Drawing Graphs as Spanners*

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Abstract. We study the problem of embedding graphs in the plane as good geometric spanners. That is, for a graph G, the goal is to construct a straight-line drawing Γ of G in the plane such that, for any two vertices u and v of G, the ratio between the minimum length of any path from u to v and the Euclidean distance between u and v is small. The maximum such ratio, over all pairs of vertices of G, is the *spanning ratio* of Γ.
First, we show that deciding whether a graph admits a straight-line drawing with spanning ratio 1, a proper straight-line drawing with spanning ratio 1, and a planar straight-line drawing with spanning ratio 1 are NP-complete, $\exists\mathbb{R}$-complete, and linear-time solvable problems, respectively. Second, we prove that, for every $\epsilon > 0$, every (planar) graph admits a proper (resp. planar) straight-line drawing with spanning ratio smaller than $1+\epsilon$. Third, we note that our drawings with spanning ratio smaller than $1+\epsilon$ have large edge-length ratio, that is, the ratio between the lengths of the longest and of the shortest edge is exponential. We show that this is sometimes unavoidable. More generally, we identify having bounded toughness as the criterion that distinguishes graphs that admit straight-line drawings with constant spanning ratio and polynomial edge-length ratio from graphs that do not.

1 Introduction

Let P be a set of points in the plane and let $\mathcal{G}$ be a geometric graph whose vertex set is P. We say that $\mathcal{G}$ is a *t-spanner* if, for every pair of points p and q in P, there exists a path from p to q in $\mathcal{G}$ whose total edge length is at most t times the Euclidean distance $\|pq\|$ between p and q. The *spanning ratio* of $\mathcal{G}$ is the smallest real number t such that $\mathcal{G}$ is a t-spanner. The problem of constructing, for a given set P of points in the plane, a sparse (and possibly planar) geometric graph whose vertex set is P and whose spanning ratio is small has received considerable attention; see, e.g., [11,12,13,14,17,20,22,38,49,50].

In this paper we look at the construction of geometric graphs with small spanning ratio from a different perspective. Namely, the problem we consider is whether it is possible to embed a given abstract graph in the plane as a geometric

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graph with small spanning ratio. That is, for a given graph, we want to construct a straight-line drawing with small spanning ratio, where the spanning ratio of a straight-line drawing is the maximum ratio, over all pairs of vertices u and v, between the total edge length of a shortest path from u to v and $\|uv\|$.

Graph embeddings in which every pair of vertices is connected by a path satisfying certain geometric properties have been the subject of intensive research. As a notorious example, a *greedy* drawing of a graph [5,7,18,23,30,35,39,41,42,47] is such that, for every pair of vertices u and v, there is a path from u to v that monotonically decreases the distance to v at every vertex. Further examples are *self-approaching* and *increasing-chord* drawings [3,19,40], *angle-monotone* drawings [9,19,36], *monotone* drawings [4,6,28,29,31,34] and *strongly-monotone* drawings [4,24,34]. While greedy, monotone, and strongly-monotone drawings might have unbounded spanning ratio, self-approaching, increasing-chord, and angle-monotone drawings are known to have spanning ratio at most 5.34 [32], at most 2.1 [43], and at most 1.42 [9], respectively. However, not all graphs, and not even all trees [35,39], admit such drawings.

Our results are the following.

First, we look at straight-line drawings with spanning ratio equal to 1, which is clearly the smallest attainable value by any graph. We prove that deciding whether a graph admits a straight-line drawing, a proper straight-line drawing (in which no vertex-vertex or vertex-edge overlaps are allowed), and a planar straight-line drawing with spanning ratio 1 are NP-complete, $\exists\mathbb{R}$-complete, and linear-time solvable problems, respectively.

Second, we show that allowing each shortest path to have a total edge length slightly larger than the Euclidean distance between its end-vertices makes it possible to draw all graphs. Namely, for every $\epsilon > 0$, every graph has a proper straight-line drawing with spanning ratio smaller than $1 + \epsilon$ and every planar graph has a planar straight-line drawing with spanning ratio smaller than $1 + \epsilon$.

Third, we address the issue that our drawings with spanning ratio smaller than $1 + \epsilon$ have poor resolution. That is, the *edge-length ratio* of these drawings, i.e., the ratio between the lengths of the longest and of the shortest edge, might be super-polynomial in the number of vertices of the graph. We show that this is sometimes unavoidable, as stars have exponential edge-length ratio in any straight-line drawing with constant spanning ratio. More in general, we present graph families for which any straight-line drawing with constant spanning ratio has edge-length ratio which is exponential in the inverse of the toughness. On the other hand, we prove that graph families with constant toughness admit proper straight-line drawings with polynomial edge-length ratio and constant spanning ratio. Finally, we prove that bounded-degree trees admit planar straight-line drawings with polynomial edge-length ratio and constant spanning ratio.

Full versions of sketched or omitted proofs can be found in the Appendix.

2 Preliminaries

For a graph G and a set S of vertices of G, we denote by $G - S$ the graph obtained from G by removing the vertices in S and their incident edges. The subgraph of G *induced by* S is the graph whose vertex set is S and whose edge set consists of every edge of G that has both its end-vertices in S. The *toughness* of a graph G is the largest real number $t > 0$ such that, for any set S such that $G - S$ consists of $k \geq 2$ connected components, we have $|S| \geq t \cdot k$.

A *drawing* of a graph maps each vertex to a point in the plane and each edge to a Jordan arc between its end-vertices. A drawing is *straight-line* if it maps each edge to a straight-line segment. Let Γ be a straight-line drawing of a graph G. The *length of a path* in Γ is the sum of the lengths of its edges. We denote by $\|uv\|_\Gamma$ (by $\pi_\Gamma(u,v)$) the Euclidean distance (resp. the length of a shortest path) between two vertices u and v in Γ; we sometimes drop the subscript Γ when the drawing we refer to is clear from the context. The *spanning ratio* of Γ is the real value $\max\limits_{u,v} \frac{\pi_\Gamma(u,v)}{\|uv\|_\Gamma}$, where the maximum is over all pairs of vertices u and v of G.

A drawing is *planar* if no two edges intersect, except at common end-vertices. A planar drawing partitions the plane into connected regions, called *faces*; the bounded faces are *internal*, while the unbounded face is the *outer face*. A graph is *planar* if it admits a planar drawing. A planar graph is *maximal* if adding any edge to it violates its planarity. In any planar drawing of a maximal planar graph every face is delimited by a 3-cycle. The *bounding box* $\mathcal{B}(\Gamma)$ of a drawing Γ is the smallest axis-parallel rectangle containing Γ in the closure of its interior. The *width* and *height* of Γ are the width and height of $\mathcal{B}(\Gamma)$.

3 Drawings with Spanning Ratio 1

In this section we study drawings with spanning ratio equal to 1.

Theorem 1. *Recognizing whether a graph admits a straight-line drawing with spanning ratio equal to* 1 *is an NP-complete problem.*

Proof sketch: The core of the proof consists of showing that a graph has a straight-line drawing with spanning ratio 1 if and only if it contains a Hamiltonian path (then the theorem follows from the NP-completeness of the problem of deciding whether a graph contains a Hamiltonian path [26,27]). In particular, let Γ be a straight-line drawing with spanning ratio 1 of a graph G and assume w.l.o.g. that no two vertices have the same x-coordinate in Γ. Let $v_1, v_2, \dots, v_n$ be the vertices of G, ordered by increasing x-coordinates. Then, for $i = 1, 2, \dots, n-1$, we have that G contains the edge $v_i v_{i+1}$, as any other path between v_i and v_{i+1} would be longer than $\|v_i v_{i+1}\|$. Hence, G contains the Hamiltonian path $(v_1, v_2, \dots, v_n)$. □

The *existential theory of the reals* problem asks whether real values exist for n variables such that a quantifier-free formula, consisting of polynomial equalities and inequalities on such variables, is satisfied. The class of problems that are complete for the existential theory of the reals is denoted by $\exists\mathbb{R}$ [44]. It is known that NP $\subseteq \exists\mathbb{R} \subseteq$ PSPACE [15], however it is not known whether $\exists\mathbb{R} \subseteq$ NP. Many geometric problems are $\exists\mathbb{R}$-complete, see, e.g., [1,37].

Theorem 2. *Recognizing whether a graph admits a proper straight-line drawing with spanning ratio equal to* 1 *is an* $\exists\mathbb{R}$*-complete problem.*

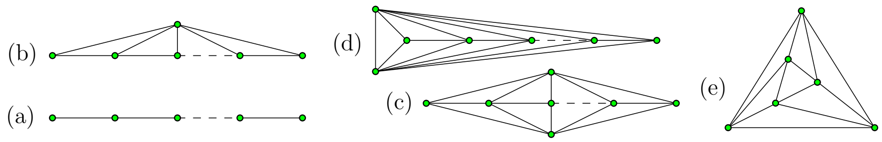

Fig. 1: The five graph classes defined in [21].

Proof sketch: Let Γ be a proper straight-line drawing with spanning ratio 1 of a graph G. Let S be the set of points at which the vertices of G are drawn. It is easy to prove that the *point visibility graph* G_S of S is isomorphic to G, where the point visibility graph G_P of a point set $P \subset \mathbb{R}^2$ has a vertex for each point $p \in P$ and has an edge between two vertices if and only if the straight-line segment between the corresponding points does not contain any point of P in its interior. The theorem follows from the fact that recognizing point visibility graphs is a problem that is $\exists\mathbb{R}$-complete [16]. $\square$

Theorem 3. *Recognizing whether a graph admits a planar straight-line drawing with spanning ratio equal to* 1 *is a linear-time solvable problem.*

Proof: Dujmović et al. [21] characterized the graphs that admit a planar straight-line drawing with a straight-line segment between every two vertices as the graphs in the five graph classes in Figure 1. Since a straight-line drawing has spanning ratio 1 if and only if every two vertices are connected by a straight-line segment, the theorem follows from the fact that recognizing whether a graph belongs to such five graph classes can be easily done in linear time. $\square$

4 Drawings with Spanning Ratio $1 + \epsilon$

In this section we study straight-line drawings with spanning ratio arbitrarily close to 1. Most of the section is devoted to a proof of the following result.

Theorem 4. *For every $\epsilon > 0$, every connected planar graph admits a planar straight-line drawing with spanning ratio smaller than $1 + \epsilon$.*

Let G be an n-vertex maximal planar graph with $n \geq 3$, let $\mathcal{G}$ be a planar drawing of G, and let (u, v, z) be the cycle delimiting the outer face of G in $\mathcal{G}$. A *canonical ordering* [8,25,33] for G is a total ordering $[v_1, \dots, v_n]$ of its vertex set such that the following hold for $k = 3, \dots, n$: (i) $v_1 = u$, $v_2 = v$, and $v_n = z$; (ii) the subgraph G_k of G induced by $v_1, \dots, v_k$ is 2-connected and the cycle $\mathcal{C}_k$ delimiting its outer face in $\mathcal{G}$ consists of the edge v_1v_2 and of a path $\mathcal{P}_k$ between v_1 and v_2; and (iii) v_k is incident to the outer face of G_k in $\mathcal{G}$. Theorem 4 is implied by the following two lemmata.

Lemma 1. *Let H be an n-vertex connected planar graph. There exist an n-vertex maximal planar graph G and a canonical ordering $[v_1, \dots, v_n]$ for G such that, for each $k \in \{1, \dots, n\}$, the subgraph H_k of H induced by $\{v_1, \dots, v_k\}$ is connected.*

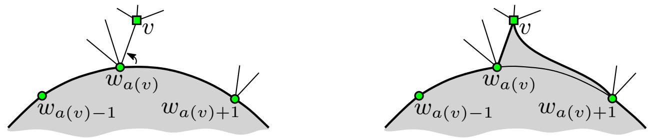

Fig. 2: Construction for the case in which $a(v) = b(v)$.

Proof sketch: For each $k = 2, \ldots, n$, we let G_k be the subgraph of G induced by $v_1, \ldots, v_k$ and L_k be the graph composed of G_k and of the vertices and edges of H that are not in G_k. Further, we define $v_1, \ldots, v_k$ and G_k so that H_k is connected, G_k is 2-connected, and L_k admits a planar drawing $\mathcal{L}_k$ such that:

1. the outer face of the planar drawing $\mathcal{G}_k$ of G_k in $\mathcal{L}_k$ is delimited by a cycle $\mathcal{C}_k$ composed of the edge v_1v_2 and of a path $\mathcal{P}_k$ between v_1 and v_2;
2. v_k is incident to the outer face of $\mathcal{G}_k$;
3. every internal face of $\mathcal{G}_k$ is delimited by a 3-cycle; and
4. the vertices and edges of H that are not in G_k lie in the outer face of $\mathcal{G}_k$.

If $k = 2$, then construct any planar drawing $\mathcal{L}_2$ of H and define v_1 and v_2 as the end-vertices of any edge v_1v_2 incident to the outer face of $\mathcal{L}_2$. Properties 1–4 are then trivially satisfied (in this case the path $\mathcal{P}_2$ is the single edge v_1v_2).

If $2 < k < n$, assume that $v_1, \ldots, v_{k-1}$ and G_{k-1} have been defined so that H_{k-1} is connected, G_{k-1} is 2-connected, and L_{k-1} admits a planar drawing $\mathcal{L}_{k-1}$ satisfying Properties 1–4. Let $\mathcal{P}_{k-1} = (u = w_1, w_2, \ldots, w_x = v)$, where $x \geq 2$.

