Enumerating Order Types for Small Point Sets with Applications*

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ABSTRACT

Order types are a means to characterize the combinatorial properties of a finite point configuration. In particular, the crossing properties of all straight-line segments spanned by a planar n-point set are reflected by its order type. We establish a complete and reliable data base for all possible order types of size $n = 10$ or less. The data base includes a realizing point set for each order type in small integer grid representation. To our knowledge, no such project has been carried out before.

We substantiate the usefulness of our data base by applying it to several problems in computational and combinatorial geometry. Problems concerning triangulations, simple polygonalizations, complete geometric graphs, and k-sets are addressed. This list of possible applications is not meant to be exhaustive. We believe our data base to be of value to many researchers who wish to examine their conjectures on small point configurations.

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Keywords

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1. INTRODUCTION

Many problems in computational and combinatorial geometry are based on finite sets of points in the Euclidean plane. For quite a large subclass, the combinatorial properties of the underlying point set S rather than its metric properties already determine the problem. In particular, the intersection properties of the line segments spanned by the set S turn out to be of importance. Examples include the concepts of triangulations, spanning trees, simple polygonalizations (crossing-free Hamiltonian cycles), k-sets, and many others. For several problems, like counting the number of triangulations of a given point set, no efficient algorithms are known. For others, like for k-sets, the combinatorial complexity is still unsettled. Sometimes even the existence of a solution has not yet been established, such as the question of whether any two given n-point sets (with the same number of extreme points) can be triangulated in an isomorphic manner.

To gain insight into the structure of hard problems, examples that are typical and/or extreme are often very helpful. To obtain these examples usually complete enumerations on all possible problem instances are performed. For the problems mentioned above this means to investigate all 'different' sets of points, where difference is with respect to the crossing properties of the complete geometric graph spanned by the set. To this end it is necessary to have a data base of all different point sets for small size n. The aim of this work is to provide such a data base for $n \leq 10$ which is both complete and correct, and to describe some improved results obtained from it. To the knowledge of the authors, no such data base has been available before.

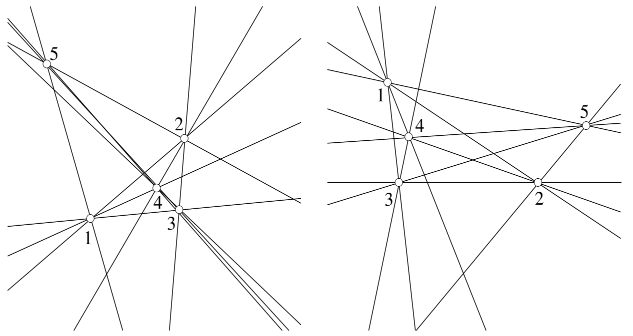

Figure 1: Two equivalent sets of 5 points

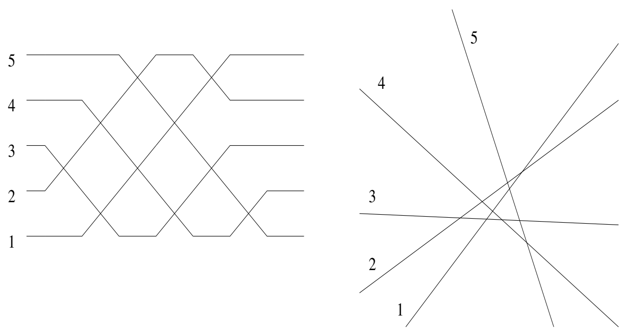

Figure 2: A wiring diagram that can be stretched

It is well known that crossing properties are exactly reflected by the order type of a point set, introduced in Goodman and Pollack [18]. The *order type* of a set $\{p_1, ..., p_n\}$ of points in general position[1] is a mapping that assigns to each ordered triple i, j, k in $\{1, ..., n\}$ the orientation (either clockwise or counter-clockwise) of the point triple p_i, p_j, p_k. Two point sets S_1 and S_2 are said to be *(combinatorially) equivalent* if they exhibit the same order types. That is, there is a bijection between S_1 and S_2 such that any triple in S_1 agrees in orientation with the corresponding triple in S_2. (See Figure 1 for an example.) Equivalently, two line segments spanned by S_1 cross if and only if the two corresponding segments for S_2 do. As a consequence, both sets have the same number of extreme points, any triangulation of S_1 is also a valid triangulation of S_2, and so on. Thus for the above-mentioned problems instead of investigating all (infinitely many) point sets it is sufficient to deal only with inequivalent sets. Still to enumerate or even to count these sets is a highly non-trivial task.

