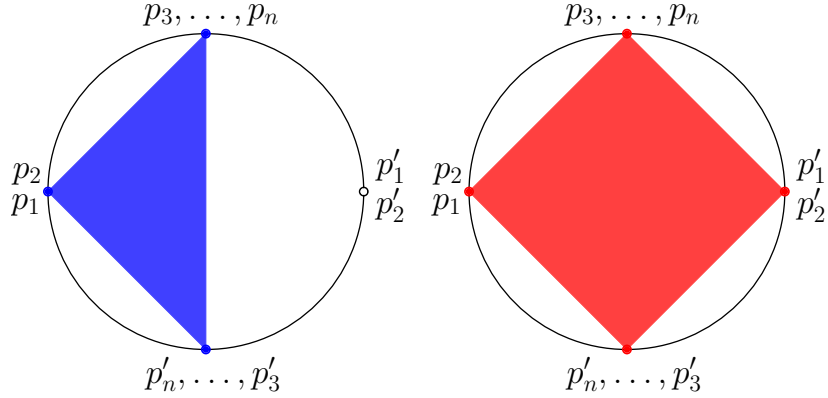


Figure 6: A counterexample.

**Proof.** Place  $p_1$  and  $p_2$  arbitrarily close to  $(1, 0)$ ; thus  $p'_1$  and  $p'_2$  are arbitrarily close to  $(-1, 0)$ . Place  $p_3, \dots, p_n$  arbitrarily close to  $(0, 1)$ ; thus  $p'_3, \dots, p'_n$  are arbitrarily close to  $(0, -1)$  as illustrated in Figure 6. Let  $P$  be the thick antipodal polygon that contains both  $p_1$  and  $p_2$  as vertices. Let  $Q$  be any non-thick antipodal polygon that contains  $p_1, p'_2, p_3$  and  $p'_4$  as vertices. Note that  $P$  is arbitrarily close to the triangle with vertices  $(0, 1)$ ,  $(0, -1)$  and  $(1, 0)$ ;  $Q$  is arbitrarily close to the quadrilateral with vertices  $(-1, 0)$ ,  $(0, 1)$ ,  $(1, 0)$ , and  $(0, -1)$ . Thus the area of  $P$  is arbitrarily close to 1, while the area of  $Q$  is arbitrarily close to 2. □

## 4 The algorithms

It is worth mentioning that the algorithmic version of the problem in which the input is a set of line segments, each connecting two points (non necessarily antipodal points) on the circle, has been proved to be NP-hard [9]. Surprisingly, the antipodal version can be easily solved by using the above characterizations.

**Theorem 4.1** *Antipodal polygons with minimum or maximum area can be found in linear time.*

**Proof.** According to Theorem 2.4, an antipodal polygon with minimum area is a thin antipodal polygon. Thus, since there exist  $O(n)$  thin antipodal polygons, we can sweep in a linear number of steps around the circle and update in constant time the area of two consecutive thin antipodal polygons. On the other hand, according to Theorem 3.3, if  $n$  is odd, there are only two thick antipodal polygons (the alternating polygons). For  $n$  even, there exists a linear number of thick antipodal polygons (having two consecutive points and the rest in alternating position). In the last case, a linear sweep around the circle can also be used to compute in linear time a thick antipodal polygon that maximizes the area. □

## 5 Higher Dimensions: Antipodal Polytopes

In this section we consider a generalization of the problem to higher dimensions. Assume therefore that all points are now placed on the unit  $d$ -dimensional sphere. Instead of antipodal polygons we

