Matching Shapes with a Reference Point

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Abstract

For two given point sets, we present a very simple (almost trivial) algorithm to translate one set so that the Hausdorff distance between the two sets is not larger than a constant factor times the minimum Hausdorff distance which can be achieved in this way. The algorithm just matches the so-called Steiner points of the two sets.

The focus of our paper is the general study of reference points (like the Steiner point) and their properties with respect to shape matching.

For more general transformations than translations, our method eliminates several degrees of freedom from the problem and thus yields good matchings with improved time bounds.

1 Introduction

This work is motivated by a problem that is typical in application areas such as computer vision or pattern recognition, namely, given two figures A, B, to determine how much they "resemble each other".

Here, a "figure" will be a union of finitely many points and line segments in $\mathbb{R}^2$ or triangles in $\mathbb{R}^3$. Note that sets of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ or surfaces in $\mathbb{R}^3$ can be approximated arbitrarily closely by these objects. As a measure for "resemblance" we will use the *Hausdorff-metric* δ_H, which is a somehow natural distance measure and gives reasonable results in practice (see [HKR]). It is defined as follows:

Definition 1 *Let A, B be compact subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then*

$$\widetilde{\delta_H}(A,B) := \max_{a\in A}\ \min_{b\in B}\ \|a-b\|,$$

where $\|\cdot\|$ is the Euclidean norm and the Hausdorff-distance is defined as

$$\delta_H(A,B) := \max\{\widetilde{\delta_H}(A,B), \widetilde{\delta_H}(B,A)\}.$$

If A and B consist of n and m line segments in the plane, their Hausdorff-distance $\delta_H(A,B)$ can be computed in time $O((n+m)\log(n+m))$ (cf. [ABB]). However, it is more natural to assume that A and B are not fixed but can be moved by a *translation*, by a *rigid motion* (translation and rotation) or even transformed by a *similarity* (scaling and rigid motion) in order to match them as well as possible and then determine the minimal Hausdorff-distance. So, in general, we have a set $\mathcal{T}$ of allowed transformations and want to determine for given figures A and B:

$$\min_{T\in\mathcal{T}}\ \delta_H(A, T(B)).$$

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Note that, for similarities, it makes a difference if we exchange the sets A and B in this problem.

This problem of finding an optimal matching has been considered for the two-dimensional case in several previous articles: In [ABB], an algorithm of running time $O((nm)\log(nm)\log^*(nm))$ for the case that $\mathcal{T}$ is the set of translations along one fixed direction is described. An $O((nm)^2\log^3(nm))$ algorithm for arbitrary translations is presented in the paper [AST], which can be improved to $O(nm(n+m)\log(nm))$ if A and B are finite sets of points [HKS]. The paper [CGHKKK] gives an algorithm of time complexity $O((nm)^3\log^2(nm))$ for arbitrary rigid motions. The latter two algorithms use sophisticated and powerful tools like parametric search and therefore do not seem to be applicable in practice.

Here, we follow a different approach which was already used in [ABB]. We do not try to find an *optimal* solution but an *approximation* to the optimal one by simpler algorithms. More precisely, if the optimal matching yields Hausdorff-distance δ our algorithms will find a matching T such that

$$\delta_H(A, T(B)) \leq a\delta$$

for some constant $a > 1$. We call such a solution a *pseudooptimal matching* with *loss factor* a.

A somehow similar approach has been taken in several articles by Heffernan and Schirra ([H,HS]) considering matching of finite sets A, B of points under rigid motions. They consider both, the *decision problem*, whether for a given ε A (or a subset of a given size k) can be mapped in a 1-1-fashion into the ε-neighborhoods of points of B, and the *optimization problem*, namely to find the smallest ε of this kind. Both problems are solved very efficiently by allowing that the optimal ε is not found exactly, but within a small tolerance interval (*approximate* algorithms).

The aim of this paper is to work out the general idea of using "reference points" for pseudooptimal algorithms. Reference points were first introduced in [ABB] and used in [CKS] to obtain an O(mn log(mn)) approximate algorithm for the decision problem for sets of line segments. Here we will also present a reference point that gives better bounds than the one in [ABB] and can be applied to similarities and problems in three dimensions as well.