Consider any vertex v in $L_{k-1} \setminus G_{k-1}$. By Properties 1 and 4 of $\mathcal{L}_{k-1}$, all the neighbors of v in G_{k-1} lie in $\mathcal{P}_{k-1}$. We say that v is a *candidate* (to be designated as v_k) *vertex* if, for some $1 \leq i \leq x$, there exists an edge w_iv such that w_iv immediately follows the edge w_iw_{i-1} in clockwise order around w_i or immediately follows the edge w_iw_{i+1} in counter-clockwise order around w_i.

For each candidate vertex v, let $w_{a(v)}$ and $w_{b(v)}$ be the neighbors of v in $\mathcal{P}_{k-1}$ such that $a(v)$ is minimum and $b(v)$ is maximum (possibly $a(v) = b(v)$). If $a(v) < b(v)$, let the *reference cycle* $\mathcal{C}(v)$ *of* v be composed of the edges $w_{a(v)}v$ and $w_{b(v)}v$ and of the subpath of $\mathcal{P}_{k-1}$ between $w_{a(v)}$ and $w_{b(v)}$. Define the *depth* of v as 0 if $a(v) = b(v)$ or as the number of candidate vertices that lie inside $\mathcal{C}(v)$ in $\mathcal{L}_{k-1}$ otherwise. We select as $v_k := v$ any candidate vertex v with depth 0.

If $a(v) = b(v)$, as in Figure 2, assume that $w_{a(v)}v$ immediately follows the edge $w_{a(v)}w_{a(v)+1}$ in counter-clockwise order around $w_{a(v)}$; the other case is symmetric. Define G_k as G_{k-1} plus the vertex v and the edges $w_{a(v)}v$ and $w_{a(v)+1}v$. Further, construct $\mathcal{L}_k$ by drawing the edge $w_{a(v)+1}v$ so that the cycle $(w_{a(v)}, w_{a(v)+1}, v)$ does not contain any vertex or edge in its interior.

If $a(v) < b(v)$, as in Figure 3, redraw each biconnected component of L_{k-1} whose vertices different from v lie inside $\mathcal{C}(v)$ planarly so that it now lies outside $\mathcal{C}(v)$; after this modification, no vertex of L_{k-1} lies inside $\mathcal{C}(v)$. Then define G_k as G_{k-1} plus the vertex v and the edges $w_{a(v)}v, w_{a(v)+1}v, \ldots, w_{b(v)}v$. Further, construct $\mathcal{L}_k$ by drawing the edges among $w_{a(v)}v, w_{a(v)+1}v, \ldots, w_{b(v)}v$ not in H so that they all lie inside $\mathcal{C}(v)$ and so that the edges $w_{a(v)}v, w_{a(v)+1}v, \ldots, w_{b(v)}v$ appear consecutively and in this counter-clockwise order around v.

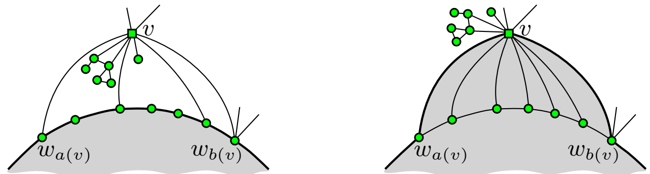

Fig. 3: Construction for the case in which $a(v) < b(v)$.

In both cases, it is easy to see that H_k is connected, G_k is 2-connected, and $\mathcal{L}_k$ satisfies Properties 1–4. See the Appendix for details.

If $k = n$, the construction is similar to the one described for the case $2 < k < n$, however further edges are added to G_n so to ensure that the outer face of $\mathcal{G}_n$ is delimited by the 3-cycle (v_1, v_2, v_n). Setting $G := G_n$ concludes the proof. $\square$

Lemma 2. *For every $k = 3, \dots, n$ and for every $\epsilon > 0$, there exists a planar straight-line drawing $\varGamma_k$ of G_k such that: (1) the outer face of $\varGamma_k$ is delimited by the cycle $\mathcal{C}_k$; further, the path $\mathcal{P}_k$ is x-monotone and lies above the edge uv, except at u and v; and (2) the restriction $\varXi_k$ of $\varGamma_k$ to the vertices and edges of H_k is a drawing with spanning ratio smaller than $1 + \epsilon$.*

Proof sketch: The proof is by induction on k. The base case $k = 3$ is trivial. Assume that, for some $k = 4, \dots, n$, a planar straight-line drawing $\varGamma_{k-1}$ of G_{k-1} has been constructed satisfying Properties 1 and 2; see Figure 4. Let δ be the diameter of a disk D enclosing $\varGamma_{k-1}$. We construct $\varGamma_k$ from $\varGamma_{k-1}$ as follows. Let $\mathcal{P}_{k-1} = (u = w_1, w_2, \dots, w_x = v)$. As proved in [25], the neighbors of v_k in G_{k-1} are the vertices in a sub-path $(w_p, \dots, w_q)$ of $\mathcal{P}_{k-1}$, where $p < q$. By Property 1 of $\varGamma_{k-1}$, we have $x(w_p) < x(w_q)$. We then place v_k at any point in the plane satisfying the following conditions: (i) $x(w_p) < x(v_k) < x(w_q)$; (ii) for every $i = p, \dots, q-1$, the y-coordinate of v_k is larger than those of the intersection points between the line through $w_i w_{i+1}$ and the vertical lines through w_p and w_q; and (iii) the distance between v_k and the point of D closest to v_k is a real value $d > \frac{k\delta}{\epsilon}$.

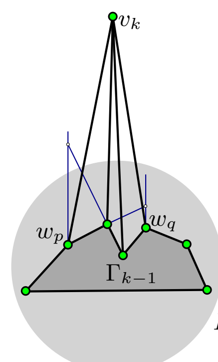

Fig. 4: Construction of $\varGamma_k$ from $\varGamma_{k-1}$.

Since $\mathcal{P}_k$ is obtained from $\mathcal{P}_{k-1}$ by substituting the path $(w_p, w_{p+1}, \dots, w_q)$ with the path (w_p, v_k, w_q), Condition (i) and the x-monotonicity of $\mathcal{P}_{k-1}$ imply that $\mathcal{P}_k$ is x-monotone. Condition (ii), the x-monotonicity of $\mathcal{P}_{k-1}$, and the planarity of $\varGamma_{k-1}$ imply that $\varGamma_k$ is planar. We now prove that the spanning ratio of $\varXi_k$ is smaller than $1 + \epsilon$. Consider any two vertices v_i and v_j. If $i < k$ and $j < k$, then $\frac{\pi_{\varXi_k}(v_i, v_j)}{\|v_i v_j\|_{\varXi_k}} \leq \frac{\pi_{\varXi_{k-1}}(v_i, v_j)}{\|v_i v_j\|_{\varXi_{k-1}}} < 1 + \epsilon$. If $i = k$, then $\|v_k v_j\|_{\varXi_k} \geq d$, by Condition (iii). Consider the path $P(v_k, v_j)$ composed of any edge $v_k v_\ell$ in H_k incident to v_k (which exists since H_k is connected) and of any path in H_{k-1} between v_ℓ and v_j (which exists since H_{k-1} is connected). The length of $P(v_k, v_j)$ is at most $d + \delta$ (by Condition (iii) and by the triangular inequality, this is an upper bound on $\|v_k v_\ell\|_{\Xi_k}$) plus $(k-2)\cdot\delta$ (this is an upper bound on the length of any path in H_{k-1}). Hence, $\frac{\pi_{\Xi_k}(v_k,v_j)}{\|v_k v_j\|_{\Xi_k}} < \frac{d+k\delta}{d} < 1+\epsilon$. This completes the induction and the proof of the lemma. $\square$

Lemmata 1 and 2 imply Theorem 4. Namely, for a connected planar graph H, by Lemma 1 we can construct a maximal planar graph G that, by Lemma 2 (with $k=n$) and for every $\epsilon > 0$, admits a planar straight-line drawing whose restriction to H is a drawing with spanning ratio smaller than $1+\epsilon$.

The following can be obtained by means of techniques similar to (and simpler than) the ones in the proof of Theorem 4; the proof is presented in the Appendix.

Theorem 5. *For every $\epsilon > 0$, every connected graph admits a proper straight-line drawing with spanning ratio smaller than $1+\epsilon$.*

5 Drawings with Small Spanning and Edge-Length Ratios

In this section we study straight-line drawings with small spanning ratio and edge-length ratio. Our main result is the following.

Theorem 6. *For every $\epsilon > 0$ and every $\tau > 0$, every n-vertex graph with toughness τ admits a proper straight-line drawing whose spanning ratio is at most $1+\epsilon$ and whose edge-length ratio is in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon\rceil)}{\log_2(2+\lceil 1/\tau\rceil)-\log_2(1+\lceil 1/\tau\rceil)}}\cdot 1/\epsilon\right)$. Further, for every $0<\tau<1$, there is a graph G with toughness τ whose every straight-line drawing with spanning ratio at most s has edge-length ratio in $2^{\Omega(1/(\tau\cdot s^2))}$.*

In order to prove Theorem 6, we study straight-line drawings of bounded-degree trees. This is because there is a strong connection between the toughness of a graph and the existence of a spanning tree with bounded degree. Indeed, if a graph G has toughness τ, then it has a spanning tree with maximum degree $\lceil 1/\tau\rceil+2$ [48]. Further, a tree has toughness equal to the inverse of its maximum degree. We start by proving the following lower bound.

Theorem 7. *For any $s\geq 1$, any straight-line drawing with spanning ratio at most s of a tree with a vertex of degree d has edge-length ratio in $2^{\Omega(d/s^2)}$.*

Proof: For any $s\geq 1$, let Γ be any straight-line drawing of T with spanning ratio at most s; refer to Figure 5(a). Let u_T be a vertex of degree d. Assume w.l.o.g. up to a scaling (which does not alter the edge-length ratio and the spanning ratio of Γ) that the length of the shortest edge incident to u_T in Γ is 1. For any integer $i\geq 0$, let $\mathcal{C}_i$ be the circle centered at u_T whose radius is $r_i = 2^i$. Further, for any integer $i>0$, let $\mathcal{A}_i$ be the closed annulus delimited by $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$. By assumption, no neighbor of u_T lies inside the open disk delimited by $\mathcal{C}_0$. We claim that, for any integer $i>0$ and for some constant c, there are at most $c\cdot s^2$ neighbors of u_T inside $\mathcal{A}_i$. This implies that at most $k\cdot c\cdot s^2$ neighbors of u_T lie inside the closed disk delimited by $\mathcal{C}_k$. Hence, if $d > k\cdot c\cdot s^2$, e.g., if $k=\lfloor\frac{d-1}{c\cdot s^2}\rfloor$, then there is a neighbor v_T of u_T outside $\mathcal{C}_k$. Then $\|u_T v_T\| > 2^k \in 2^{\Omega(d/s^2)}$. Hence, the theorem follows from the claim.

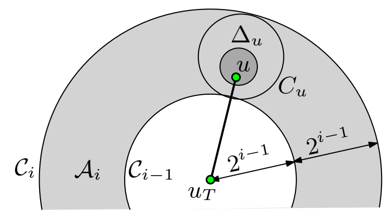

Fig. 5: Illustration for the proof of Theorem 7.

It remains to prove the claim. For each neighbor u of u_T inside $\mathcal{A}_i$, let Δ_u be a closed disk such that: (i) u lies inside Δ_u; (ii) Δ_u lies inside $\mathcal{A}_i$; and (iii) the diameter of Δ_u is $\delta_i = 2^{i-2}/s$. The existence of Δ_u can be proved as follows. Consider the circle C_u whose antipodal points are the intersection points of $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ with the ray from u_T through u. Note that C_u lies inside $\mathcal{A}_i$ and has diameter $2^{i-1} > \delta_i = 2^{i-2}/s$. Then Δ_u is any disk with diameter δ_i that contains u and that lies inside the closed disk delimited by C_u.

Suppose, for a contradiction, that there exist two neighbors u and v of u_T inside $\mathcal{A}_i$ such that the disks Δ_u and Δ_v intersect. Then $\pi_\Gamma(u,v) \geq 2^i$, since both the edges uu_T and vu_T are longer than $r_{i-1} = 2^{i-1}$. By the triangular inequality, $||uv||_\Gamma \leq 2 \cdot \delta_i = 2^{i-1}/s$. Hence $\frac{\pi_\Gamma(u,v)}{||uv||_\Gamma} \geq 2s$, while the spanning ratio of Γ is at most s. This contradiction proves that, for any two neighbors u and v of u_T inside $\mathcal{A}_i$, the disks Δ_u and Δ_v do not intersect. The area of $\mathcal{A}_i$ is $\pi \cdot (r_i^2 - r_{i-1}^2) = \pi \cdot (2^{2i} - 2^{2i-2}) = 3\pi \cdot (2^{2i-2})$. Since each disk Δ_u lying inside $\mathcal{A}_i$ has area $\pi \cdot (2^{2i-6}/s^2)$ and does not intersect any different disk Δ_v, it follows that $\mathcal{A}_i$ contains at most $\frac{3\pi\cdot(2^{2i-2})\cdot s^2}{\pi\cdot 2^{2i-6}} = 48 \cdot s^2$ distinct disks Δ_u and hence at most $48 \cdot s^2$ neighbors of u_T. This proves the claim and hence the theorem. $\square$

Corollary 1. *Let S be an n-vertex star. For any $s \geq 1$, any straight-line drawing of S with spanning ratio at most s has edge-length ratio in $2^{\Omega(n/s^2)}$.*

The lower bound of Theorem 6 follows from Theorem 7 and from the fact that a tree with maximum degree d has toughness $1/d$. On the other hand, the upper bound of Theorem 6 is obtained by means of the following.