Let us point out some situations were the complete enumeration of all order types for small n leads to results for general problem size n.

The obvious case is when a counterexample can be provided that generalizes to larger n. There might exist counterexamples too large to be found by hand though small enough to be detected by checking all order types. For instance, a question raised by Arkin et al. [4] asks for how 'convex' the best polygonalization of an n-point set has to be in the worst case. More precisely, what is the minimum number $r(n)$ such that every n-point set admits a crossing-free (Hamiltonian) cycle with at most $r(n)$ reflex angles? The best known bounds are $\frac{n}{4} \leq r(n) \leq \frac{n}{2}$, and the lower bound example seems improvable by considering all sets of up to 10 points.

On the other hand, the non-existence of small counterexamples gives some evidence for the truth of a conjecture. For example, our tests did not lead to a counterexample for the isomorphism conjecture for triangulations mentioned above. As a result, any two sets of 8 points (or less) admit an isomorphic triangulation provided their number of convex hull points is the same.

[1] All point sets considered in this paper are assumed to be in *general position*, i.e., no three points are collinear. In addition, all line arrangements are assumed to be *simple*, i.e., no three lines pass through the same point and no two lines are parallel.

As another example, case analyses for problem instances of constant size are often encountered when proving some combinatorial property. This is particularly true for induction proofs if a sufficiently large induction basis need to be found. Consider the problem of finding a function $t(n)$ such that every set of n points exhibits $\Omega(t(n))$ triangulations. (The general position assumption is crucial here to avoid trivialities.) For $n \leq 10$, the minimum number of triangulations can be determined exactly by counting them for each possible order type. This serves for obtaining initial values for a recurrence relation for $t(n)$ in Aichholzer et al. [3] which then evaluates to $t(n) > 2^n$. This improves the best known results on $t(n)$. The point is that the quality of the initial values affects the asymptotic behavior of $t(n)$.

The recurrence relation itself exploits a new result on crossing families which has also been obtained from order types: every set of $n \geq 10$ points spans a crossing family of size 3 (i.e., three pairwise crossing line segments), and this bound is tight in n. This drastically improves on the previous best bound $n \geq 37$ obtainable from Tóth and Valtr's [24] formula for the size of subsets in convex position, and is a result of interest in its own right.

Our approach to generate all order types makes use of the duality[2] of point sets (in general position) and (simple) line arrangements in the Euclidean plane. A *line arrangement* is the dissection of the plane induced by a set of n straight lines. As no direct way to enumerate these structures is known, we first produce all non-isomorphic arrangements of so-called pseudolines. A set of *pseudolines* is a set of simple curves which pairwise cross at exactly one point. Handling pseudolines is relatively easy in view of their equivalent description by wiring diagrams; see, e.g., Goodman [17]. We can read off a corresponding *pseudo order type* from each pseudoline arrangement. (The intersection orders on all the pseudolines uniquely determine the orientation of all element triples). Back in the primal setting, this leads to a list of candidates guaranteed to contain all different order types.

The main problem, however, is to identify all the *realizable* order types in this list, that is, those which can actually be realized by a set of points. As a known phenomenon, a pseudoline arrangement need not be *stretchable*, i.e., isomorphic to some straight line arrangement. There exist non-stretchable arrangements already for 8 pseudolines; see, e.g.,

[2] Any of the well-known duality transforms may serve this purpose, although none of them leads to a bijection in order type.

Björner et al. [8]. As a consequence, our candidate list will contain non-realizable pseudo order types. Moreover, even if realizability has been decided for a particular candidate, how can we find a corresponding point set? We will provide answers to these questions, and a comparison to known related results, in the following section.

In some cases there is no need to identify the realizable order types. For instance, if a particular conjecture for point sets is true for all pseudo order types then it must be true for the subset of order types which actually can be realized by some set of points.