2 Reference Point Methods

Like in [ABB] pseudooptimal algorithms use suitable *reference points*, which we define for arbitrary dimension d as follows:

Definition 2 *Let C^d denote the set of compact subsets of $\mathbb{R}^d$, and let $\mathcal{T}$ be a set of transformations on $\mathbb{R}^d$. A mapping $s\colon C^d \to \mathbb{R}^d$ is called a* reference point *with respect to $\mathcal{T}$ if the following two conditions are satisfied:*

(a) s is equivariant *with respect to $\mathcal{T}$, i.e., for all $A, B \in C^d$and $T \in \mathcal{T}$ we have*

$$s(T(A)) = T(s(A)).$$

(b) There exists some constant $c \geq 0$ such that if for all $A, B \in C^d$,

$$\|s(A) - s(B)\| \leq c \cdot \delta_H(A, B).$$

In other words, s is a Lipschitz-continuous mapping between the metric spaces (C^d, δ_H) and $(\mathbb{R}^d, \|\cdot\|)$ with Lipschitz constant c. We call c the quality *of the reference point s.*

Based on the existence of a reference point for $\mathcal{T}$ we obtain the following algorithms for pseudooptimal matchings where $\mathcal{T}$ is the set of translations, rigid motions, and similarity transformations, respectively:

Algorithm T

1. Compute $s(A)$ and $s(B)$ and translate B by $s(A)-s(B)$ (so that $s(B)$ is mapped onto $s(A)$). Let B' be the image of B.
2. Output B' as the pseudooptimal solution (together with the Hausdorff-distance $\delta_H(A, B')$).

Algorithm R

1. as in Algorithm T.
2. Find an optimal matching of A and B' under rotations of B' around $s(A)$.
3. Output the solution B'' and the Hausdorff-distance $\delta_H(A, B'')$.

Algorithm S

1. as in Algorithm T.
2. determine the diameters $d(A)$ and $d(B)$ and scale B' by $\alpha := d(A)/d(B)$ around the center $s(A)$.
3. as Step 2 in algorithm R with the scaled image of B'.
4. as Step 3 in algorithm R.

As the algorithms R and S are formulated, they look only for *proper* rigid motions and positive similarities, respectively. Reflections can be included by simply running the algorithm a second time with a reflected copy of A.

These algorithms are simpler than the ones for finding the optimal solutions, since after Step 1 the matchings are restricted to ones leaving the reference point invariant. In d dimensions this eliminates d degrees of freedom. The qualities and running times of these algorithms are as follows:

Theorem 1 *Suppose that we can determine a reference point of quality c for the sets of transformations $\mathcal{T}$ in the Algorithms T, R, and S in linear time. In the case of similarity transformations also assume that $s(A)$ always lies within the convex hull $conv(A)$.*

(a) Algorithm T finds a pseudooptimal matching for translations with loss factor $a = c + 1$.

(b) Algorithm R finds a pseudooptimal matching for rigid motions with loss factor $a = c + 1$.

(c) Algorithm S finds a pseudooptimal matching for similarity transformations with loss factor $a = c + 3$.

In the plane, the running times for two sets of n and m points and line segments are $O((n+m)\log(n+m))$ for Algorithm T and $O(nm\log(nm)\log^*(nm))$ for Algorithms R and S. In three dimensions, where A and B are sets of triangles, the running times become $O(mn)$ for Algorithm T and $O((nm)^3 H(n,m))$ for Algorithms R and S. Here $H(n,m)$ is the time to compute the Hausdorff distance.

Note that an upper bound of $O((n^2m + nm^2)\, log^2(nm))$ for $H(n,m)$ is known [AG].

For the proof of the theorem we need the following lemmas, which can be shown by elementary geometrical considerations.