Theorem 8. *For every $\epsilon > 0$, every n-vertex tree T with maximum degree d admits a proper straight-line drawing such that no three vertices are collinear, the spanning ratio is at most $1+\epsilon$, the distance between any two vertices is at least 1, and the width, the height, and the edge-length ratio are in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon \rceil)}{\log_2(d/(d-1))}} \cdot 1/\epsilon\right)$.*

Theorem 8 proves the upper bound in Theorem 6 and hence concludes its proof. Namely, let G be an n-vertex graph with toughness τ and let $\epsilon > 0$; then G has a spanning tree T with maximum degree $d = \lceil 1/\tau \rceil + 2$ [48]. Apply Theorem 8 to construct a straight-line drawing Γ_T of T. Construct a straight-line drawing Γ_G of G from Γ_T by drawing the edges of G not in T as straight-line segments. Then Γ_G is proper, as no three vertices are collinear in Γ_T. Further, the spanning ratio of Γ_G is at most the one of Γ_T, hence it is at most $1+\epsilon$. Finally, the edge-length ratio of Γ_G is in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon \rceil)}{\log_2(d/(d-1))}} \cdot 1/\epsilon\right)$, given that the distance between any two vertices in Γ_T (and hence in Γ_G) is at least 1 and given that the width and height of Γ_T (and hence of Γ_G) are in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon \rceil)}{\log_2(d/(d-1))}} \cdot 1/\epsilon\right)$.

We defer the proof of Theorem 8 to the Appendix and present a proof of Theorem 9, in which it is shown that trees with bounded maximum degree even admit *planar* straight-line drawings with constant spanning ratio and polynomial edge-length ratio. The cost of planarity is found in the dependence on the maximum degree, which is worse than in Theorem 8.

Theorem 9. *For every $\epsilon > 0$, every n-vertex tree T with maximum degree d admits a planar straight-line drawing whose spanning ratio is at most $1+\epsilon$ and whose edge-length ratio is in $\mathcal{O}\left((2n)^{2+(d-2)\cdot\log_2(1+\lceil \frac{2}{\epsilon} \rceil)} \cdot \log_2 n\right)$.*

Proof sketch: Let $\gamma = \lceil \frac{2}{\epsilon} \rceil$. If $d \leq 2$, then T is a path and the desired drawing is trivially constructed. We can hence assume that $d \geq 3$. Root T at any leaf r; this ensures that every vertex of T has at most $d-1$ children. In order to avoid some technicalities in the upcoming algorithm, we also assume that every non-leaf vertex of T has at least two children. This is obtained by inserting a new child for each vertex of T with just one child; note that the *size* of the tree, i.e., its number of vertices, is less than doubled by this modification. We again call T the tree after this modification and by n its size.

Our construction is a "well-spaced" version of an algorithm by Shiloach [46]. We construct a planar straight-line drawing Γ of T in which (i) r is at the top-left corner of $\mathcal{B}(\Gamma)$, and (ii) for every vertex u of T, the path from u to r in T is (non-strictly) xy-monotone.

If $n = 1$, then Γ is obtained by placing r at any point in the plane. If $n > 1$, then let $r_1, \dots, r_k$ be the children of r, where $k \leq d-1$, let $T_1, \dots, T_k$ be the subtrees of T rooted at $r_1, \dots, r_k$, whose sizes are $n_1, \dots, n_k$, respectively. Assume, w.l.o.g. up to a relabeling, that $n_1 \leq \dots \leq n_k$; hence, $n_i \leq n/2$ for $i = 1, 2, \dots, k-1$. Refer to Figure 6. Place r at any point in the plane. Inductively construct planar straight-line drawings $\Gamma_1, \dots, \Gamma_k$ of $T_1, \dots, T_k$, respectively. Position Γ_1 so that r_1 is on the same vertical line as r, one unit below it; let d_1 be the width of Γ_1. Then, for $i = 2, \dots, k$, position Γ_i so that r_i is one unit below r and $\gamma \cdot (d_{i-1} + \log_2 n)$ units to the right of the right side of $\mathcal{B}(\Gamma_{i-1})$; denote by d_i the width of the bounding box of the drawings $\Gamma_1, \dots, \Gamma_i$. Finally, move Γ_k one unit above, so that r_k is on the same horizontal line as r.

We now analyze the properties of Γ. By construction Γ is a straight-line drawing. The planarity of Γ is easily proved by exploiting the fact that r_i is at the top-left corner of $\mathcal{B}(\Gamma_i)$ and that $r_1, r_2, \dots, r_{k-1}$ all lie one unit below r.

Height. Let $h(n)$ be the maximum height of a drawing of an n-vertex tree constructed by the algorithm. The same analysis as in [46] shows that $h(n) \leq \log_2 n$, given that $h(1) = 0$ and $h(n) \leq \max\{h(\frac{n}{2}) + 1, h(n-1)\}$ for $n \geq 2$.

Spanning ratio. We prove that, for any two vertices u and v that do not belong to the same subtree T_i, it holds true that $\frac{\pi_\Gamma(u,v)}{\|uv\|_\Gamma} \leq \frac{\gamma+2}{\gamma}$. This suffices

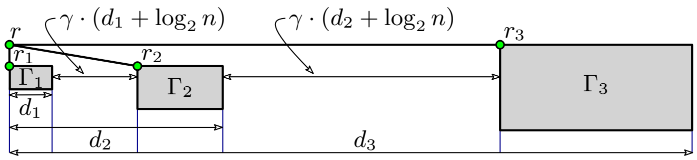

Fig. 6: Inductive construction of Γ. In this example $k = 3$.

to prove that the spanning ratio of Γ is at most $\frac{\gamma+2}{\gamma}$. Suppose w.l.o.g. that u belongs to a subtree T_i and v belongs to a subtree T_j, with $i < j$.

First, we have $\|uv\| \geq x_v + \gamma \cdot (d_{j-1} + \log_2 n)$, where x_v denotes the distance between v and the left side of $\mathcal{B}(\Gamma_j)$, while the second term is the distance between the left side of $\mathcal{B}(\Gamma_j)$ and the right side of $\mathcal{B}(\Gamma_{j-1})$.

Clearly, we have $\pi_\Gamma(u, v) = \pi_\Gamma(u, r) + \pi_\Gamma(r, v)$. The path between u and r (between v and r) is xy-monotone, hence $\pi_\Gamma(u, r)$ (resp. $\pi_\Gamma(v, r)$) is upper bounded by the horizontal distance plus the vertical distance between u and r (resp. between v and r). The vertical distance between u and r (between v and r) is at most $\log_2(n)$, since the height of Γ is at most $\log_2(n)$. The horizontal distance between u and r is at most $d_i \leq d_{j-1}$, while the one between v and r is $x_v + \gamma \cdot (d_{j-1} + \log_2 n) + d_{j-1}$. Hence, $\pi_\Gamma(u, v) \leq (d_{j-1} + \log_2 n) + (x_v + \gamma \cdot (d_{j-1} + \log_2 n) + d_{j-1} + \log_2 n) = x_v + (\gamma + 2) \cdot (d_{j-1} + \log_2 n)$. Thus:

$$\frac{\pi_\Gamma(u, v)}{\|uv\|_\Gamma} \leq \frac{(\gamma + 2) \cdot (\frac{x_v}{\gamma} + d_{j-1} + \log_2 n)}{\gamma \cdot (\frac{x_v}{\gamma} + d_{j-1} + \log_2 n)} \leq \frac{\gamma + 2}{\gamma} \leq 1 + \epsilon.$$

Width. Let $w_1, \dots, w_k$ be the widths of $\Gamma_1, \dots, \Gamma_k$. By construction, $d_1 = w_1$ and, for each $j = 2, \dots, k$, we have $d_j = d_{j-1} + \gamma \cdot (d_{j-1} + \log_2 n) + w_j = (\gamma + 1) \cdot d_{j-1} + \gamma \cdot \log_2 n + w_j$. Hence, by induction on j, we have $d_j = (\gamma + 1)^{j-1} \cdot w_1 + (\gamma + 1)^{j-2} \cdot w_2 + \ldots + (\gamma + 1) \cdot w_{j-1} + w_j + ((\gamma + 1)^{j-1} - 1) \cdot \log_2 n$. In particular, the width of Γ is equal to d_k and hence to:

$$\sum_{i=1}^{k} ((\gamma + 1)^{k-i} \cdot w_i) + ((\gamma + 1)^{k-1} - 1) \cdot \log_2 n. \tag{1}$$

Let $w(n)$ be the maximum width of a drawing of an n-vertex tree constructed by the algorithm. By construction $w(1) = 0$. For $n \geq 2$, by Equality 1, we get:

$$w(n) \leq (\gamma + 1)^{d-2} \cdot \sum_{i=1}^{k-1} w(n_1) + w(n_k) + (\gamma + 1)^{d-2} \cdot \log_2 n. \tag{2}$$

Recall that $n_1, \dots, n_{k-1} \leq n/2$. On the other hand, n_k might be larger than $n/2$; if that is so, Inequality 2 is used to replace the term $w(n_k)$ into Inequality 2 itself. The repetition of this substitution eventually results in the following (see the Appendix for details):

$$w(n) \leq (\gamma + 1)^{d-2} \cdot \sum_{i,j} w(n_{i,j}) + (\gamma + 1)^{d-2} \cdot (n - 1) \cdot \log_2 n, \tag{3}$$

where the terms $n_{i,j}$ denote the sizes of distinct subtrees of T (hence $\sum n_{i,j} \leq n-1$), each of which has at most $n/2$ nodes (hence $n_{i,j} \leq n/2$).

We prove, by induction on n, that $w(n) \leq f(n) := \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot n^2 \cdot \log_2 n$. This is trivial when $n = 1$, given that $w(1) = 0$. Assume now that $n > 1$. By Inequality 3 and by induction, we get $w(n) \leq (\gamma+1)^{d-2} \cdot \sum_{i,j} \left(\left((\gamma+1)^{d-2}\right)^{\log_2 n_{i,j}} \cdot n_{i,j}^2 \cdot \log_2 n_{i,j}\right) + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n$. Since $n_{i,j} \leq n/2 < n$, we get $w(n) \leq (\gamma+1)^{d-2} \cdot \left((\gamma+1)^{d-2}\right)^{\log_2 (n/2)} \cdot \sum_{i,j} n_{i,j}^2 \cdot \log_2 n + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n = \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot \sum_{i,j} n_{i,j}^2 \cdot \log_2 n + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n$. Since $\sum n_{i,j} \leq n-1$, we have $\sum_{i,j} n_{i,j}^2 \leq (n-1)^2$. Thus, $w(n) \leq \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot ((n-1)^2 + (n-1)) \cdot \log_2 n \leq \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot n^2 \cdot \log_2 n$. This completes the induction and the analysis of the width of Γ.

Edge-length ratio. By construction, the length of each edge connecting r to a child is larger than or equal to 1, hence the same is true for every edge of T. Thus, the edge-length ratio of Γ is upper bounded by the maximum length of an edge of T. In turn, this is at most the sum of the height plus the width of Γ, which is in $\mathcal{O}\left(\left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot n^2 \cdot \log_2 n\right)$, as proved above. The factor $\left((\gamma+1)^{d-2}\right)^{\log_2 n}$ can be rewritten as $n^{(d-2)\cdot\log_2(\gamma+1)}$. The bound claimed in the statement is then obtained by substituting $\gamma = \lceil \frac{2}{\epsilon} \rceil$ and by observing that the value of n used in the calculations is at most twice the size of the initial tree. $\square$

6 Open Problems

Our research raises a number of open problems which might be worth studying.

First, it would be interesting to tighten the bounds in Theorem 6 relating the toughness to the edge-length ratio of a drawing with constant spanning ratio.

Second, there is still much to be understood about the edge-length ratio of planar straight-line drawings with constant spanning ratio. Theorem 9 shows that planar straight-line drawings with constant spanning ratio and polynomial edge-length ratio exist for bounded-degree trees. We also observe that every n-vertex 2-connected outerplanar graph G admits a planar straight-line drawing with spanning ratio at most $\sqrt{2}$ and edge-length ratio in $\mathcal{O}(n^{1.5})$; this can be achieved by placing the vertices of G, in the order given by the Hamiltonian cycle of G, at the vertices of a lattice xy-monotone polygonal curve; see, e.g., [2]. Further, Schnyder drawings are known to be 2-spanners [12]. Hence, n-vertex 3-connected planar graphs admit planar straight-line drawings with spanning ratio at most 2 and edge-length ratio in $\mathcal{O}(n)$ [10,45]; do they admit planar straight-line drawings with spanning ratio smaller than 2 (and possibly arbitrarily close to 1) and polynomial edge-length ratio? Can Theorem 6 be extended to prove that a planar straight-line drawing with constant spanning ratio and polynomial edge-length ratio exists for planar graphs with bounded toughness?

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Appendix: Full Version of "Drawing Graphs as Spanners"*

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Abstract. We study the problem of embedding graphs in the plane as good geometric spanners. That is, for a graph G, the goal is to construct a straight-line drawing Γ of G in the plane such that, for any two vertices u and v of G, the ratio between the minimum length of any path from u to v and the Euclidean distance between u and v is small. The maximum such ratio, over all pairs of vertices of G, is the *spanning ratio* of Γ.