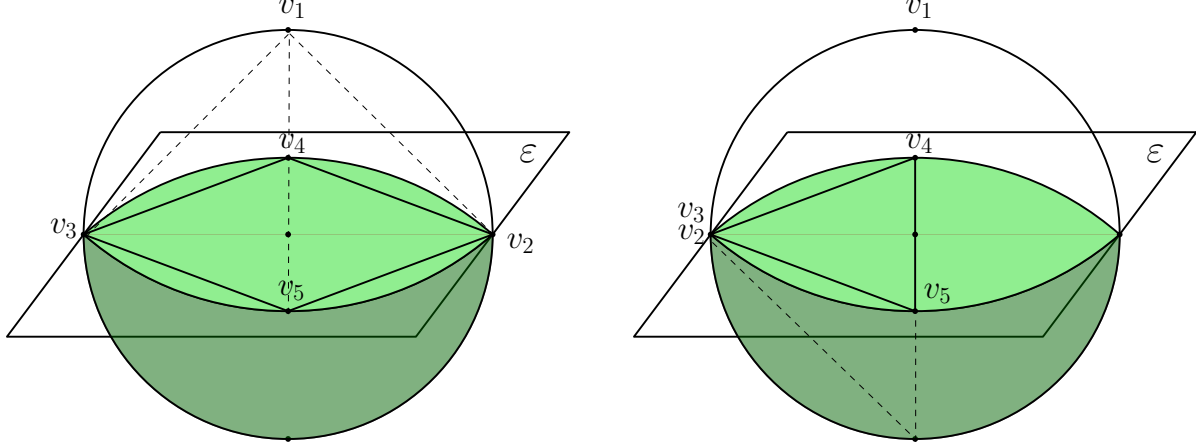


Figure 7: A three dimensional example proving that Theorem 2.4 does not hold.

thus have antipodal polytopes.

The natural generalization for higher dimensions is as follows. Let  $S := \{p_1, p'_1, p_2, p'_2, \dots, p_n, p'_n\}$  be a set of  $n$  antipodal pairs on the unit  $d$ -dimensional sphere centered at the origin. A *thin* antipodal polytope is an antipodal polytope whose vertices all lie on one side of some hyperplane passing through the origin. Note that this definition generalizes the thin antipodal polygon's notion.

In dimension 3 or greater Theorem 2.4 does not hold. There are antipodal point sets  $S \subset \mathbb{R}^d$  such that there exists an antipodal thin polytope with greater  $d$ -dimensional volume than a non-thin antipodal polytope on  $S$ . We start by providing a three dimensional example and then argue how to generalize it to higher dimensions.

For some small  $\varepsilon > 0$ , let  $\delta = \sqrt{1 - 2\varepsilon^2}$  and consider the set  $S_1$  of the five points  $v_1 := (0, 0, 1)$ ,  $v_2 := (\delta, \varepsilon, \varepsilon)$ ,  $v_3 := (-\delta, \varepsilon, \varepsilon)$ ,  $v_4 := (\varepsilon, \delta, \varepsilon)$ , and  $v_5 := (\varepsilon, -\delta, \varepsilon)$ . Let  $S$  be the antipodal point set consisting of  $S_1$  and all its antipodal points. The convex hull of  $S_1$  is a pyramid with a square base (with corners  $v_2, \dots, v_5$ ) which lies in the horizontal plane just  $\varepsilon$  above the origin. Refer to Figure 7 top. The top of the pyramid is at height 1. Thus, this pyramid does not contain the origin in its interior, and for  $\varepsilon \rightarrow 0$  the volume of the pyramid converges to  $2/3$ .

To obtain our second polyhedra first flip the vertex  $v_1$  to  $v'_1 := (0, 0, -1)$ . This gives a similar upside-down pyramid, which contains the origin in its interior. By also flipping  $v_2$  to  $v'_2 := (-\delta, -\varepsilon, -\varepsilon)$ , we essentially halve the base of the pyramid to be a triangle. We denote the resulting point set by  $S_2 = \{v'_1, v'_2, v_3, v_4, v_5\} \subset S$ . Note that  $v'_2$  and  $v_3$  are rather close together. As the triangle  $v_3, v_4, v_5$  lies above the origin, the convex hull of  $S_2$  still contains the origin in its interior. See Figure 7 bottom. Moreover, the volume of the convex hull of  $S_2$  converges to  $1/3$  for  $\varepsilon \rightarrow 0$ , and thus towards half of the volume of the convex hull of  $S_1$ .

So together these two polyhedra constitute an example which shows that Theorem 2.4 can not be generalized to higher dimensions:  $S$  is a set of five antipodal pairs of points on the surface of the 3-dimensional unit sphere such that the convex hull of  $S_1$  does not contain the origin, while the convex hull of  $S_2$  does. But in the limit the volume of the convex hull of  $S_1$  becomes twice as large as the volume of the convex hull of  $S_2$ .