Let us point out that the situation gets conceptually and computationally easier in the projective plane where – unlike in the Euclidean plane – inequivalent order types directly correspond to non-isomorphic line arrangements, and isomorphism classes of pseudoline arrangements coincide with (reorientation classes of) rank 3 oriented matroids. Complete enumerations of the non-realizable classes of these matroids have been done for sizes $n \leq 10$; see [8, 17]. The Euclidean order types can be derived from their projective counterparts. Combinatorial descriptions of the latter are not available to us, however. Apart from that, the hard problem is to construct their geometric realizations.

2. ORDER TYPE ENUMERATION

The first part of this section describes the approach we chose for generating the order type data base for all point sets of size $n \leq 10$. The second part gives a quick view of the output and reports on the representation of the realizing point sets as well as on the reliability of our data base.

2.1 The approach

Our strategy for computing all different order types proceeds in the following three steps. Each step is further detailed and explained below.

- Generating a candidate list $\mathcal{C}$ which is guaranteed to contain each pseudo order type – and thus each realizable order type – exactly once. We utilize wiring diagrams to calculate a unique representation (by a λ-matrix) for each order type.
- Grouping the members of $\mathcal{C}$ into equivalence classes by correspondence to the same projective order type. In every class, either each or no order type is realizable. The numbers P_n of realizable projective classes of size n are known from the literature for $n \leq 10$.
- Realizing all realizable order types in $\mathcal{C}$. For each member of $\mathcal{C}$ we try to recover a realizing point set from its λ-matrix. A counter is kept for the number of realizable projective classes all of whose members have been realized already. The process is terminated when this number reaches P_n.

2.1.1 *Generating the candidate list*

Let S be a set of n points in the Euclidean plane. It is well known that the order type of S can be encoded by a so-called λ-*matrix*. For some fixed labelling of S, each entry $\lambda(i, j)$ of this matrix gives the number of points in S which lie to the left of the directed line ℓ_{ij} through points i and j. Goodman and Pollack [18] introduced λ-matrices and showed that they are valid representations of order types: to know how many points lie on a fixed side of each line ℓ_{ij} is sufficient for knowing which points these are.

As the λ-matrix (but not the order type) of a set S depends on its labelling, attention may be restricted to a fixed one. For technical reasons, the smallest λ-matrix of S in lexicographical order is chosen. We call the corresponding labelling $p_1, \ldots, p_n$ of S a *natural ordering*, because p_1 then is a point on the convex hull of S and $p_2, \ldots, p_n$ appear in clockwise order as seen from p_1. For a point set in general position (and we only consider such sets in this paper) its natural ordering is unique – up to self-symmetry of the point set. Our first aim is to generate the λ-matrices for all naturally ordered sets of fixed size $n \leq 10$.

Using a suitable point-line duality, any naturally ordered set $\{p_1, \ldots, p_n\}$ can be transformed into an ordered set $\{g_1, \ldots, g_n\}$ of directed lines with the following properties. (a) No line crossings lie to the left of g_1. (b) Each line g_i for $i > 1$ crosses g_1 from left to right. (c) For $1 \leq i < j \leq n$, the number of lines which have the crossing $g_i \cap g_j$ to their left is exactly $\lambda(i, j)$, the corresponding entry in the λ-matrix of $\{p_1, \ldots, p_n\}$. As a consequence, this λ-matrix can be read off the line arrangement formed by $\{g_1, \ldots, g_n\}$.

More precisely, the order σ_i in which each directed line g_i crosses all other lines already determines the λ-matrix. Note that for g_1 this order is fixed, $\sigma_1 = 2, 3, \ldots, n$. (This follows from properties (a) to (c)). So, in order to generate all desired λ-matrices for n points, it mainly suffices to generate all arrangements of $n-1$ directed lines which exhibit different crossing orders $\sigma_2, \ldots, \sigma_n$. A superset thereof, namely all arrangements of $n-1$ directed *pseudolines* with different crossing orders, can be computed relatively easily by means of their equivalent representation as *wiring diagrams*. We omit precise definitions of these concepts and refer to [18] and Figure 2 instead. Note that a pseudoline arrangement may as well be represented by a λ-matrix by using property (c) above.