First, we show that deciding whether a graph admits a straight-line drawing with spanning ratio 1, a proper straight-line drawing with spanning ratio 1, and a planar straight-line drawing with spanning ratio 1 are NP-complete, $\exists\mathbb{R}$-complete, and linear-time solvable problems, respectively, where a drawing is proper if no two vertices overlap and no edge overlaps a vertex.

Second, we show that moving from spanning ratio 1 to spanning ratio $1+\epsilon$ allows us to draw every graph. Namely, we prove that, for every $\epsilon > 0$, every (planar) graph admits a proper (resp. planar) straight-line drawing with spanning ratio smaller than $1+\epsilon$.

Third, our drawings with spanning ratio smaller than $1+\epsilon$ have large edge-length ratio, that is, the ratio between the length of the longest edge and the length of the shortest edge is exponential. We show that this is sometimes unavoidable. More generally, we identify having bounded toughness as the criterion that distinguishes graphs that admit straight-line drawings with constant spanning ratio and polynomial edge-length ratio from graphs that require exponential edge-length ratio in any straight-line drawing with constant spanning ratio.

1 Introduction

Let P be a set of points in the plane and let $\mathcal{G}$ be a geometric graph whose vertex set is P. We say that $\mathcal{G}$ is a *t-spanner* if, for every pair of points p and q in P, there exists a path from p to q in $\mathcal{G}$ whose total edge length is at most t times the Euclidean distance $\|pq\|$ between p and q. The *spanning ratio* of $\mathcal{G}$ is the smallest real number t such that $\mathcal{G}$ is a t-spanner. The problem of constructing, for a given set P of points in the plane, a sparse (and possibly planar) geometric graph whose vertex set is P and whose spanning ratio is small has received considerable attention; see, e.g., [12,13,14,18,21,23,39]. We cite here the fact that the Delaunay triangulation of a point set has spanning ratio at least 1.593 [52] and at most 1.998 [51], and refer to the survey of Bose and Smid [15] for more results.

In this paper we look at the construction of geometric graphs with small spanning ratio from a different perspective. Namely, the problem we consider is whether it is possible to embed a given abstract graph in

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the plane as a geometric graph with small spanning ratio. That is, for a given graph, we want to construct a straight-line drawing with small spanning ratio, where the spanning ratio of a straight-line drawing is the maximum ratio, over all pairs of vertices u and v, between the total edge length of a shortest path from u to v and $\|uv\|$.

Graph embeddings in which every pair of vertices is connected by a path satisfying certain geometric properties have been the subject of intensive research. As the most notorious example, a *greedy* drawing of a graph [5,7,19,24,31,36,40,42,43,49] is such that, for every pair of vertices u and v, there is a path from u to v that monotonically decreases the distance to v at every vertex. More restricted than greedy drawings are *self-approaching* and *increasing-chord* drawings [3,20,41]. In a self-approaching drawing, for every pair of vertices u and v, there is a *self-approaching* path from u to v, i.e., a path P such that $\|ac\| > \|bc\|$, for any three points a, b, and c in this order along P; in an increasing-chord drawing, for every pair of vertices u and v, there is a path from u to v which is self-approaching both from u to v and from v to u. Even more restricted are *angle-monotone* drawings [10,20,37] in which, for every pair of vertices u and v, there is a path from u to v such that the angles of any two edges of the path differ by at most $90°$. Finally, *monotone* drawings [4,6,29,30,32,35] and *strongly-monotone* drawings [4,25,35] require, for every pair of vertices u and v, that a path from u to v exists that is monotone with respect to some direction or with respect to the direction of the straight line through u and v, respectively. While greedy, monotone, and strongly-monotone drawings might have unbounded spanning ratio, self-approaching, increasing-chord, and angle-monotone drawings are known to have spanning ratio at most 5.34 [33], at most 2.1 [44], and at most 1.42 [10], respectively. However, not all graphs, and not even all trees [36,40], admit self-approaching, increasing-chord, or angle-monotone drawings.

Our results are the following.

- First, we look at straight-line drawings with spanning ratio equal to 1, which is clearly the smallest attainable value by any graph. We prove that deciding whether a graph admits a straight-line drawing, a proper straight-line drawing (in which no vertex-vertex or vertex-edge overlaps are allowed), and a planar straight-line drawing with spanning ratio 1 are NP-complete, $\exists\mathbb{R}$-complete, and linear-time solvable problems, respectively.
- Second, we show that allowing each shortest path to have a total edge length slightly larger than the Euclidean distance between its end-vertices makes it possible to draw all graphs. Namely, we prove that, for every $\epsilon > 0$, every graph admits a proper straight-line drawing with spanning ratio smaller than $1+\epsilon$ and every planar graph admits a planar straight-line drawing with spanning ratio smaller than $1+\epsilon$.
- Third, we address the issue that our drawings with spanning ratio smaller than $1+\epsilon$ have poor resolution. That is, the *edge-length ratio* of these drawings, i.e., the ratio between the lengths of the longest and of the shortest edge, might be super-polynomial in the number of vertices of the graph. We show that this is sometimes unavoidable, as stars have exponential edge-length ratio in any straight-line drawing with constant spanning ratio. More in general, we show that there exist graph families such that any straight-line drawing with constant spanning ratio has edge-length ratio which is exponential in the inverse of the toughness. On the other hand, we prove that graph families with constant toughness admit proper straight-line drawings with polynomial edge-length ratio and constant spanning ratio. Finally, we prove that trees with bounded degree admit planar straight-line drawings with polynomial edge-length ratio and constant spanning ratio.

2 Preliminaries

For a graph G and a set S of vertices of G, we denote by $G - S$ the graph obtained from G by removing the vertices in S and their incident edges. The subgraph of G *induced by* S is the graph whose vertex set is S and whose edge set consists of every edge of G that has both its end-vertices in S. For a vertex v, a $\{v\}$-*bridge* of G is the subgraph of G induced by v and by the vertices of a connected component of $G - \{v\}$. The *toughness* of a graph G is the largest real number $t > 0$ such that, for every integer $k \geq 2$, G cannot be split into k connected components by the removal of fewer than $t \cdot k$ vertices; that is, for any set S such that $G - S$ consists of $k \geq 2$ connected components, we have $|S| \geq t \cdot k$.

A *drawing* of a graph maps each vertex to a distinct point in the plane and each edge to a Jordan arc between its end-vertices. A drawing is *straight-line* if it maps each edge to a straight-line segment. Let Γ be a straight-line drawing of a graph G. The *length of a path* in Γ is the sum of the lengths of its edges. We denote by $\|uv\|_\Gamma$ (by $\pi_\Gamma(u,v)$) the Euclidean distance (resp. the length of a shortest path) between two vertices u and v in Γ; we sometimes drop the subscript Γ when the drawing we refer to is clear from the context. The *spanning ratio* of Γ is the real value $\max\limits_{u,v} \frac{\pi_\Gamma(u,v)}{\|uv\|_\Gamma}$, where the maximum is over all pairs of vertices u and v of G.

A drawing is *planar* if no two edges intersect, except at common end-vertices. A planar drawing partitions the plane into connected regions, called *faces*; the bounded faces are *internal*, while the unbounded face is the *outer face*. A graph is *planar* if it admits a planar drawing. A planar graph is *maximal* if adding any edge to it violates its planarity. In any planar drawing of a maximal planar graph every face is delimited by a 3-cycle.

The *bounding box* $\mathcal{B}(\Gamma)$ of a drawing Γ is the smallest axis-parallel rectangle containing Γ in the closure of its interior. We denote by $\mathcal{B}_l(\Gamma)$ and $\mathcal{B}_r(\Gamma)$ the left and right side of $\mathcal{B}(\Gamma)$, respectively. The *width* and *height* of Γ are the width and height of $\mathcal{B}(\Gamma)$.

3 Drawings with Spanning Ratio 1

In this section we study straight-line drawings with spanning ratio equal to 1. We characterize the graphs that admit straight-line drawings, proper straight-line drawings, and planar straight-line drawings with spanning ratio equal to 1 and, consequently, derive results on the complexity of recognizing such graphs. We start with the following.

Lemma 1. *A graph admits a straight-line drawing with spanning ratio equal to* 1 *if and only if it contains a Hamiltonian path.*

Proof: ($\Longrightarrow$) Suppose that a graph G admits a straight-line drawing Γ with spanning ratio 1. Assume, w.l.o.g. up to a rotation of the Cartesian axes, that no two vertices of G have the same x-coordinate in Γ. Let $v_1, v_2, \dots, v_n$ be the vertices of G, ordered by increasing x-coordinates. Then, for $i = 1, 2, \dots, n-1$, we have that G contains the edge $v_i v_{i+1}$, as otherwise any path between v_i and v_{i+1} would be longer than $\|v_i v_{i+1}\|$. Hence, G contains the Hamiltonian path $(v_1, v_2, \dots, v_n)$.

($\Longleftarrow$) A straight-line drawing with spanning ratio 1 of a graph containing a Hamiltonian path $(v_1, v_2, \dots, v_n)$ can be constructed by placing v_i at $(i, 0)$, for $i = 1, \dots, n$. □

Theorem 1. *Recognizing whether a graph admits a straight-line drawing with spanning ratio equal to* 1 *is an NP-complete problem.*

Proof: The theorem follows by Lemma 1 and from the fact that deciding whether a graph contains a Hamiltonian path is an NP-complete problem [27,28]. □

A graph G is a *point visibility graph* if there exists a finite point set $P \subset \mathbb{R}^2$ such that: (i) G has a vertex for each point in P; and (ii) G has an edge between two vertices if and only if the straight-line segment between the corresponding points does not contain any point of P in its interior; see [9, Chapter 15]. We have the following.

Lemma 2. *A graph admits a proper straight-line drawing with spanning ratio equal to* 1 *if and only if it is a point visibility graph.*

Proof: ($\Longrightarrow$) Suppose that a graph G admits a proper straight-line drawing Γ with spanning ratio 1. Let v_Γ be the point at which a vertex v of G is drawn in Γ. Let $P := \{v_\Gamma \in \mathbb{R}^2 | v \in V(G)\}$ and let G_P be the point visibility graph of P. We claim that an edge uv belongs to G if and only if the edge $u_\Gamma v_\Gamma$ belongs to G_P; the claim implies that G_P is isomorphic to G and hence that G is a point visibility graph. First, if uv

belongs to G, then Γ contains the straight-line segment $\overline{u_\Gamma v_\Gamma}$. Since Γ is proper, no vertex of G lies in the interior of $\overline{u_\Gamma v_\Gamma}$, hence G_P contains the edge $u_\Gamma v_\Gamma$. Conversely, if G_P contains the edge $u_\Gamma v_\Gamma$, then no point in P lies in the interior of the straight-line segment $\overline{u_\Gamma v_\Gamma}$. Hence, the edge uv belongs to G, as otherwise the length of any path between u and v would be larger than $\|uv\|_\Gamma$.

($\Longleftarrow$) Suppose that a graph G is the visibility graph of a point set P. For any point $p \in P$, let v_p be the corresponding vertex of G. Let Γ be the straight-line drawing of G that maps each vertex v_p to the point p. Consider any edge $v_p v_q$ of G. No vertex v_r lies in the interior of the straight-line segment $\overline{pq}$ in Γ, as otherwise $v_p v_q$ would not belong to G; it follows that Γ is proper. Further, consider any two vertices v_p and v_q of G and let $v_p = v_{r_1}, v_{r_2}, \dots, v_{r_k} = v_q$ be the sequence of vertices of G lying on the straight-line segment $\overline{pq}$ in Γ, ordered as they occur from p to q. Then G contains the path $(v_p = v_{r_1}, v_{r_2}, \dots, v_{r_k} = v_q)$, whose length in Γ is $\|v_p v_q\|_\Gamma$. It follows that the spanning ratio of Γ is 1. □

The *existential theory of the reals* problem asks whether real values exist for n variables such that a quantifier-free formula, consisting of polynomial equalities and inequalities on such variables, is satisfied. The class of problems that are complete for the existential theory of the reals is denoted by $\exists\mathbb{R}$ [45]. It is known that NP $\subseteq \exists\mathbb{R} \subseteq$ PSPACE [16], however it is not known whether $\exists\mathbb{R} \subseteq$ NP. Many geometric problems are $\exists\mathbb{R}$-complete, see, e.g., [1,38].

Theorem 2. *Recognizing whether a graph admits a proper straight-line drawing with spanning ratio equal to* 1 *is an* $\exists\mathbb{R}$*-complete problem.*

Proof: The theorem follows by Lemma 2 and from the fact that recognizing point visibility graphs is a problem that is $\exists\mathbb{R}$-complete [17]. □

We conclude the section by presenting the following.

Theorem 3. *Recognizing whether a graph admits a planar straight-line drawing with spanning ratio equal to* 1 *is a linear-time solvable problem.*

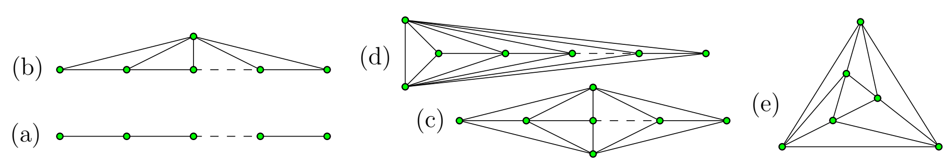

Fig. 1: The five graph classes defined in [22].