Lemma 2 *Let $B \subset \mathbb{R}^d$ be a compact set with diameter $d(B)$, and let p be a point in its convex hull $conv(B)$. Let τ_1, τ_2 be homotheties (scalings) with center p and ratios (scaling factors) α_1 and α_2, respectively. Then*

$$\delta_H(\tau_1(B), \tau_2(B)) \leq |(\alpha_1 - \alpha_2)d(B)|.$$

Lemma 3 *If $A, B \subset \mathbb{R}^d$ are compact sets with diameters $d(A)$ and $d(B)$, respectively, then*

$$|d(A) - d(B)| \leq 2\delta_H(A, B).$$

Proof: This holds because B is contained in the δ-neighborhood of A and vice versa. $\square$

Proof of Theorem 1: We prove only (c) which implicitly contains the proofs for (a) and (b). Consider an optimal similarity transformation S_{opt}. It can be written as $S_{\text{opt}} = \tau_{\text{opt}} \circ T_{\text{opt}}$, where T_{opt} a rigid motion and τ_{opt} is a homothety with ratio α'. Let δ be the optimal Hausdorff-distance $\delta = \delta_H(A, S_{\text{opt}}(B))$. Then

$$\|s(A) - s(S_{\text{opt}}(B))\| \leq c\delta.$$

Let t be the translation by $s(A) - s(S_{\text{opt}}(B))$; then $\widetilde{S} = t \circ S_{\text{opt}}$ is a similarity transformation mapping $s(B)$ onto $s(A)$ and

$$(1) \qquad \delta_H(A, \widetilde{S}(B)) \leq (c+1)\delta.$$

Write $\widetilde{S}$ as $\widetilde{S} = \widetilde{\tau} \circ \widetilde{T}$, where $\widetilde{T}$ a rigid motion mapping $s(B)$ onto $s(A)$ and $\widetilde{\tau}$ is a homothety with center $s(A)$ and ratio α'. Let $\alpha = d(A)/d(B)$ as in Algorithm S, τ the homothety with center $s(A)$ and ratio α, and $S = \tau \circ \widetilde{T}$. Then

$$(2) \ \delta_H(A, S(B)) \leq \delta_H(A, \widetilde{S}(B)) + \delta_H(\widetilde{S}(B), S(B))$$

Now,

$$\begin{aligned}
(3) \ \delta_H(\widetilde{S}(B), S(B)) &= \delta_H\left(\widetilde{\tau}(\widetilde{T}(B)),\ \tau(\widetilde{T}(B))\right) \\
&\leq \left|(\alpha' - \alpha)d(\widetilde{T}(B))\right|, \text{ by Lemma 2} \\
&= |(\alpha' - \alpha)d(B)| = |\alpha' d(B) - d(A)| \\
&= \left|d(S_{\text{opt}}(B)) - d(A)\right|, \\
&\qquad \text{(since } \alpha' \text{ was the ratio of } S_{\text{opt}}) \\
&\leq 2 \cdot \delta_H(S_{\text{opt}}(B), A) = 2\delta,
\end{aligned}$$

by Lemma 3. From (1), (2), and (3) we have

$$(4) \qquad \delta_H(A, S(B)) \leq (c+3)\delta$$

for a similarity transformation S composed of a rigid motion that maps $s(B)$ onto $s(A)$ and a homothety with center $s(A)$ and ratio α. Since Algorithm S finds the optimum among these similarity transformations the bound (4) holds for it, as well.

For the time bound we observe that Step 1 can be done in linear time, and Step 2 in $O(n\log n + m\log m)$. (In three dimensions, the diameters can

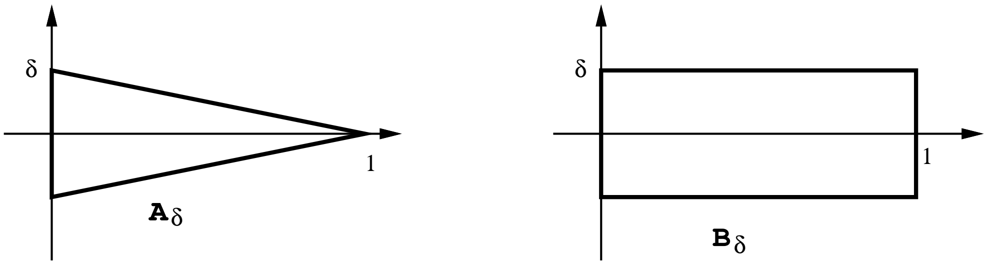

Figure 1: $s(A_\delta) = (1/3, 0)$, $s(B_\delta) = (1/2, 0)$, where s is either the center of the smallest enclosing ellipse or the centroid of the figure. This can be seen by applying an affine transformation which maps A_δ to an equilateral triangle or B_δ to a square.