In summary, unique λ-matrix representations for all pseudo order types of size n are obtained efficiently in this way.

2.1.2 *Grouping into projective classes*

Given some point set realization on the sphere of a *projective* order type, a realizing point set for each Euclidean order type which belongs to that projective class can be obtained by rotation of the sphere and central projection on the plane. To group our order type list $\mathcal{C}$ into projective classes we use a combinatorial simulation of this process.

To this end, a *rotation* of a given (Euclidean) order type T is defined for each index i that corresponds to an extreme point for T. A rotation for i simply reverses the orientation of all index triples that contain i. Its implementation in λ-matrix representation is straightforward. To be able to recognize identity of order types we lexicographically minimize each λ-matrix obtained from a rotation. Comparison of λ-matrices for identity then can be done quickly for each pair. All order types belonging to the same projective class can be obtained by rotation from each other. We use backtracking to ensure an exhaustive enumeration within each class. In this way, each member of the list $\mathcal{C}$ is assigned to its projective class by comparison of λ-matrices.

These classes exactly correspond to the (reorientation classes of) rank 3 oriented matroids whose number is known for sizes $n \leq 10$; see Table 1. Let us mention at this place

n	4	5	6	7	8	9	10
Projective Pseudo Order Types	1	1	4	11	135	4 382	312 356
– thereof non-realizable						1	242
= Projective Order Types	1	1	4	11	135	4 381	312 114
Euclidean Pseudo Order Types	2	3	16	135	3 315	158 830	14 320 182
– thereof non-realizable						13	10 635
= Euclidean Order Types	2	3	16	135	3 315	158 817	14 309 547

Table 1: Number of different order types of size n

that several multiple purpose methods for generating oriented matroids have been developed, the most recent ones being by Bokowski and Guedes de Oliveira [9] and by Finschi and Fukuda [14]. The advantages of the method we chose are its simplicity, transparency, and efficiency.

2.1.3 *Finding the realizing point sets*

Realizing a given order type means reconstructing a point set from a given λ-matrix that represents this order type. Deciding realizability (for order types in particular and oriented matroids in general) is an intriguing problem which is known to be NP-hard. Several heuristics have been developed; see [17] and references therein. For our case (rank 3 oriented matroids) a singly exponential algorithm exists but turned out to be too slow for our purposes. Instead, we used the following combination of methods with success.

In a first step, we applied an insertion strategy to obtain realizations of size n from realizations of size $n-1$, for $n \leq 10$. Suppose that, for each (realizable) order type T of size $n-1$, a corresponding point set $S(T)$ is available. Consider the arrangement $A(T)$ formed by $\binom{n-1}{2}$ lines, each passing through a pair of points in $S(T)$. Point sets of size n are now generated from $S(T)$ by placing an additional point in a cell of $A(T)$, for all possible cells. Carrying out this process for all order types of size $n-1$ leads to realizations for certain order types of size n. It is well known [8] that, in general, not all desired realizations are obtainable in this way; a principal difficulty with this approach is that the *geometry* of an $(n-1)$-point set $S(T)$ critically affects the *set of order types* of size n which actually get realized. To increase effectiveness we restarted the insertion method after random (but order type preserving) perturbations of the $(n-1)$-point sets.

We continue with scanning through the projective classes our candidate list $\mathcal{C}$ has been grouped into. If the number of classes having got at least one member realized coincides with P_n (the number of realizable projective classes) we continue to the next step. Else we decide upon each completely unrealized class by a different method: we try to realize one of its members directly, starting from scratch with a simulated annealing method. (Only the case $n = 10$, and there only a very small fraction of classes – including the non-realizable ones, of course – needed this treatment; see Subsection 2.2.) Moreover, and most important, we realized at least one member for each of the P_n realizable classes in one or the other way, and thus succeeded in distinguishing them from the non-realizable ones.

Finally, for each projective class T^o found to be realizable but still containing unrealized members, we do the following. There is a member T of T^o whose realizing point set $S(T)$ has been computed. We calculate from $S(T)$ a spherical point set $S(T^o)$ realizing T^o, and derive realizing point sets for the yet unrealized members of T^o by rotation and projection of $S(T^o)$.