Proof: Dujmović et al. [22] characterized the graphs that admit a planar straight-line drawing with a straight-line segment between every two vertices as the graphs in the five graph classes in Figure 1. Since a straight-line drawing has spanning ratio 1 if and only if every two vertices are connected by a straight-line segment, the theorem follows from the fact that recognizing whether a graph belongs to such five graph classes can be easily done in linear time. □

4 Drawings with Spanning Ratio $1 + \epsilon$

In this section we study straight-line drawings with spanning ratio arbitrarily close to 1. Most of the section is devoted to a proof of the following result.

Theorem 4. *For every* $\epsilon > 0$*, every connected planar graph admits a planar straight-line drawing with spanning ratio smaller than* $1 + \epsilon$*.*

Let G be an n-vertex maximal planar graph with $n \geq 3$, let $\mathcal{G}$ be a planar drawing of G, and let (u, v, z) be the cycle delimiting the outer face of G in $\mathcal{G}$. A *canonical ordering* [8,26,34] for G is a total ordering $\sigma_G = [v_1, v_2, \ldots, v_n]$ of its vertex set such that the following hold for $k = 3, \ldots, n$: (i) $v_1 = u$, $v_2 = v$, and $v_n = z$; (ii) the subgraph G_k of G induced by $v_1, v_2, \ldots, v_k$ is 2-connected and the cycle $\mathcal{C}_k$ delimiting its outer face in $\mathcal{G}$ consists of the edge v_1v_2 and of a path $\mathcal{P}_k$ between v_1 and v_2; and (iii) v_k is incident to the outer face of G_k in $\mathcal{G}$. Theorem 4 is implied by the following two lemmata.

Lemma 3. *Let H be any n-vertex connected planar graph. There exist an n-vertex maximal planar graph G and a canonical ordering $\sigma_G = [v_1, v_2, \ldots, v_n]$ for G such that, for each $k \in \{1, 2, \ldots, n\}$, the subgraph H_k of H induced by $[v_1, v_2, \ldots, v_k]$ is connected.*

Proof: For $k = 2, 3, \ldots, n$, let G_k be the subgraph of G induced by $v_1, v_2, \ldots, v_k$ and let L_k be the graph composed of G_k and of the vertices and edges of H that are not in G_k.

For each $k = 2, 3, \ldots, n$, we define $v_1, v_2, \ldots, v_k$ and G_k so that H_k is connected, G_k is 2-connected, and L_k admits a planar drawing $\mathcal{L}_k$ such that:

1. the outer face of the planar drawing $\mathcal{G}_k$ of G_k in $\mathcal{L}_k$ is delimited by a cycle $\mathcal{C}_k$ composed of the edge v_1v_2 and of a path $\mathcal{P}_k$ between v_1 and v_2;
2. v_k is incident to the outer face of $\mathcal{G}_k$;
3. every internal face of $\mathcal{G}_k$ is delimited by a 3-cycle; and
4. the vertices and edges of H that are not in G_k lie in the outer face of $\mathcal{G}_k$.

If $k = 2$, then construct any planar drawing $\mathcal{L}_2$ of H and define v_1 and v_2 as the end-vertices of any edge v_1v_2 incident to the outer face of $\mathcal{L}_2$. Properties 1–4 are then trivially satisfied (in this case the path $\mathcal{P}_2$ is the single edge v_1v_2).

If $2 < k < n$, assume that $v_1, v_2, \ldots, v_{k-1}$ and G_{k-1} have been defined so that H_{k-1} is connected, G_{k-1} is 2-connected, and L_{k-1} admits a planar drawing $\mathcal{L}_{k-1}$ such that Properties 1–4 above are satisfied. Let $\mathcal{P}_{k-1} = (u = w_1, w_2, \ldots, w_x = v)$, where $x \geq 2$.

Consider any vertex v that is in L_{k-1} and that is not in G_{k-1}. By Properties 1 and 4 of $\mathcal{L}_{k-1}$, all the neighbors of v in G_{k-1} lie in $\mathcal{P}_{k-1}$. We say that v is a *candidate* (to be designated as v_k) *vertex* if, for some $1 \leq i \leq x$, there exists an edge w_iv such that w_iv immediately follows the edge w_iw_{i-1} in clockwise order around w_i or immediately follows the edge w_iw_{i+1} in counter-clockwise order around w_i; see Figure 2.

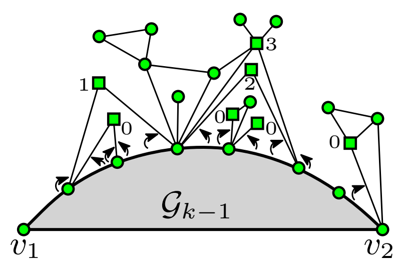

Fig. 2: The drawing $\mathcal{L}_{k-1}$ of L_{k-1}, where the interior of $\mathcal{G}_{k-1}$ is colored gray. Each candidate vertex is represented by a square and labeled with its depth.

For each candidate vertex v, let $w_{a(v)}$ and $w_{b(v)}$ be the neighbors of v in $\mathcal{P}_{k-1}$ such that $a(v)$ is minimum and $b(v)$ is maximum (possibly $a(v) = b(v)$). If $a(v) < b(v)$, define the *reference cycle* $\mathcal{C}(v)$ *of* v as the cycle composed of the edges $w_{a(v)}v$ and $w_{b(v)}v$ and of the subpath of $\mathcal{P}_{k-1}$ between $w_{a(v)}$ and $w_{b(v)}$. Define the *depth* $d(v)$ of v as 0 if $a(v) = b(v)$ or as the number of candidate vertices that lie inside $\mathcal{C}(v)$ in $\mathcal{L}_{k-1}$ otherwise.

We claim that there exists a candidate vertex with depth 0. Consider a candidate vertex v with minimum depth and assume, for a contradiction, that $d(v) > 0$; then there exists a candidate vertex u that lies inside

$\mathcal{C}(v)$ in $\mathcal{L}_{k-1}$. By the planarity of $\mathcal{L}_{k-1}$, the candidate vertices that lie inside $\mathcal{C}(u)$ form a subset of those that lie inside $\mathcal{C}(v)$; moreover, there is at least one candidate vertex, namely u, that lies inside $\mathcal{C}(v)$ and not inside $\mathcal{C}(u)$, hence $d(u) < d(v)$. This contradicts the assumption that v has minimum depth and proves the claim.

Consider a candidate vertex v with $d(v) = 0$. We let $v_k := v$ and distinguish two cases.

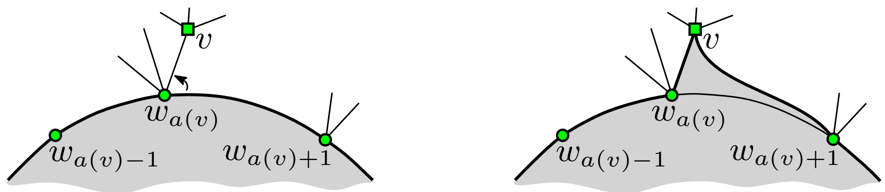

Fig. 3: (a) A candidate vertex v with $d(v) = 0$ and $a(v) = b(v)$. (b) The drawing $\mathcal{L}_k$ of L_k obtained by drawing the edge $w_{a(v)+1}v$ in $\mathcal{L}_{k-1}$.

If $a(v) = b(v)$, assume that $w_{a(v)}v$ immediately follows the edge $w_{a(v)}w_{a(v)+1}$ in counter-clockwise order around $w_{a(v)}$, the other case is symmetric; refer to Figure 3. Define G_k as G_{k-1} plus the vertex v and the edges $w_{a(v)}v$ and $w_{a(v)+1}v$. Then H_k is connected because H_{k-1} is connected and the edge $w_{a(v)}v$ belongs to H. Further, G_k is 2-connected because G_{k-1} is 2-connected and v is adjacent to two distinct vertices of G_{k-1}. Define $\mathcal{L}_k$ by drawing the edge $w_{a(v)+1}v$ so that the cycle $(w_{a(v)}, w_{a(v)+1}, v)$ does not contain any vertex or edge in its interior. Property 1 is satisfied by $\mathcal{L}_k$ with $\mathcal{P}_k = (u = w_1, w_2, \ldots, w_{a(v)}, v, w_{a(v)+1}, \ldots, w_x = v)$; note that v has no neighbor in G_k other than $w_{a(v)}$ and $w_{a(v)+1}$, since $a(v) = b(v)$. Property 2 is satisfied by $\mathcal{L}_k$ since $\mathcal{L}_{k-1}$ satisfies Property 4 and by construction. Since the cycle $(w_{a(v)}, w_{a(v)+1}, v)$ does not contain any vertex in its interior and since $\mathcal{L}_{k-1}$ satisfies Properties 3 and 4, it follows that $\mathcal{L}_k$ also satisfies Properties 3 and 4.

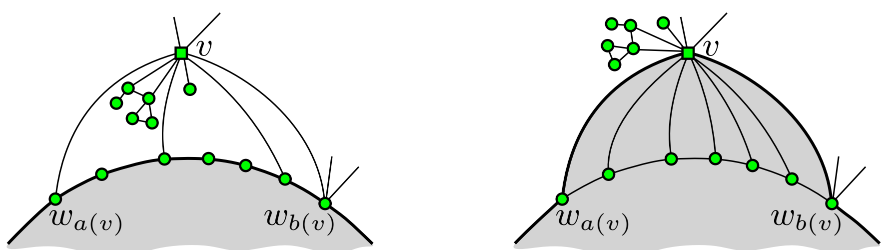

Fig. 4: (a) A candidate vertex v with $d(v) = 0$ and $a(v) < b(v)$. (b) The drawing $\mathcal{L}_k$ of L_k obtained by moving out of $\mathcal{C}(v)$ each $\{v\}$-bridge of L_{k-1} whose vertices different from v lie inside $\mathcal{C}(v)$ and by drawing the edges among $w_{a(v)}v, w_{a(v)+1}v, \ldots, w_{b(v)+1}v$ not in H planarly inside $\mathcal{C}(v)$.

Next, we consider the case in which $a(v) < b(v)$; refer to Figure 4. We claim that the only edges incident to vertices in the path $(w_{a(v)}, w_{a(v)+1}, \ldots, w_{b(v)})$ and lying inside $\mathcal{C}(v)$ in $\mathcal{L}_{k-1}$ are those connecting such vertices to v. Suppose, for a contradiction, that an edge $w_i u$ with $u \neq v$ lies inside $\mathcal{C}(v)$. If $a(v) < i < b(v)$, then there exists an edge $w_i z$ with $z \neq v$ that immediately follows $w_i w_{i-1}$ in clockwise order around w_i or that immediately follows $w_i w_{i+1}$ in counter-clockwise order around w_i; hence, z is a candidate vertex. Further, by the planarity of $\mathcal{L}_{k-1}$, we have that $w_i z$ lies inside $\mathcal{C}(v)$, except at w_i, however this contradicts $d(v) = 0$. The proof for the cases in which $i = a(v)$ or $i = b(v)$ is analogous.

It follows from the previous claim that v is the only vertex of $\mathcal{C}(v)$ which might have incident edges that lie inside $\mathcal{C}(v)$ in $\mathcal{L}_{k-1}$ and that have one end-vertex not in $\mathcal{C}(v)$. We redraw each $\{v\}$-bridge of L_{k-1} whose vertices different from v lie inside $\mathcal{C}(v)$ planarly so that it now lies outside $\mathcal{C}(v)$; after this modification, no vertex of L_{k-1} lies inside $\mathcal{C}(v)$.

Define G_k as G_{k-1} plus the vertex $v_k := v$ and the edges $w_{a(v)}v, w_{a(v)+1}v, \dots, w_{b(v)}v$. Then H_k is connected, because H_{k-1} is connected and the edge $w_{a(v)}v$ belongs to H. Further, G_k is 2-connected, because G_{k-1} is 2-connected and v is adjacent to at least two distinct vertices of G_{k-1}. Define $\mathcal{L}_k$ by drawing the edges among $w_{a(v)}v, w_{a(v)+1}v, \dots, w_{b(v)}v$ that do not belong to H so that they all lie inside $\mathcal{C}(v)$, except at their end-vertices, and so that the edges $w_{a(v)}v, w_{a(v)+1}v, \dots, w_{b(v)}v$ appear consecutively and in this counter-clockwise order around v. Property 1 is satisfied by $\mathcal{L}_k$ with $\mathcal{P}_k = (u = w_1, w_2, \dots, w_{a(v)}, v, w_{b(v)}, w_{b(v)+1}, \dots, w_x = v)$. Property 2 is satisfied by $\mathcal{L}_k$ by construction and since $\mathcal{L}_{k-1}$ satisfies Property 4. Every internal face of $\mathcal{L}_k$ that is not an internal face of $\mathcal{L}_{k-1}$ is delimited by a 3-cycle (w_i, w_{i+1}, v), for some $a(v) \leq i < b(v)$; hence $\mathcal{L}_k$ satisfies Property 3 since $\mathcal{L}_{k-1}$ does. Finally, $\mathcal{L}_k$ satisfies Property 4 since every vertex or edge of H that is not in G_k lies outside $\mathcal{G}_{k-1}$ since $\mathcal{L}_{k-1}$ satisfies Property 4 and lies outside $\mathcal{C}(v)$ by construction.