be computed trivially in $O(n^2 + m^2)$ time.) Step 3, finding the optimal matching under rotations, can be done in time $O(nm \log(nm) \log^*(nm))$ in the plane by the algorithm of [ABB], whose analysis relies on Davenport-Schinzel sequences. This algorithm explicitly computes the Hausdorff-distance, so Step 4 is for free. In 3-space, we have rotations in $\mathbb{R}^3$ around a fixed center, which is an optimal matching problem with 2 degrees of freedom. It can be solved in time $O((nm)^3 H(n, m))$ by methods of [ABB]. $\boxminus$

3 The Steiner Point

The previous section would be useless if it were not possible to find suitable reference points. In [ABB] it was observed that in the two-dimensional case the point $s(A) = (x_{\max}, y_{\max})$, where $x_{\max}$ and $y_{\max}$ are the maximal x- and y-coordinates of points in A, is a reference point of quality $\sqrt{2}$ for translations.

For rigid motions the situation is not as easy. We will first list a few points that come to mind but turn out not to be reference points. In fact, for arbitrary small $\delta > 0$, we can construct figures A_δ, B_δ for which $\delta_H(A_\delta, B_\delta) \le \delta$, but $\|s(A_\delta) - s(B_\delta)\|$ is not in $O(\delta)$ or does not even converge to 0 for $\delta \to 0$. Some of these counterexamples are from [B], some are new:

$s(A)$ is the centroid of the	$\|s(A_\delta) - s(B_\delta)\|$
a) vertices of the convex hull	$\Omega(1)$
b) convex hull	$\Omega(1)$
c) smallest enclosing circle	$\Omega(\sqrt{\delta})$
d) smallest enclosing ellipse	$\Omega(1)$
e) smallest enclosing rectangle	$\Omega(1)$

Figure 1, for example, shows A_δ, B_δ for cases b) and d). In [ABB] also a positive example of a reference point for rigid motions in two dimensions was given: the centroid of the boundary of the convex hull. It was shown that this reference point is of quality at most $4\pi + 3 \approx 15.57$. Here, we will give a reference point which works even for similarity transformations, is easy to compute, can be generalized to higher dimensions, and whose quality is $4/\pi \approx 1.27$.

First we observe that we can without loss of generality restrict our attention to *convex* figures. In fact, in [ABB] it was shown that for any two compact sets A, B:

$$\delta_H(conv(A), conv(B)) \le \delta_H(A, B).$$

From that it follows easily that a reference point for the convex hull of a compact set A is a reference point for A, as well.

Our candidate for a reference point is the so-called *Steiner point*, which has been investigated intensively in the field of convex geometry [G, S, Sch].

Definition 3 *We denote by B^d the d-dimensional unit ball and by S^{d-1} its boundary, the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$.*

Let A be a convex body (convex and compact subset) in $\mathbb{R}^d$. The support function $h_A\colon \mathbb{R}^d \to \mathbb{R}$ *of A is given by*

$$h_A(\mathbf{u}) = \max_{\mathbf{a} \in A} \langle \mathbf{a}, \mathbf{u} \rangle$$

(see Figure 2).