2.2 Output and reliability

Table 1 gives a quick overview of our results in comparison with known other results. Lines 1 to 3 were taken from [17]. Order types that are reorientations of each other (that is, obtainable from each other by reversing the orientation for all index triples) are counted only once. Recall that the number of projective (pseudo) order types is just the number of isomorphism classes of projective (pseudo)line arrangements. The tremendous growth of Euclidean order types can be read off the lower part of the table. By the nature of our approach, we computed a combinatorial description for each of the objects counted in Table 1, along with a geometric representation of the object if it is realizable.

Table 2 lists the numbers of Euclidean order types according to the size h of the convex hull of the realizing point sets. The last line coincides with the respective line in Table 1. Table 3 refers to counting each reorientation pair of order types twice. Note that reorientation of an order type may or may not lead to its identical copy; thus reorientation leads to an increase only in the latter case.

It took 17 minutes on a 500 MHz Pentium III to generate all Euclidean pseudo order types of size $n = 9$, and to find realizing point sets for all but 13 of them, by using the insertion strategy followed by rotation within each projective class. The remaining 13 candidates turned out to belong to the same projective class (which therefore has to be the only existing non-realizable projective class), witnessing completeness of the task. As had to be expected, the situation turned out to be more complex for size $n = 10$. Generation of all pseudo order types plus partial realization by means of insertion took about 36 hours and left some 200.000 Euclidean pseudo order types unrealized. Most of the corresponding projective classes got some member realized, however, and could be completed quickly by applying rotation. In particular, only 251 projective classes remained without any realized member. To try to realize a first member, we had to invoke our simulated annealing routine for all these classes, as we had no information on which are the 242 classes known to be non-realizable from theory. We were successful for 9 classes within 60 hours which finally completed this task.

Whenever computing realizing point sets, care was taken to avoid large coordinates. In addition, emphasis was laid on finding point coordinate descriptions more handsome than being calculated by the basic algorithm. In particular, the simulated annealing routine was used to post-process the point sets. In their final form, our point sets enjoy the following properties.

h / n	4	5	6	7	8	9	10
3	1	1	6	49	1 178	55 235	4 876 476
4	1	1	6	59	1 468	70 475	6 319 019
5		1	3	22	570	28 232	2 628 738
6			1	4	90	4 552	450 176
7				1	8	311	33 969
8					1	11	1 146
9						1	22
10							1
Σ	2	3	16	135	3 315	158 817	14 309 547

Table 2: Number of Euclidean order types classified by extreme points

h / n	4	5	6	7	8	9	10
3	1	1	8	92	2 296	110 336	9 750 002
4	1	1	8	107	2 862	140 593	12 633 467
5		1	3	37	1 081	56 300	5 254 263
6			1	5	156	8 973	898 682
7				1	9	591	67 400
8					1	15	2 186
9						1	29
10							1
Σ	2	3	20	242	6 405	316 809	28 606 030

Table 3: As Table 2, but respecting reorientation

- *Compact grid representation.* For $n \leq 8$ all point sets are expressed with 1 byte coordinates, and for $n = 9$ and $n = 10$ with 16 bit (unsigned) integer coordinates. This should be contrasted with known negative results on the efficient grid embeddability of order types; see Goodman et al. [19]. Small integer coordinates ensure the efficiency and numerical stability of calculations based on our data base.
- *Coordinate uniqueness.* No two points in a set share the same x-coordinate or the same y-coordinate. Thus sorting the points by one of their coordinates is easy and unique. Moreover, the application of common duality transforms will not create parallel lines within a set.
- *Resolution.* The minimum Euclidean distance between two points in a set is greater than 4.
- *'Very' general position.* Apart from being in general position, every set guarantees a normal distance of at least 1 between each point and each line through two other points. Thus any circle passing through three points in a set will have a small radius.
- *No cocircularities.* No four points in a set lie on a common circle. Therefore, in-circle tests as, e.g., being used for Delaunay triangulation based algorithms, can be carried out uniquely.

We close this section by addressing the issue of reliability of our data base. The theoretical correctness of our approach has been argued for already. This raises the question of how reliable are our implementations. The algorithms used in Subsections 2.1.1 and 2.1.2 for generating all Euclidean pseudo order types, and for grouping them into projective classes respectively, are of purely combinatorial nature. Evidence for their correct implementation is gained from the correctly computed total numbers of projective pseudo order types (which have been known before; see Table 1). As a byproduct, the size (number of members) of each projective class is computed in a reliable way.