If $k = n$, the construction slightly differs from the one described for the case $2 < k < n$, as we also require that the outer face of $\mathcal{G}_n$ is delimited by the 3-cycle (v_1, v_2, v_n). Hence, if $a(v) = b(v)$ (resp. if $a(v) < b(v)$), then G_n also contains the edges w_1v, w_2v, $\dots$, $w_{a(v)-1}v$, $w_{a(v)+2}v$, $w_{a(v)+3}v$, $\dots$, w_xv (resp. the edges w_1v, w_2v, $\dots$, $w_{a(v)-1}v$, $w_{b(v)+1}v$, $w_{b(v)+2}v$, $\dots$, w_xv); further, the edges $w_1v, w_2v, \dots, w_xv$ are drawn in $\mathcal{L}_n$ in such a way that they appear in this counter-clockwise order around v, and so that the outer face of $\mathcal{L}_n$ is delimited by the 3-cycle $(w_1 = v_1, w_x = v_2, v = v_n)$. The proof that $\mathcal{L}_n$ satisfies Properties 1–4 is similar, and in fact simpler, than the one described above.

The above construction implies the statement of the lemma. Namely, H_k is connected for $k = 3, 4, \dots, n$. Further, G is a maximal planar graph by Property 3 and by the additional requirement for the case $k = n$. We now prove that $[v_1, v_2, \dots, v_n]$ is a canonical ordering for G. By Properties 1 and 2 of $\mathcal{L}_n$, we have that v_1, v_2, and v_n are incident to the outer face of $\mathcal{L}_n$; further, for $k = 3, 4, \dots, n$, we have that G_k is 2-connected and its outer face in $\mathcal{G}_k$ is delimited by the edge v_1v_2 and by a path $\mathcal{P}_k$ between v_1 and v_2, by Property 1 of $\mathcal{L}_k$; finally, v_k is incident to the outer face of $\mathcal{G}_k$ for $k = 3, 4, \dots, n$, by Property 2 of $\mathcal{L}_k$. □

Lemma 4. *For every $k = 3, \dots, n$ and for every $\epsilon > 0$, there exists a planar straight-line drawing Γ_k of G_k such that:*

1. *the outer face of Γ_k is delimited by the cycle $\mathcal{C}_k$; further, the path $\mathcal{P}_k$ is x-monotone and lies above the edge uv, except at u and v; and*
2. *the restriction Ξ_k of Γ_k to the vertices and edges of H_k is a drawing with spanning ratio smaller than $1 + \epsilon$.*

Proof: The proof is by induction on k. If $k = 3$, then a planar straight-line drawing Γ_3 of G_3 is constructed by drawing the 3-cycle $v_1v_2v_3$ as an isosceles triangle in which v_1v_2 is horizontal and has length $\epsilon/2$, while v_1v_3 and v_2v_3 have length 1, with v_3 above the edge v_1v_2. By Lemma 3, the graphs H_2 and H_3 are connected, hence the edge v_1v_2 belongs to them and at least one of the edges v_1v_3 and v_2v_3 belongs to H_3. Hence, we have $\frac{\pi(v_1,v_2)}{\|v_1v_2\|} = \frac{\|v_1v_2\|}{\|v_1v_2\|} = 1 < 1 + \epsilon$. Further, $\frac{\pi(v_1,v_3)}{\|v_1v_3\|} \leq \max\{\|v_1v_3\|, \|v_1v_2\| + \|v_2v_3\|\} = 1 + \epsilon/2 < 1 + \epsilon$. Analogously, $\frac{\pi(v_2,v_3)}{\|v_2v_3\|} < 1 + \epsilon$.

Now assume that, for some $k = 4, \dots, n$, a planar straight-line drawing Γ_{k-1} of G_{k-1} has been constructed satisfying Properties 1 and 2; refer to Figure 5. Let δ be the diameter of a disk D containing Γ_{k-1} in its interior. We construct Γ_k from Γ_{k-1} by placing v_k in the plane as follows. Let $\mathcal{P}_{k-1} = (u = w_1, w_2, \dots, w_x = v)$. As proved in [26], the neighbors of v_k in G_{k-1} are the vertices in a sub-path $(w_p, w_{p+1}, \dots, w_q)$ of $\mathcal{P}_{k-1}$, where $1 \leq p < q \leq x$. By Property 1 of Γ_{k-1}, we have $x(w_p) < x(w_{p+1}) < \dots < x(w_q)$. We then place v_k at any point in the plane such that the following conditions are satisfied: (i) $x(w_p) < x(v_k) < x(w_q)$; (ii) for every $i = p, \dots, q-1$, the y-coordinate of v_k is larger than the y-coordinates of the intersection points between the line through w_iw_{i+1} and the vertical lines through w_p and w_q; and (iii) the distance between v_k and the point of D closest to v_k is a real value $d > \frac{k\delta}{\epsilon}$.

Since $\mathcal{P}_k$ is obtained from $\mathcal{P}_{k-1}$ by substituting the path $(w_p, w_{p+1}, \dots, w_q)$ with the path (w_p, v_k, w_q), Condition (i) and the x-monotonicity of $\mathcal{P}_{k-1}$ imply that $\mathcal{P}_k$ is x-monotone. Since Γ_{k-1} is planar, in order to prove the planarity of Γ_k it suffices to prove that no edge incident to v_k intersects any distinct edge of

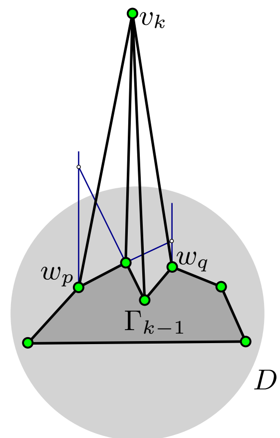

Fig. 5: Construction of Γ_k from Γ_{k-1}.

G_k, except at common end-vertices. Condition (ii) implies that the edges incident to v_k lie in the outer face of Γ_{k-1}, hence they do not intersect any edge of G_{k-1}, except at common end-vertices. Again Condition (ii) and the x-monotonicity of $\mathcal{P}_{k-1}$ imply that no two edges incident to v_k intersect each other, except at v_k. We now prove that the spanning ratio of Ξ_k is smaller than $1+\epsilon$. Consider any two vertices v_i and v_j. If $i<k$ and $j<k$, then $\frac{\pi_{\Xi_k}(v_i,v_j)}{\|v_iv_j\|_{\Xi_k}} \leq \frac{\pi_{\Xi_{k-1}}(v_i,v_j)}{\|v_iv_j\|_{\Xi_{k-1}}} < 1+\epsilon$. If $i=k$, then $\|v_kv_j\|_{\Xi_k} \geq d$, by Condition (iii). Consider the path $P(v_k,v_j)$ composed of any edge v_kv_ℓ in H_k incident to v_k (which exists since H_k is connected) and of any path in H_{k-1} between v_ℓ and v_j (which exists since H_{k-1} is connected). The length of $P(v_k,v_j)$ is at most $d+\delta$ (by Condition (iii) and by the triangular inequality, this is an upper bound on $\|v_kv_\ell\|_{\Xi_k}$) plus $(k-2)\cdot\delta$ (this is an upper bound on the length of any path in H_{k-1}). Hence, $\frac{\pi_{\Xi_k}(v_k,v_j)}{\|v_kv_j\|_{\Xi_k}} < \frac{d+k\delta}{d} < 1+\epsilon$. This completes the induction and the proof of the lemma. □

Lemmata 3 and 4 imply Theorem 4. Namely, for any connected planar graph H, by Lemma 3 we can construct a maximal planar graph G that, by Lemma 4 (with $k=n$) and for every $\epsilon>0$, admits a planar straight-line drawing whose restriction to the vertices and edges of H is a drawing with spanning ratio smaller than $1+\epsilon$.

The following can be obtained by means of techniques similar to (and simpler than) the ones employed in the proof of Theorem 4.

Theorem 5. *For every $\epsilon>0$, every connected graph admits a proper straight-line drawing with spanning ratio smaller than $1+\epsilon$.*

Proof: Consider any n-vertex graph G and let T be any spanning tree of G. Let $v_1,v_2,\dots,v_n$ be any total ordering for the vertex set of T such that the subtree T_k of T induced by $v_1,v_2,\dots,v_k$ is connected, for each $k=1,2,\dots,n$.

For $k=1,2,\dots,n$, we construct a straight-line drawing Γ_k of T_k with spanning ratio smaller than $1+\epsilon$ and such that no three vertices lie on a straight line. If $k=1$, then Γ_1 is constructed by placing v_1 at any point in the plane. Now assume that a straight-line drawing Γ_{k-1} of T_{k-1} has been constructed with spanning ratio smaller than $1+\epsilon$ and such that no three vertices lie on a straight line. Let δ be the diameter of a disk D containing Γ_{k-1} in its interior. We construct Γ_k from Γ_{k-1} by placing v_k at any point in the plane such that: (1) v_k does not lie on any straight line through two vertices of T_{k-1}; and (2) the distance between v_k and the point of D that is closest to v_k is a real value $d>\frac{k\delta}{\epsilon}$.

By Property (1), no three vertices lie on a straight line in Γ_k. We prove that the spanning ratio of Γ_k is smaller than $1+\epsilon$. Consider any two vertices v_i and v_j. If $i<k$ and $j<k$, then $\frac{\pi_{\Gamma_k}(v_i,v_j)}{\|v_iv_j\|_{\Gamma_k}} \leq \frac{\pi_{\Gamma_{k-1}}(v_i,v_j)}{\|v_iv_j\|_{\Gamma_{k-1}}} < 1+\epsilon$. If $i=k$, then $\|v_kv_j\|_{\Gamma_k} \geq d$, by Property (2). Further, $\pi_{\Gamma_k}(v_k,v_j)$ is at most $d+\delta$ (by Property (2) and by the triangular inequality, this is an upper bound on the length of the edge of T_k incident to v_k) plus $(k-2)\cdot\delta$ (this is an upper bound on the length of any path in T_{k-1}). Hence, $\frac{\pi_{\Gamma_k}(v_k,v_j)}{\|v_k v_j\|_{\Gamma_k}} < \frac{d+k\delta}{d} < 1+\epsilon$.

A drawing Γ of G is obtained from the drawing Γ_n of $T = T_n$ by drawing the edges that are not in T as straight-line segments. Then Γ is proper, as no three vertices of T lie on a straight line in Γ_n, and has spanning ratio smaller than $1+\epsilon$, as the same is true for Γ_n. □

5 Drawings with Small Spanning Ratio and Edge-Length Ratio

In this section we study straight-line drawings with small spanning ratio and edge-length ratio. Our main result is the following.

Theorem 6. *For every $\epsilon > 0$ and $\tau > 0$, every n-vertex graph with toughness τ admits a proper straight-line drawing whose spanning ratio is at most $1+\epsilon$ and whose edge-length ratio is in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon\rceil)}{\log_2(2+\lceil 1/\tau\rceil)-\log_2(1+\lceil 1/\tau\rceil)}}\cdot 1/\epsilon\right)$.*

Further, for every $0 < \tau < 1$, there is a graph G with toughness τ such that every straight-line drawing of G with spanning ratio at most s has edge-length ratio in $2^{\Omega(1/(\tau\cdot s^2))}$.

In order to prove Theorem 6, we study straight-line drawings of bounded-degree trees. This is because there is a strong connection between the toughness of a graph and the existence of a spanning tree with bounded degree. Indeed, if a graph G has toughness τ, then it has a spanning tree with maximum degree $\lceil 1/\tau\rceil+2$ [50]. Further, a tree has toughness equal to the inverse of its maximum degree. We start by proving the following upper bound.

Theorem 7. *For every $\epsilon > 0$, every n-vertex tree T with maximum degree d admits a proper straight-line drawing such that no three vertices are collinear, the spanning ratio is at most $1+\epsilon$, the distance between any two vertices is at least 1, and the width, the height, and the edge-length ratio are in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon\rceil)}{\log_2(d/(d-1))}}\cdot 1/\epsilon\right)$.*

Proof: Let $\gamma = \lceil \frac{2}{\epsilon}\rceil$. Root T at any vertex r. For any two vertices p and q of T, let P_{pq} be the path in T from p to q. We prove that, for an arbitrary real value $\eta > 0$, T admits a proper straight-line drawing Γ that, in addition to the properties in the statement of the theorem, satisfies the following: (1) r is at the top-left corner of $\mathcal{B}(\Gamma)$; (2) for every vertex z of T, the path P_{zr} monotonically decreases in the x-direction and monotonically increases in the y-direction from z to r; and (3) the height of Γ is at most η.

If $n = 1$, then Γ is obtained by placing r at any point in the plane. If $n > 1$, then there exists an edge uv whose removal separates T into two trees T_1 and T_2, each with at most $\frac{d-1}{d}n$ vertices [48]. Assume, w.l.o.g., that T_1 contains r and u, while T_2 contains v. Then T_1 is rooted at r and T_2 is rooted at v. Inductively construct proper straight-line drawings Γ_1 and Γ_2 of T_1 and T_2, respectively, with parameter $\eta/3$ satisfying Properties 1–4. Let w_1 and w_2 be the widths of Γ_1 and Γ_2, respectively. Refer to Figure 6.

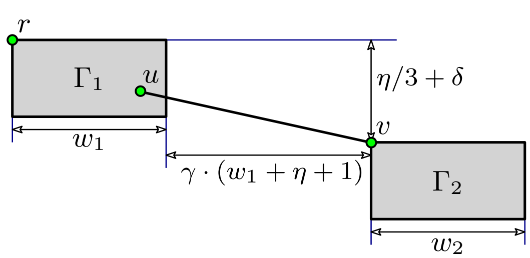

Fig. 6: Illustration for the construction in Theorem 7.