The Steiner point $s(A)$ *of A is defined as*

$$s(A) = \frac{d}{\mathrm{Vol}(S^{d-1})} \int_{S^{d-1}} h_A(\mathbf{u}) \mathbf{u} \, d\omega(\mathbf{u}),$$

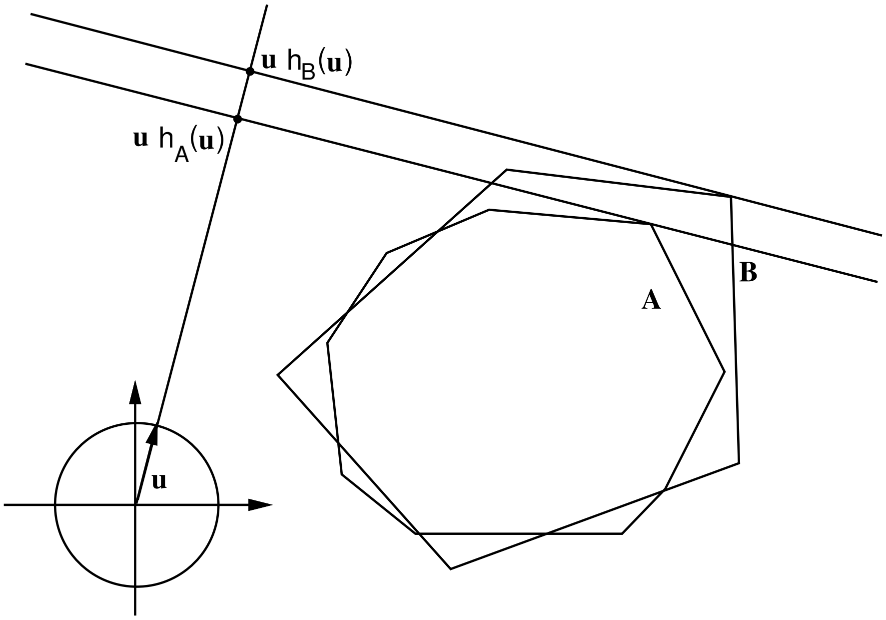

Figure 2: The support functions $h_A(\mathbf{u})$, $h_B(\mathbf{u})$ of two convex bodies A, B.

where $d\omega(\mathbf{u})$ is the surface element of S^{d-1}.

For a non-convex compact set $A \in \mathbb{R}^d$, we define the support function and hence the Steiner point by the same definitions. They coincide with the support function and the Steiner point of the convex hull conv(A).

For the Steiner point we obtain

Theorem 4 *The Steiner point is a reference point for similarity transformations in arbitrary dimension d. For $d = 2$, its quality is $4/\pi$, for $d = 3$ it is $3/2$, for arbitrary d it is less than $\sqrt{2/\pi} \cdot \sqrt{d+1}$.*

Proof: The equivariance of the Steiner point under similarity transformations is well known [G, Sch]. For the bound on the quality, we observe that for two convex bodies A and B, $\|h_A(\mathbf{u}) - h_B(\mathbf{u})\| \leq \delta_H(A, B) =: \delta$ for any $\mathbf{u} \in S^{d-1}$ (see Figure 2).

Now let $\mathbf{p} = s(A) - s(B)$, and consider the inner product of $\mathbf{p}$ with an arbitrary unit vector $\mathbf{e}$. Without loss of generality we assume that $\mathbf{e} = (0, \ldots, 0, 1)$ is the unit vector in the d-th coordinate direction.

$$
\begin{aligned}
\langle \mathbf{p}, \mathbf{e} \rangle &= \langle s(A) - s(B), \mathbf{e} \rangle \\
&= \frac{d}{\mathrm{Vol}(S^{d-1})} \int_{S^{d-1}} (h_A(\mathbf{u}) - h_B(\mathbf{u})) \langle \mathbf{u}, \mathbf{e} \rangle \, d\omega(\mathbf{u}) \\
&\leq \frac{d}{\mathrm{Vol}(S^{d-1})} \cdot \Bigg[\int_{\substack{S^{d-1} \\ u_d \geq 0}} (+\delta) \langle \mathbf{u}, \mathbf{e} \rangle \, d\omega(\mathbf{u}) + \int_{\substack{S^{d-1} \\ u_d \leq 0}} (-\delta) \langle \mathbf{u}, \mathbf{e} \rangle \, d\omega(\mathbf{u}) \Bigg] \\
&= \delta d \cdot \Bigg[\frac{1}{\mathrm{Vol}(S^{d-1})/2} \cdot \int_{\substack{S^{d-1} \\ u_d \geq 0}} \langle \mathbf{u}, \mathbf{e} \rangle \, d\omega(\mathbf{u}) \Bigg]
\end{aligned}
$$