The critical part are the geometric computations in Subsection 2.1.3. We checked their correctness as follows. For each point set in its final grid form, we recalculated its λ-matrix and minimized it lexicographically. (In fact, we considered both the set and its reflected counterpart, and selected the smaller one of the two minimized matrices.) This involves only numerically reliable integer manipulations. By sorting the obtained matrices lexicographically, and checking for identity of neighbors, we ensured that our point sets represent pairwise different order types. Finally, we checked that the number of Euclidean order types we have realized is correct, i.e., that no order type is missing. To this end, we added up the sizes (pre-computed as mentioned before) of all projective classes which have some member realized as a grid point set.

In summary a complete, user-friendly, and reliable data base for all order types of sizes $n \leq 10$ has been obtained. The data base is made public on the web at `http://www.igi.TUGraz.at/oaich/triangulations/ordertypes.html`. Due to space limitations, the grid point sets of size 10 are not accessible on-line but rather have been stored on a CD which is available from us upon request.

3. APPLICATIONS AND FIRST RESULTS

The present section reports on applications and first results obtained from our order type data base. The problems addressed are mostly related to triangulations but the list may easily be extended to questions of a different flavour now that the data base is available. We briefly address some of them at the end of this section.

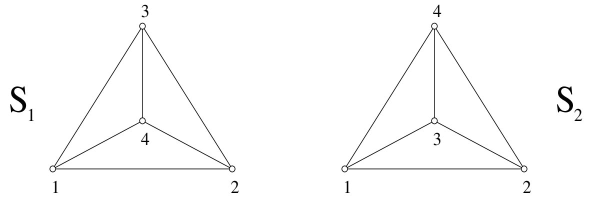

Figure 3: Edge-isomorphism is not 'hull-honest'

3.1 Isomorphism conjecture for planar triangulations

Informally speaking, the isomorphism conjecture for triangulations deals with the question of whether two point sets can be triangulated in the same fashion. Consider two equal-sized point sets $S_1 = \{p_1, \ldots, p_n\}$ and $S_2 = \{q_1, \ldots, q_n\}$ and two triangulations $T(S_1)$ and $T(S_2)$ thereof. Then $T(S_1)$ and $T(S_2)$ are called *(edge) isomorphic* iff for every edge $(p_i, p_j) \in T(S_1)$ there exists the edge $(q_i, q_j) \in T(S_2)$. Clearly, isomorphic triangulations live on sets with the same number of extreme points because the number of triangulation edges just depends on the size of the convex hull. Still, an edge isomorphism may map extreme points to non-extreme ones, as in Figure 3. (If this effect is undesirable in a particular application, like morphing, the definition may be strengthened to an isomorphism between the face lattices formed by the triangles, edges, and vertices of the two triangulations.)

It is known that for *pre-assigned* point indices there are labelled sets S_1 and S_2 such that no isomorphic triangulations exist, but it is an open problem whether this can be decided in polynomial time. See Saalfeld [23] where some necessary conditions are given. The problem becomes easier if S_1 and S_2 represent the cyclic order of the vertices of two simple polygons. In this case, the existence can be decided in time $O(n^3)$, and isomorphic triangulability can be forced by adding $O(n^2)$ extra points in either polygon; see Aronov et al. [5]. An improved algorithm sensitive to the number of reflex angles of the polygons is given in Kranakis and Urrutia [21].

On the other hand, for general point sets S_1 and S_2 it is conjectured that if the order of indices may be *rearranged* for one set, then there always exist isomorphic triangulations. A few results for general n are known. Krasser [22] proved the conjecture true if the number of non-extreme points in either set is restricted to three or less. Moreover, it is not too hard to prove the existence of a *universal* point set, which is triangulable isomorphically with respect to every other set of the same size and hull. A general affirmative answer would be a deep result, showing that any two point sets are 'topologically equivalent' in this sense. For results on the isomorphism conjecture known so far see Aichholzer et al. [2].