Translate Γ_1 so that r lies at $(0,0)$. Further, translate Γ_2 so that v lies at $(w_1+\gamma\cdot(w_1+\eta+1), -\eta/3-\delta)$, where $0 < \delta < \eta/3$ is a real value chosen so that no line through two vertices in the same tree T_i, with

$i \in \{1,2\}$, overlaps a vertex in the tree T_j, with $j \in \{1,2\}$ and $j \neq i$. Note that, since Γ_1 and Γ_2 are proper and since $\mathcal{B}(\Gamma_1)$ and $\mathcal{B}(\Gamma_2)$ are disjoint, there are finitely many values of δ for which the line through two vertices in a tree T_i overlaps a vertex in a different tree T_j, hence such a value δ always exists.

We now analyze the properties of Γ. By construction, Γ is a straight-line drawing of T.

By induction, no three vertices are collinear in each of Γ_1 and Γ_2; further, by construction, Γ_1 and Γ_2 are arranged so that no line through two vertices in the same tree T_i, with $i \in \{1,2\}$, overlaps a vertex in the tree T_j, with $j \in \{1,2\}$ and $j \neq i$. Hence, no three vertices are collinear in Γ, and in particular. Γ is proper.

Property (1) is satisfied by Γ given that r is at the top-left corner of $\mathcal{B}(\Gamma_1)$, by induction, and given that every vertex of T_2 lies to the right and below r in Γ, by construction.

In order to prove that Γ satisfies Property (2), consider any vertex z of T. If z belongs to T_1, then P_{zr} monotonically decreases in the x-direction and monotonically increases in the y-direction from z to r, since Γ_1 satisfies Property (2). If z belongs to T_2, then P_{zr} is composed of the path P_{zv}, of the edge vu, and of the path P_{ur}. The paths P_{zv} and P_{ur} monotonically decrease in the x-direction and monotonically increase in the y-direction from z to v and from u to r, respectively, since Γ_2 and Γ_1 satisfy Property (2). Further, the x-coordinate of u is smaller than the one of v and the y-coordinate of u is larger than the one of v; the latter follows from the fact that every vertex of T_1 has y-coordinate in $[-\eta/3, 0]$, while every vertex of T_2 has y-coordinate smaller than $-\eta/3$.

The height of Γ is at most $2\eta/3 + \delta$, which is smaller than η, hence Γ satisfies Property (3).

We now discuss the spanning ratio of Γ. We prove that, for any vertex w of T_1 and any vertex z of T_2, it holds true that $\frac{\pi_\Gamma(w,z)}{\|wz\|_\Gamma} \leq \frac{\gamma+2}{\gamma}$. This suffices to prove that the spanning ratio of Γ is at most $\frac{\gamma+2}{\gamma}$, since the drawings of T_1 and T_2 in Γ are the ones inductively constructed by the algorithm. The distance between w and z is larger than or equal to $\gamma \cdot (w_1 + \eta + 1) + x_z$, where $\gamma \cdot (w_1 + \eta + 1)$ is the horizontal distance between $\mathcal{B}_r(\Gamma_1)$ and $\mathcal{B}_l(\Gamma_2)$, while x_z denotes the distance between $\mathcal{B}_l(\Gamma_2)$ and z. Clearly, we have $\pi_\Gamma(w,z) = \pi_\Gamma(w,r) + \pi_\Gamma(z,r)$. The path P_{wr} is monotone in the x- and y-directions, hence $\pi_\Gamma(w,r)$ is upper bounded by the horizontal distance between w and r, which is at most w_1, plus the vertical distance between w and r, which is at most $\eta/3$. Analogously, $\pi_\Gamma(z,r)$ is upper bounded by the horizontal distance between z and r, which is $w_1 + \gamma \cdot (w_1 + \eta + 1) + x_z$, plus the vertical distance between z and r, which is at most η. Hence, $\pi_\Gamma(w,z) < (\gamma + 2) \cdot (w_1 + \eta + 1) + x_z$. Thus:

$$\frac{\pi_\Gamma(u,v)}{\|uv\|_\Gamma} < \frac{(\gamma+2) \cdot (w_1 + \eta + 1 + \frac{x_z}{\gamma})}{\gamma \cdot (w_1 + \eta + 1 + \frac{x_z}{\gamma})} \leq \frac{\gamma+2}{\gamma} \leq 1 + \epsilon.$$

Finally, we analyze the edge-length ratio of Γ. Note that the distance between any vertex of T_1 and any vertex of T_2 is larger than 1, hence the same is true for every pair of vertices of T. In particular, the length of every edge is larger than 1. Thus, the edge-length ratio of Γ is upper bounded by the maximum length of an edge of T. In turn, this is at most the height plus the width of Γ. By Property (3), the height of Γ is at most η. By construction, the width of Γ is equal to $w_1 + \gamma \cdot (w_1 + \eta + 1) + w_2$. Denote by $w(n)$ the maximum width of a drawing of an n-vertex tree constructed by the above algorithm. Since each of T_1 and T_2 has at most $\frac{d-1}{d}n$ vertices, we get that $w(n) \leq (\gamma+2) \cdot (w(\frac{d-1}{d}n) + \eta + 1)$. Repeatedly substituting this inequality into itself and recalling that $w(n) = 0$ for $n \leq 1$, we get $w(n) \leq (1+\eta) \cdot (\gamma+2) + (1+\eta) \cdot (\gamma+2)^2 + \cdots + (1+\eta) \cdot (\gamma+2)^{\lceil \log_{\frac{d}{d-1}}(n) \rceil} \leq (1+\eta) \cdot \frac{\gamma+2}{\gamma+1} \cdot (\gamma+2)^{\log_{\frac{d}{d-1}}(n)+1} = (1+\eta) \cdot \frac{\gamma+2}{\gamma+1} \cdot (\gamma+2) \cdot n^{\frac{\log_2(\gamma+2)}{\log_2(d/(d-1))}}$. We have $\frac{\gamma+2}{\gamma+1} \leq 2$, given that $\gamma = \lceil \frac{2}{\epsilon} \rceil \geq 1$; further, we can set η to be any constant, say $\eta = 1$. Thus, we get $w(n) \in \mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon \rceil)}{\log_2(d/(d-1))}} \cdot 1/\epsilon\right)$ and the same holds true for the edge-length ratio of Γ. □

We can now prove the upper bound in Theorem 6. Consider an n-vertex graph G with toughness τ and let $\epsilon > 0$; then G has a spanning tree T with maximum degree $d = \lceil 1/\tau \rceil + 2$ [50]. Apply Theorem 7 in order to construct a straight-line drawing Γ_T of T. Construct a straight-line drawing Γ_G of G from Γ_T by representing the edges of G not in T as straight-line segments. This results in a proper drawing of G, given that no three vertices are collinear in Γ_T. Further, the spanning ratio of Γ_G is at most the one of Γ_T, hence it is at most $1 + \epsilon$. Finally, the edge-length ratio of Γ_G is in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon \rceil)}{\log_2(d/(d-1))}} \cdot 1/\epsilon\right)$, given that the distance between any two vertices in Γ_T (and hence in Γ_G) is at least 1 and given that the width and height of Γ_T (and hence of Γ_G) are in $\mathcal{O}\left(n^{\frac{\log_2(2+\lceil 2/\epsilon \rceil)}{\log_2(d/(d-1))}} \cdot 1/\epsilon\right)$. Substituting the value $d = \lceil 1/\tau \rceil + 2$ provides us with the upper bound in Theorem 6.

The lower bound in Theorem 6 comes from the following theorem.

Theorem 8. *Let T be a tree with a vertex of degree d. For any $s \geq 1$, any straight-line drawing of T with spanning ratio at most s has edge-length ratio in $2^{\Omega(d/s^2)}$.*

Proof: For any $s \geq 1$, let Γ be any straight-line drawing of T with spanning ratio at most s; refer to Figure 7. Let u_T be a vertex of degree d. Assume w.l.o.g. up to a scaling (which does not alter the edge-length ratio and the spanning ratio of Γ) that the length of the shortest edge incident to u_T in Γ is 1. For any integer $i \geq 0$, let $\mathcal{C}_i$ be the circle centered at u_T whose radius is $r_i = 2^i$. Further, for any integer $i > 0$, let $\mathcal{A}_i$ be the closed annulus delimited by $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$. By assumption, no neighbor of u_T lies inside the open disk delimited by $\mathcal{C}_0$. We claim that, for any integer $i > 0$ and for some constant c, there are at most $c \cdot s^2$ neighbors of u_T inside $\mathcal{A}_i$. This implies that at most $k \cdot c \cdot s^2$ neighbors of u_T lie inside the closed disk delimited by $\mathcal{C}_k$. Hence, if $d > k \cdot c \cdot s^2$, e.g., if $k = \lfloor \frac{d-1}{c \cdot s^2} \rfloor$, then there is a neighbor v_T of u_T outside $\mathcal{C}_k$. Then $\|u_T v_T\| > 2^k \in 2^{\Omega(d/s^2)}$. Hence, the theorem follows from the claim.

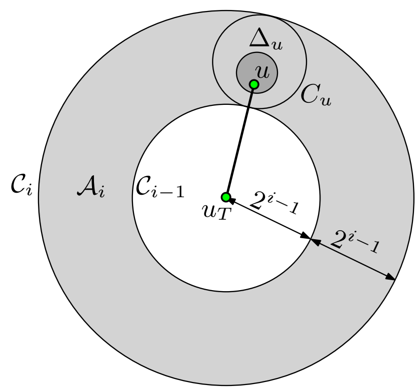

Fig. 7: Illustration for the proof of Theorem 8.

It remains to prove the claim. For each neighbor u of u_T inside $\mathcal{A}_i$, let Δ_u be a closed disk such that: (i) u lies inside Δ_u; (ii) Δ_u lies inside $\mathcal{A}_i$; and (iii) the diameter of Δ_u is $\delta_i = 2^{i-2}/s$. The existence of Δ_u can be proved as follows. Consider the circle C_u whose antipodal points are the intersection points of $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ with the ray from u_T through u. Note that C_u lies inside $\mathcal{A}_i$ and has diameter $2^{i-1} > \delta_i = 2^{i-2}/s$. Then Δ_u is any disk with diameter δ_i that contains u and that lies inside the closed disk delimited by C_u.

Suppose, for a contradiction, that there exist two neighbors u and v of u_T inside $\mathcal{A}_i$ such that the disks Δ_u and Δ_v intersect. Then $\pi_\Gamma(u,v) \geq 2^i$, since both the edges uu_T and vu_T are longer than $r_{i-1} = 2^{i-1}$. By the triangular inequality, $||uv||_\Gamma \leq 2 \cdot \delta_i = 2^{i-1}/s$. Hence $\frac{\pi_\Gamma(u,v)}{||uv||_\Gamma} \geq 2s$, while the spanning ratio of Γ is at most s. This contradiction proves that, for any two neighbors u and v of u_T inside $\mathcal{A}_i$, the disks Δ_u and Δ_v do not intersect.

The area of $\mathcal{A}_i$ is $\pi \cdot (r_i^2 - r_{i-1}^2) = \pi \cdot (2^{2i} - 2^{2i-2}) = 3\pi \cdot (2^{2i-2})$. Since each disk Δ_u lying inside $\mathcal{A}_i$ has area $\pi \cdot (2^{2i-6}/s^2)$ and does not intersect any different disk Δ_v, it follows that $\mathcal{A}_i$ contains at most $\frac{3\pi \cdot (2^{2i-2}) \cdot s^2}{\pi \cdot 2^{2i-6}} = 48 \cdot s^2$ distinct disks Δ_u and hence at most $48 \cdot s^2$ neighbors of u_T. This proves the claim and concludes the proof of the theorem. □

Corollary 1. *Let S be an n-vertex star. For any $s \geq 1$, any straight-line drawing of S with spanning ratio at most s has edge-length ratio in $2^{\Omega(n/s^2)}$.*

The lower bound of Theorem 6 follows from Theorem 8 and from the fact that a tree with maximum degree d has toughness $1/d$. This concludes the proof of Theorem 6.

We now prove that trees with bounded maximum degree admit planar straight-line drawings with constant spanning ratio and polynomial edge-length ratio. The cost of planarity is found in the dependence on the maximum degree, which is worse than in Theorem 7.

Theorem 9. *For every $\epsilon > 0$, every n-vertex tree T with maximum degree d admits a planar straight-line drawing whose spanning ratio is at most $1+\epsilon$ and whose edge-length ratio is in $\mathcal{O}\left((2n)^{2+(d-2)\cdot\log_2(1+\lceil\frac{2}{\epsilon}\rceil)}\cdot\log_2 n\right)$.*

Proof: Let $\gamma = \lceil\frac{2}{\epsilon}\rceil$. If $d \leq 2$, then T is a path and a planar straight-line drawing with spanning ratio 1 and edge-length ratio 1 is trivially constructed. We can hence assume that $d \geq 3$. Root T at any leaf r; this ensures that every vertex of T has at most $d-1$ children. In order to avoid some technicalities in the upcoming algorithm, we also assume that every non-leaf vertex of T has at least two children. This is obtained by inserting a new child for each vertex of T with just one child; note that the *size* of the tree, i.e., its number of vertices, is less than doubled by this modification. We again call T the tree after this modification and by n its size. Clearly, no path between two vertices of the initial tree uses the newly inserted vertices, hence removing the inserted vertices together with their incident edges from a drawing with spanning ratio smaller than or equal to $1+\epsilon$ of the modified tree results in a drawing with spanning ratio smaller than or equal to $1+\epsilon$ of the initial tree.