The expression in brackets in the last line is nothing but the d-th coordinate of the center of gravity of the upper unit half-sphere. If we compute the integral by projecting away the d-th coordinate and integrate over $(x_1, x_2, \ldots, x_{d-1}) \in B^{d-1} \subseteq \mathbb{R}^{d-1}$, a straightforward calculation gives that the surface element is transformed by

$$
d\omega = \frac{1}{\langle \mathbf{u}, \mathbf{e} \rangle} dx_1 dx_2 \cdots dx_{d-1}.
$$

Thus the integral turns out to be just the volume of B^{d-1}, and we get the following bound on $\langle \mathbf{p}, \mathbf{e} \rangle$:

$$
\begin{aligned}
\langle \mathbf{p}, \mathbf{e} \rangle \leq \delta d \cdot \frac{\mathrm{Vol}(B^{d-1})}{\mathrm{Vol}(S^{d-1})/2} &= \delta 2d \cdot \frac{\pi^{(d-1)/2} / \Gamma\left(\frac{d+1}{2}\right)}{d\pi^{d/2} / \Gamma\left(\frac{d+2}{2}\right)} \\
&= \delta \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)},
\end{aligned}
$$

using the formulas for the volume and surface area of higher-dimensional balls and spheres. This last expression is a bound on the length of $\mathbf{p}$ since $\mathbf{e}$ was in fact an arbitrary unit vector and therefore can be substituted by $\mathbf{p}/\|\mathbf{p}\|$. The values $\Gamma(3/2) = \sqrt{\pi}/2$, $\Gamma(2) = 1$, and $\Gamma(5/2) = 3\sqrt{\pi}/4$ give the claimed quality bounds for $d = 2$ and $d = 3$. The quotient of the two Γ-functions is between $\sqrt{d/2}$ and $\sqrt{(d+1)/2}$, and this gives the general bound. $\square$

From this proof one can see how to construct an example showing that the bound cannot be improved: S^{n-1} must be divided into two half-spheres, and $h_A(\mathbf{u}) - h_B(\mathbf{u})$ will ideally always be equal to $+\delta$ or $-\delta$, depending on the half-sphere in which $\mathbf{u}$ lies. Figure 3 shows two two-dimensional

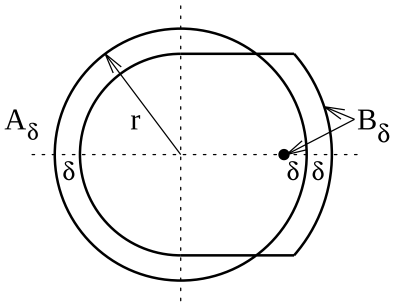

Figure 3: $\|s(A_\delta) - s(B_\delta)\|$ is close to $4/\pi \cdot \delta$.

point sets A_δ and B_δ. A_δ is just a circle of radius r, and B_δ consists of a "distorted" circle and an additional point. If we allow to apply any similarity to B_δ in order to minimize the Hausdorff distance from A_δ, the optimal position is as shown in Figure 3, and the Hausdorff distance is δ. The distance of the Steiner point $s(B_\delta)$ from the center $s(A_\delta)$ of the circle can be calculated as $2/\pi \cdot \sqrt{r\delta} \arccos \frac{r-\delta}{r+\delta}$, which approaches $4/\pi \cdot \delta$ as r goes to ∞. If one lets the two Steiner points coincide, the Hausdorff distance rises by this amount, showing that $1+4/\pi$ is indeed the loss factor of Algorithm T. Since A_δ is rotation-symmetric, this holds also for Algorithm R. The above construction generalizes easily to higher dimensions.