It is straight forward to observe that this example can be generalized to any dimension  $d \geq 4$ . There we have  $2d - 1$  antipodal pairs of points, where we set  $\delta = \sqrt{1 - (d - 1)\varepsilon^2}$  and every point has one coordinate at  $\pm\delta$  and the remaining coordinates at  $\pm\varepsilon$ , analogous to the 3-dimensional case. For

$d-1$  of the coordinate axes two such pairs are 'aligned' as in the 3-dimensional example, and for the last axis there is only one such pair. The resulting polytope does not contain the origin. Flipping the vertex of the singular pair and one vertex for all but one aligned pairs results in a polytope which contains the origin, but has a volume of only  $1/2^{d-2}$  of the first polytope.

We call a  $d$ -dimensional antipodal polytope *thick* if the number of vertices in any half-space defined by a hyperplane through the origin contains at least  $\lceil \frac{n-d}{2} \rceil$  points of the polytope. Note that in the two dimensional case, a thick antipodal polygon satisfies that at least  $\lceil \frac{n-2}{2} \rceil$  of its vertices lie in both open half-planes defined by any given line through the origin<sup>1</sup>.

It is not clear that for a given antipodal set in  $\mathbb{R}^d$  an antipodal thick polytope should exist. However, for every  $n \geq d$ , there exist antipodal sets in  $\mathbb{R}^d$  that admit an antipodal thick polytope. We use the following Lemma.

**Lemma 5.1 (Gale's Lemma [10]).** *For every  $d \geq 0$  and every  $k \geq 1$ , there exists a set  $X \subset S^d$  of  $2k + d$  points such that every open hemisphere of  $S^d$  contains at least  $k$  points of  $X$ .*

From the proof of Gale's Lemma in [11] (page 64), it follows that the provided set does not contain an antipodal pair of points. Recall that  $S^{d-1} \subset \mathbb{R}^d$ ; let  $X$  be the subset of  $S^{d-1}$  provided by Gale's Lemma for  $k = \lceil \frac{n-d+1}{2} \rceil$ . If necessary remove a point from  $X$  so that  $X$  consists of exactly  $n$  points. Let  $X'$  be the set of antipodal points of  $X$ . Set  $S := X \cup X'$ . Let  $P$  be the antipodal polytope on  $S$  with  $X$  as a vertex set. It follows from Gale's Lemma that  $P$  is thick.

## 6 Open problems

Let us assume that we are given a circular lattice with an antipodal set of  $2n$  points (evenly spaced) and we would like to compute an extremal antipodal  $k$ -polygon with  $k < n$  vertices. This problem is significantly different to the considered case  $k = n$ . Recall that, for  $k = n$ , the linear algorithms proposed in this paper are strongly based on the simple characterization for the extremal antipodal polygons. Namely, the minimal thin antipodal polygon has consecutive vertices and the thick one has an alternating configuration. It is not difficult to come up with examples for which that characterization does not hold in the general case  $k < n$ . On the other hand, finding the extremal antipodal  $(n-1)$ -polygon, called  $(2n, n-1)$ -problem for short, can be easily reduced to solve  $O(n)$  times the  $(2(n-1), n-1)$ -problem. To see this, observe that in the  $(2n, n-1)$ -problem an antipodal pair is not selected and can thus be removed from the input. This approach gives a simple  $O(n^{n-k+1})$  time algorithm for solving the general  $(2n, k)$ -problem. This leaves as open problem to prove if the  $(2n, k)$ -problem can be solve in  $o(n^k)$  time.

Instead of area, it is also interesting to consider other extremal measures, like perimeter or the sum of inter-point distances. Finally, for higher dimensions, we leave the existence of thick polytopes for arbitrary antipodal point sets as an open problem. For future research it will be interesting to consider alternative definitions of thin and thick antipodal polytopes to extend the problem to higher dimensions.

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<sup>1</sup>This property is not "if and only if" because there exist non-thick polygons fulfilling the property.

## 7 Acknowledgments

The problems studied here were introduced and partially solved during a visit to the University of La Havana, Cuba. We thank the project COFLA: Computational analysis of the Flamenco music (FEDER P09-TIC-4840 and FEDER P12-TIC-1362) for posing us the basic problem studied in this paper.

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