Our construction is a "well-spaced" version of an algorithm by Shiloach [47]. Namely, we construct a planar straight-line drawing Γ of T in which (i) r is at the top-left corner of $\mathcal{B}(\Gamma)$, and (ii) for every vertex u of T, the path from u to r in T is (non-strictly) xy-monotone.

If $n = 1$, then Γ is obtained by placing r at any point in the plane. If $n > 1$, then let $r_1, r_2, \dots, r_k$ be the children of r, where $k \leq d-1$, let $T_1, T_2, \dots, T_k$ be the subtrees of T rooted at $r_1, r_2, \dots, r_k$, and let $n_1, n_2, \dots, n_k$ be the size of $T_1, T_2, \dots, T_k$, respectively. Assume, w.l.o.g. up to a relabeling, that $n_1 \leq n_2 \leq \dots \leq n_k$; hence, $n_i \leq n/2$ for $i = 1, 2, \dots, k-1$. Refer to Figure 8. Place r at any point in the plane. Inductively construct planar straight-line drawings $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ of $T_1, T_2, \dots, T_k$, respectively. Position Γ_1 so that r_1 is on the same vertical line as r, one unit below it; let d_1 be the width of Γ_1. Then, for $i = 2, \dots, k$, position Γ_i so that r_i is one unit below r and $\gamma \cdot (d_{i-1} + \log_2 n)$ units to the right of $\mathcal{B}_r(\Gamma_{i-1})$; denote by d_i the width of the bounding box of the drawings $\Gamma_1, \Gamma_2, \dots, \Gamma_i$. Finally, move Γ_k one unit above, so that r_k is on the same horizontal line as r.

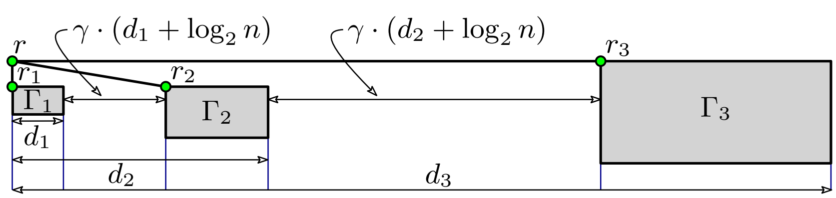

Fig. 8: Inductive construction of Γ. In this example $k = 3$.

We now analyze the properties of Γ. By construction, we have that Γ is a straight-line drawing. The planarity of Γ is easily proved by exploiting the fact that r_i is at the top-left corner of $\mathcal{B}(\Gamma_i)$ and that $r_1, r_2, \dots, r_{k-1}$ all lie one unit below r.

Height. Denote by $h(n)$ the maximum height of a drawing of an n-vertex tree constructed by the above algorithm. The same analysis as in [47] shows that $h(n) \leq \log_2 n$. This comes from $h(1) = 0$ and $h(n) \leq \max\{h(\frac{n}{2}) + 1, h(n-1)\}$ for $n \geq 2$.

Spanning ratio. We prove that, for any two vertices u and v that do not belong to the same subtree T_i, it holds true that $\frac{\pi_\Gamma(u,v)}{\|uv\|_\Gamma} \leq \frac{\gamma+2}{\gamma}$. This suffices to prove that the spanning ratio of Γ is at most $\frac{\gamma+2}{\gamma}$. Suppose that u belongs to a subtree T_i and v belongs to a subtree T_j, with $i < j$; the case in which one of u and v is r can be discussed analogously.

First, we have $\|uv\| \geq x_v + \gamma \cdot (d_{j-1} + \log_2 n)$, where x_v denotes the distance between v and $\mathcal{B}_l(\Gamma_j)$, while the second term is the distance between $\mathcal{B}_l(\Gamma_j)$ and $\mathcal{B}_r(\Gamma_{j-1})$.

Clearly, we have $\pi_\Gamma(u,v) = \pi_\Gamma(u,r) + \pi_\Gamma(r,v)$. The path between u and r (between v and r) is xy-monotone, hence $\pi_\Gamma(u,r)$ (resp. $\pi_\Gamma(v,r)$) is upper bounded by the horizontal distance plus the vertical distance between u and r (resp. between v and r). The vertical distance between u and r (between v and r) is at most $\log_2(n)$, since the height of Γ is at most $\log_2(n)$. The horizontal distance between u and r is at most $d_i \leq d_{j-1}$, while the one between v and r is $x_v + \gamma \cdot (d_{j-1} + \log_2 n) + d_{j-1}$. Hence, $\pi_\Gamma(u,v) \leq (d_{j-1} + \log_2 n) + (x_v + \gamma \cdot (d_{j-1} + \log_2 n) + d_{j-1} + \log_2 n) = x_v + (\gamma+2) \cdot (d_{j-1} + \log_2 n)$. Thus:

$$\frac{\pi_\Gamma(u,v)}{\|uv\|_\Gamma} \leq \frac{(\gamma+2) \cdot (\frac{x_v}{\gamma} + d_{j-1} + \log_2 n)}{\gamma \cdot (\frac{x_v}{\gamma} + d_{j-1} + \log_2 n)} \leq \frac{\gamma+2}{\gamma} \leq 1 + \epsilon.$$

Width. Let $w_1, \dots, w_k$ be the widths of $\Gamma_1, \dots, \Gamma_k$. By construction, $d_1 = w_1$ and, for each $j = 2, \dots, k$, we have $d_j = d_{j-1} + \gamma \cdot (d_{j-1} + \log_2 n) + w_j = (\gamma+1) \cdot d_{j-1} + \gamma \cdot \log_2 n + w_j$. Hence, by induction on j, we have $d_j = (\gamma+1)^{j-1} \cdot w_1 + (\gamma+1)^{j-2} \cdot w_2 + \dots + (\gamma+1) \cdot w_{j-1} + w_j + ((\gamma+1)^{j-1} - 1) \cdot \log_2 n$. In particular, the width of Γ is equal to d_k and hence to:

$$\sum_{i=1}^{k} ((\gamma+1)^{k-i} \cdot w_i) + ((\gamma+1)^{k-1} - 1) \cdot \log_2 n. \tag{1}$$

For the reminder of the proof, we introduce the notation $k_1 = k$ and $n_{1,i} = n_i$, for $i = 1, 2, \dots, k_1$. Recall that $k_1 \leq d-1$. Denote by $w(n)$ the maximum width of a drawing of an n-vertex tree constructed by the above algorithm. By construction, we have $w(1) = 0$. For $n \geq 2$, by Equality 1, we get:

$$w(n) \leq (\gamma+1)^{d-2} \cdot \sum_{i=1}^{k_1-1} w(n_{1,i}) + w(n_{1,k_1}) + (\gamma+1)^{d-2} \cdot \log_2 n. \tag{2}$$

Let $r_{2,1}, r_{2,2}, \dots, r_{2,k_2}$ be the children of r_k, where $k_2 \leq d-1$, and let $n_{2,1}, n_{2,2}, \dots, n_{2,k_2}$ be size of the subtrees $T_{2,1}, T_{2,2}, \dots, T_{2,k_2}$ of T rooted $r_{2,1}, r_{2,2}, \dots, r_{2,k_2}$, respectively. Assume, w.l.o.g., that $n_{2,1} \leq n_{2,2} \leq \dots \leq n_{2,k_2}$; hence, $n_{2,i} \leq n/2$ for $i = 1, 2, \dots, k_2 - 1$. By the same argument used to derive Inequality 2, we get that the term $w(n_{1,k_1})$ in Inequality 2 can be replaced by $(\gamma+1)^{d-2} \cdot \sum_{i=1}^{k_2-1} w(n_{2,i}) + w(n_{2,k_2}) + (\gamma+1)^{d-2} \cdot \log_2 n$, hence we get

$$w(n) \leq (\gamma+1)^{d-2} \cdot \left(\sum_{i=1}^{k_1-1} w(n_{1,i}) + \sum_{i=1}^{k_2-1} w(n_{2,i}) \right) + w(n_{2,k_2}) + 2 \cdot (\gamma+1)^{d-2} \cdot \log_2 n. \tag{3}$$

Again, the term $w(n_{2,k_2})$ in Inequality 3 can be replaced by $(\gamma+1)^{d-2} \cdot \sum_{i=1}^{k_3-1} w(n_{3,i}) + w(n_{3,k_3}) + (\gamma+1)^{d-2} \cdot \log_2 n$, where $n_{3,1}, n_{3,2}, \dots, n_{3,k_3}$ are the sizes of the subtrees of T rooted at the children of r_{2,k_2}, with $k_3 \leq d-1$ and $n_{3,i} \leq n/2$ for $i = 1, 2, \dots, k_3 - 1$. The repetition of this argument eventually leads to the inequality:

$$w(n) \leq (\gamma+1)^{d-2} \cdot \left(\sum_{i=1}^{k_1-1} w(n_{1,i}) + \sum_{i=1}^{k_2-1} w(n_{2,i}) + \dots + \sum_{i=1}^{k_t-1} w(n_{t,i}) \right) + t \cdot (\gamma+1)^{d-2} \cdot \log_2 n,$$

where t is the index at which r_{t,k_t} has no children, hence $n_{t,k_t} = 1$ and $w(n_{t,k_t}) = 0$. Since $t \leq n-1$, we get

$$w(n) \leq (\gamma+1)^{d-2} \cdot \sum_{i,j} w(n_{j,i}) + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n, \tag{4}$$

where the sum is defined over all pair of integers j and i such that $1 \leq j \leq t$ and $1 \leq i \leq k_j - 1$.

We prove, by induction on n, that $w(n) \leq f(n) := \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot n^2 \cdot \log_2 n$. This is trivial when $n = 1$, given that $w(1) = 0$. Assume now that $n > 1$. By Inequality 4 and by induction, we get $w(n) \leq (\gamma+1)^{d-2} \cdot \sum_{j,i} \left(\left((\gamma+1)^{d-2}\right)^{\log_2 n_{j,i}} \cdot n_{j,i}^2 \cdot \log_2 n_{j,i}\right) + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n$. Since $n_{j,i} \leq n/2 < n$, we get $w(n) \leq (\gamma+1)^{d-2} \cdot \left((\gamma+1)^{d-2}\right)^{\log_2 (n/2)} \cdot \sum_{j,i} n_{j,i}^2 \cdot \log_2 n + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n = \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot \sum_{j,i} n_{j,i}^2 \cdot \log_2 n + (\gamma+1)^{d-2} \cdot (n-1) \cdot \log_2 n$. Since any two subtrees $T_{j,i}$ are disjoint, we get that $\sum n_{j,i} \leq n-1$, and hence $\sum_{j,i} n_{j,i}^2 \leq (n-1)^2$. Thus, $w(n) \leq \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot ((n-1)^2 + (n-1)) \cdot \log_2 n \leq \left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot n^2 \cdot \log_2 n$. This completes the induction and the analysis of the width of Γ.

Edge-length ratio. By construction, the length of each edge connecting r to a child is larger than or equal to 1, hence the same is true for every edge of T. Thus, the edge-length ratio of Γ is upper bounded by the maximum length of an edge of T. In turn, this is at most the sum of the height plus the width of Γ, which is in $\mathcal{O}\left(\left((\gamma+1)^{d-2}\right)^{\log_2 n} \cdot n^2 \cdot \log_2 n\right)$, as proved above. The factor $\left((\gamma+1)^{d-2}\right)^{\log_2 n}$ can be rewritten as $n^{(d-2)\cdot \log_2(\gamma+1)}$. The bound claimed in the statement is then obtained by substituting $\gamma = \lceil \frac{2}{\epsilon} \rceil$ and by observing that the value of n used in the calculations is at most twice the size of the initial tree. □

6 Open Problems

Our research raises a number of open problems which might be worth studying.

First, the bounds in Theorem 6 relating the toughness to the edge-length ratio of a drawing with constant spanning ratio are not tight; it would hence be interesting to improve them.

Second, we believe that there is still much to be understood about the edge-length ratio of planar straight-line drawings with constant spanning ratio. Theorem 9 shows that planar straight-line drawings with constant spanning ratio and polynomial edge-length ratio exist for bounded-degree trees. We also observe that every n-vertex 2-connected outerplanar graph G admits a planar straight-line drawing with spanning ratio at most $\sqrt{2}$ and edge-length ratio in $\mathcal{O}(n^{1.5})$; this can be achieved by placing the vertices of G, in the order given by the Hamiltonian cycle of G, at the vertices of a lattice xy-monotone polygonal curve; see, e.g., [2]. Further, it is known that Schnyder drawings are 2-spanners [13], hence every n-vertex 3-connected planar graph admits a planar straight-line drawing with spanning ratio at most 2 and edge-length ratio in $\mathcal{O}(n)$; see [11,46]. Do 3-connected planar graphs (or even just maximal planar graphs) admit planar straight-line drawings with spanning ratio smaller than 2 (and possibly arbitrarily close to 1) and polynomial edge-length ratio? Is it possible to extend Theorem 6 by proving that a planar straight-line drawing with constant spanning ratio and polynomial edge-length ratio exists for every planar graph with bounded toughness?

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