The following theorem is well-known [G, S].

Theorem 5 *The Steiner point of a convex polytope is the weighted sum of its vertices, where the weight of vertex v is that fraction of the surface of the unit sphere that lies between the unit vectors normal to the hyperplanes meeting at v (the normalized exterior angle at v). (See Figure 4 for a two-dimensional example.)*

For smooth convex bodies, the Steiner point can also be defined as the centroid of a non-uniform mass distribution on the boundary, where the density is the (Gaussian) curvature.

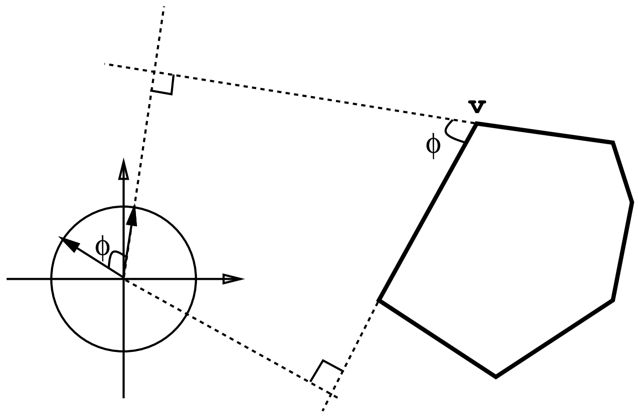

Figure 4: The weight of vertex v is $\frac{\phi}{2\pi}$.

Combining Theorems 1, 4, and 5 we get:

Theorem 6 *Let A and B be sets of n and m line segments in $d = 2$ dimensions or n and m triangles in $d = 3$ dimensions. Then pseudooptimal matchings can be found for A and B applying the corresponding algorithms of Section 2 as indicated in the following table.*

$\mathcal{T}$	running time	loss factor
translations		
$d = 2$	$O((n+m)\log(n+m))$	$4/\pi + 1$
$d = 3$	$O(H(n,m))$	2.5
rigid motions		
$d = 2$	$O(nm\log(nm)\log^*(nm))$	$4/\pi + 1$
$d = 3$	$O((nm)^3 H(n,m))$	2.5
similarities		
$d = 2$	$O(nm\log(nm)\log^*(nm))$	$4/\pi + 3$
$d = 3$	$O((nm)^3 H(n,m))$	4.5

Proof: For the proof note that the Steiner point for a convex polygon or polytope can be computed in linear time because of Theorem 5, after the convex hulls have been constructed in $O(n\log n + m\log m)$ time. The bound of $O(H(n,m))$ (as in