It is an interesting challenge to investigate the conjecture for sufficiently small n numerically. We can make use of the following property: if the sets S_1 and S_2 exhibit the same order type then every triangulation of S_1 has its counterpart in S_2. In principle, for each pair of order types resp. their realizations S_1, S_2 all (exponentially many) possible triangulations have to be computed and tested for isomorphism. In addition, the point indices have to be varied, which again gives an exponential growth of possibilities. This is doable for sizes $n \leq 8$ utilizing the results we obtained in Section 2. For $n = 9$ we tried out a heuristic for deciding isomorhpic triangulability but did not obtain complete results yet. For the moment, we have to content ourselves with the following result.

OBSERVATION 1. *Let S_1 and S_2 be two n-point sets in general position in the plane and with the same number of extreme points. Then for $n \leq 8$ there exist isomorphic triangulations for S_1 and S_2. This remains true even if the cyclic correspondence between the extreme points of S_1 and S_2 is prescribed.*

3.2 How many triangulations every point set must have

Efficiently counting the number of triangulations of a given set of n points in the plane is an interesting open problem. The currently fastest method for counting is based on the concept of so-called triangulation paths, recently introduced in Aichholzer [1]. But still the running time shows exponential growth and computations are limited to $n \leq 30$.

On the other hand, no tight asymptotic bounds on the number of triangulations are known. The best examples for maximizing this number yield $\Theta(2^{3n})$ triangulations whereas the best known upper bound is much larger, approximately $O(2^{8n})$; see Denny and Sohler [11].

Even for small values of n no exact numbers have been known. In view of the fact that point sets of the same order type admit equally many triangulations, we computed these numbers for all order types of size $n \leq 10$, using the method of [1]. We stress that this method actually needs the geometry of the realizing point sets (rather than just the λ-matrices) – a fact which is also true for other enumeration methods like reverse search; see Avis and Fukuda [7].

OBSERVATION 2. *Let S be a set of n points in general position in the plane. From $n = 3$ to 10, Table 4 gives exact lower and upper bounds on the number of triangulations of S.*

n	minimum	maximum
3	1	1
4	1	2
5	2	5
6	4	14
7	11	42
8	30	150
9	89	780
10	250	4550

Table 4: Exact extremum numbers of triangulations

So, for example, every set of 8 points must have at least 30 triangulations, and there actually is a set having that few triangulations. (The general position assumption – which has been adopted throughout – is crucial here.) This shows that point sets in convex position, whose number of triangulations is given by the Catalan numbers $C(n-2)$ (yielding 132 for $n = 8$), do not lead to the minimum. This conjecture would have been plausible as it is true for crossing-free matchings and crossing-free spanning trees; see García et al. [16]. Though the conjecture was known to be false before, the results in Table 4 may be useful in proving or disproving the extremal behavior of other point configurations. For instance, we could observe that the point sets leading to the lower bounds in Table 4 obey a rather special structure.

In fact, and surprisingly, no good general lower bounds are known. We therefore seek for functions $t(n)$ such that every set of n points exhibits $\Omega(t(n))$ triangulations. A quick answer is $t(n) = 2^{(n-4)/6}$. This results from the fact that any triangulation on n points contains at least $(n-4)/6$ edges that can be flipped (i.e., replaced by the other diagonal of the corresponding convex quadrilateral) in an independent way; see Galtier et al. [15]. A substantial improvement is based on the assertion on crossing families below which has been found by checking all order types for 10 points. A *crossing family* of size k of a point set S is a set of k line segments spanned by S which pairwise cross each other.

OBSERVATION 3. *Every set of $n \geq 10$ points in general position in the plane admits a crossing family of size 3. There exists a set of 9 points which does not have this property.*

This result improves over previously known lower and upper bounds and seems to be of separate interest. Aronov et al. [6] proved the existence of crossing families of size $\sqrt{n/12}$ for every set of n points. This evaluates to $n \geq 108$ for size 3. A smaller bound, $n \geq 37$, derives from a result by Tóth and Valtr [24]: among any $\binom{2c-5}{c-2}+2$ points there are at least c points in convex position. Thus there must be a crossing family of size $\lfloor \frac{c}{2} \rfloor$.