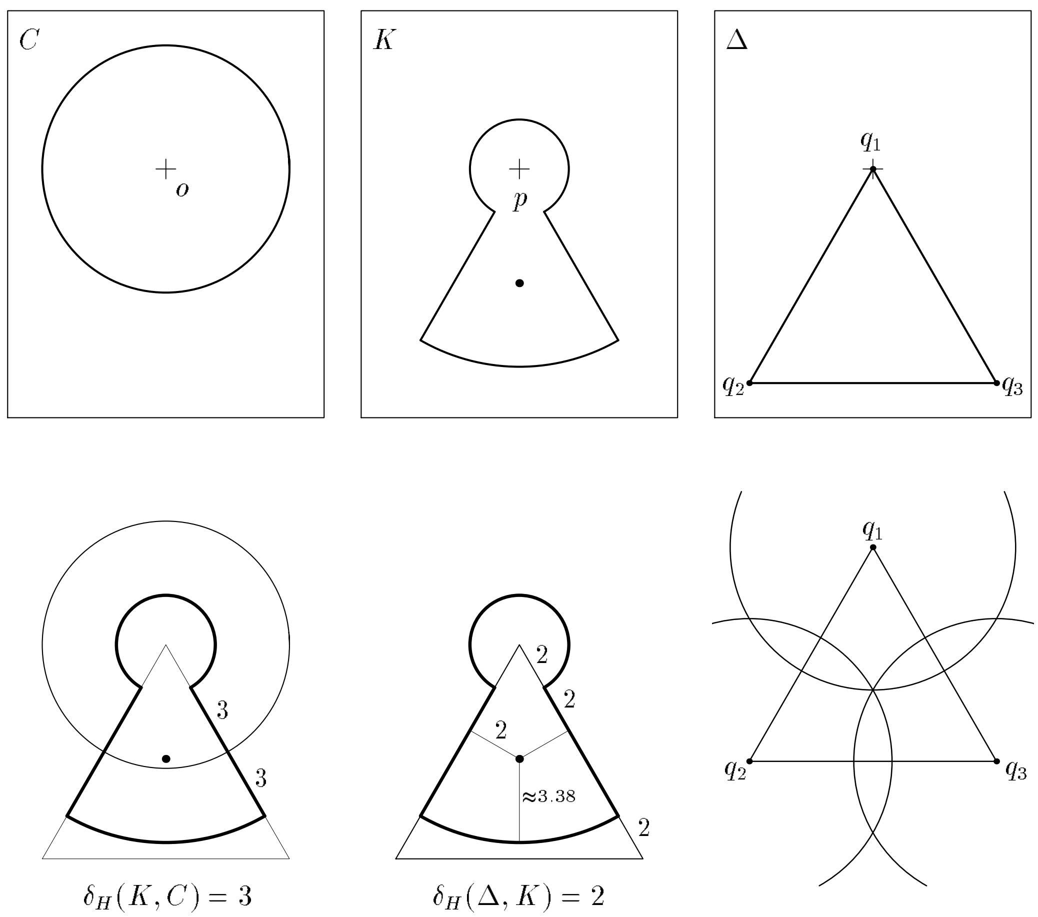

Figure 5: A lower bound for the quality of a reference point.

Theorem 1) for translations in three dimensions comes from the final computation of the Hausdorff distance. Just finding the pseudooptimal translation takes only $O(n \log n + m \log m)$ time. □

4 Lower Bounds

For translations in two dimensions the following lower bound on the quality of reference points can be shown:

Theorem 7 *The quality c of any reference point for translations (Algorithm T) in two (or higher) dimensions cannot be better (i.e., smaller) than $\sqrt{4/3}$.*

Proof: We consider three point sets (see the upper part of Figure 5): A circle C with center o and radius 5; a keyhole-shaped figure K with two circular arcs of radii 2 and 8 and opening angles of $5\pi/3$ and $\pi/3$, respectively, and with an additional point inside; and finally an equilateral triangle $\Delta = q_1 q_2 q_3$ with side length 10. The optimal matching between C and K and between K and Δ is shown in the lower part of Figure 5. It is obtained by superimposing the points o and p (marked by crosses) or the points p and q_1, respectively. The dimensions given in the lower middle part of Figure 5 also exhibit the position of the extra point in the keyhole K.

Let us first assume that the reference point $s(C)$ of the circle is its center o. Since $\delta_H(C,K) = 3$, the reference point $s(K)$ of K must lie in a circle of radius $3c$ around p. Similarly, $s(\Delta)$ must lie in a circle of radius $2c$ around $s(K)$, i.e., in a circle of radius $5c$ around q_1. If we turn K by 120° in both ways, we can conclude in a similar way that $s(\Delta)$ must lie in circles of radius $5c$ around q_2 and around q_3. By equivariance with respect to translations, the triangle Δ has only a single reference point, and therefore the three circles must intersect, as shown in the lower right part of Figure 5. This means that c must be at least $\sqrt{4/3} \approx 1.155$.

If the reference point $s(C)$ is not the center o, the only difference is that the centers of the final three circles will be translated by the respective amount. $\square$

The above construction can be generalized to d dimensions to give a lower bound of $\sqrt{2d/(d+1)}$ on the quality c.

Comparing $\sqrt{4/3} \approx 1.155$ with the upper bound of $4/\pi \approx 1.27$ for the Steiner point, we note that the gap is quite small. We are quite sure that the lower bound can be improved, and we believe that the Steiner point is *the* optimal reference point with the best quality guarantee.

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