Observation 3 is exploited in Aichholzer et al. [3] to obtain the relation $t(n) \geq 3 \cdot t(n_1) \cdot t(n_2)$ for $n_1 + n_2 = n + 2$. For any base α, the function $t(n) = \frac{1}{3} \cdot \alpha^{n-2}$ is a solution. To get the recurrence started, values of α as large as possible are sought by examining all instances of small constant size. This is done by utilizing Table 4, in combination with other methods. The best values achieved up to now are $\alpha = 2 + \varepsilon$ for $\varepsilon \leq \frac{1}{80}$, that is, $t(n) \geq \frac{1}{12} \cdot (2+\varepsilon)^n$.

3.3 Some more applications

Applicability of our data base is by no means restricted to triangulation problems but rather includes various other questions concerning the crossing properties of a complete geometric graph. A few are briefly mentioned below. We plan to elaborate on these and other related questions in detail in a forthcoming paper.

The *crossing number problem* asks for the number $c(n)$ of (straight line edge) crossings every complete geometric graph K_n on n points in the plane must have. This problem has been studied first in Erdös and Guy [12]. In particular, the exact values of $c(n)$ have been known for $n \leq 9$. We were able to obtain $c(10) = 62$ which disproves a conjecture in [12]. Note that each crossing of K_n bijectively corresponds to a convex quadrilateral spanned by the underlying point set. Thus we have found the minimum number of convex quadrilaterals in an n-point set as well.

Related is the optimal *Hamiltonian cycle problem*, which refers to the maximum number $h(n)$ of crossing-free (rectilinear) Hamiltonian cycles a K_n can realize. In other words, what is the largest number of polygonalizations an n-point set does allow? Hayward [20] cites prior work on $h(n)$ and improves the lower bound. However, exact values of $h(n)$ have been unknown for $n > 6$. We calculated these values for n up to 10 from our data base. As a result, the lower bounds given in [20] for $n = 7, 8$ were found to be sharp while the contrary was true for $n = 9, 10$.

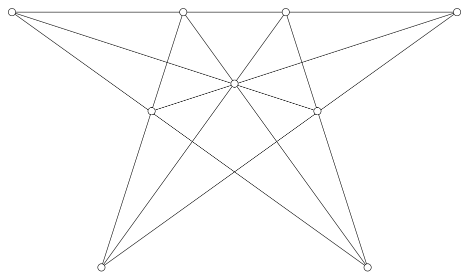

Figure 4: Order type with no grid representation

A *k-set* of a planar n-point set S consists of k points which can be separated from S by a straight line. Finding upper and lower bounds on the maximum number $f_k(n)$ of k-sets is an intriguing problem which has been studied extensively; see, e.g., Dey [10] and references therein. It is not hard to see that point sets of the same order type realize the same numbers of k-sets. For $n \leq 10$ the exact numbers $f_k(n)$ can be computed using the data base. The maximum number of halving lines (or $\frac{n}{2}$-sets) for 10 points, $f_5(10) = 13$, has recently been determined in Felsner [13].

4. CONCLUDING REMARKS

The goal of this work has been to provide a complete, valid, and user-friendly data base for order types of $n \leq 10$ points in the Euclidean plane, and to substantiate its usefulness to problems in computational and combinatorial geometry. Improved results on triangulation problems and on other questions have been obtained. We believe our data base to be of use to many researchers who wish to examine their conjectures on small point configurations. As in geometrical algorithms numerical stability is an important issue, we put particular emphasis on generating small and nice grid representations of the realizing point sets.

Our approach for computing the data base uses the fact that, for sizes $n \leq 10$, the numbers (though not the combinatorial descriptions) of all non-realizable projective pseudo order types are known from the literature. These numbers are still unknown for larger n. So, apart from the obvious time and space limitations, a complete data base for $n = 11$ currently is out of our reach.

It is for the same reasons that our approach does not work in case of collinearities, except for very small instances: no realizability results are known for $n > 8$, and the number of possible order types increases tremendously compared to the case of general position. Moreover, it is theoretically impossible to obtain integer grid representations for all order types if the general position assumption is dropped, in view of the existence of non-rational oriented matroids for sizes $n > 8$; see [8] where Figure 4 is taken from.

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