On Compatible Triangulations of Point Sets

Oswin Aichholzer*
Franz Aurenhammer
Hannes Krasser†

Institute for Theoretical Computer Science
Graz University of Technology, Graz, Austria
e-mail: {oaich,auren,hkrasser}@igi.tu-graz.ac.at

Abstract

Two conjectures on compatible triangulations for planar point sets are stated and proved for small sets and for special sets of arbitrary size.

1 Introduction

Let S_1 and S_2 be two finite sets of points in the Euclidean plane. Two triangulations T_1 of S_1 and T_2 of S_2 are called *compatible* if the face lattices formed by their triangles, edges, and vertices (points) are isomorphic. The problem of triangulating two given point sets compatibly comes in two flavors, namely where the correspondence between the points of S_1 and S_2 is either fixed or variable. The case of fixed correspondence is a well-known problem, see [5], and it is not hard to see that compatible triangulations do not always exist. The challenging problem left open is to determine the complexity of the related decision problem, which so far it is neither known to be NP-complete nor to be polynomially solvable. The problem becomes easier if S_1 and S_2 represent the cyclic order of the vertices of two simple n-gons. In this case, the existence can be decided in time $O(n^3)$, and compatible triangulability can be achieved by adding $O(n^2)$ extra points in either polygon; see [3].

In this work we consider the problem of finding compatible triangulations *without* fixed correspondence. To our knowledge this problem has not been studied before. The main question is, of course, under which propositions compatible triangulations do exist. It is clear that in both sets S_1 and S_2

*Research supported by APART [Austrian Programme for Advanced Research and Technology] of the Austrian Academy of Sciences, and the EU Improving Human Potential Programme HPRI-1999-CT-00071

†Research supported by the FWF [Austrian Fonds zur Förderung der Wissenschaftlichen Forschung]

the number of points must be the same, as has to be the number of points on the convex hulls $CH(S_1)$ and $CH(S_2)$. In fact, convex hull points necessarily map to convex hull points, which need *not* be the case if only graph isomorphism rather than compatibility of the triangulations is required. We will assume that both point sets are in general position, meaning that no three points lie on a common straight line. There are examples of point sets not in general position that do not admit compatible triangulations under the propositions mentioned above, see [4]. We also consider an extended version, where the correspondence between the points on the convex hulls is prescribed. The status of this extended problem is still unresolved. We have the following two conjectures.

Conjecture 1.1 *(Main Conjecture on Compatible Triangulations)*
Let S_1 and S_2 be two sets of points without fixed correspondence. Then compatible triangulations for S_1 and S_2 exist if S_1 and S_2 fulfill the following properties:

- $|S_1| = |S_2|$,
- $|CH(S_1)| = |CH(S_2)|$, *and*
- S_1 *and* S_2 *are in general position.*

Conjecture 1.2 *(Extended Conjecture on Compatible Triangulations)*
Conjecture 1.1 still holds if the correspondence between the points on the convex hulls of S_1 and S_2 is given by a cyclic labeling.

Below we prove Conjecture 1.2 for point sets with at most three interior points.

2 Constructions

We first focus on the following class of point sets. A generalization of this class has been considered in [1].

Lemma 2.1 *Let S_1 be a set of points so that there exists a point $p \in CH(S_1)$ with $S_1 \setminus \{p\}$ in convex position. Then Conjecture 1.2 is true.*

Proof: Consult Figure 1. Draw edges from p to all interior points of S_1 and then connect these points with edges in sorted order as seen from p. Do the same for p's counterpart in S_2 and label the interior points of both sets in a compatible manner. For S_1 the untriangulated area is a convex polygon C, whereas for S_2 a simple polygon P remains untriangulated. But every triangulation of P is also a valid triangulation of C. $\square$

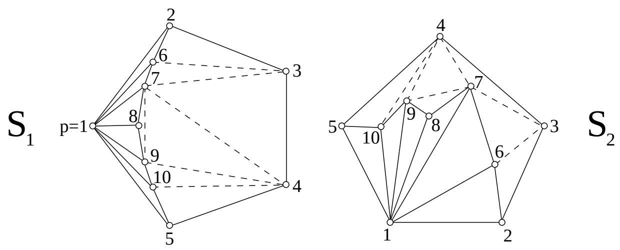

Figure 1: Triangulating compatibly with a special point set.

Lemma 2.2 *Let S_1 and S_2 be two sets of points with $|S_1 \setminus CH(S_1)| = |S_2 \setminus CH(S_2)| \leq 3$. Then Conjecture 1.2 is true.*

Sketch of proof: If S_1 and S_2 are both in convex position then every triangulation of S_1 also leads to a triangulation of S_2. If both sets have exactly one interior point then star-like triangulations always exist and they are compatible. In the case of two interior points, it is always possible to find two points labeled a and b in $CH(S_1)$ such that (1) a and b lie on different sides of the straight line through the two interior points of S_1, and (2) the points labeled a and b in $CH(S_2)$ lie on different sides of the straight line through the two interior points of S_2. We build two triangles by connecting a and b to both interior points, and connect the remaining hull points to either a or b accordingly. Doing this for both point sets results in compatible triangulations. The case of three interior points is more intriguing. It can be handled either by Lemma 2.1 or in a way similar to two interior points. A case analysis of all essential positions of the three interior points shows that compatible triangulations exist, although the triangle formed by them need not be part of the solution; see Figure 2. For further details see [4]. $\square$

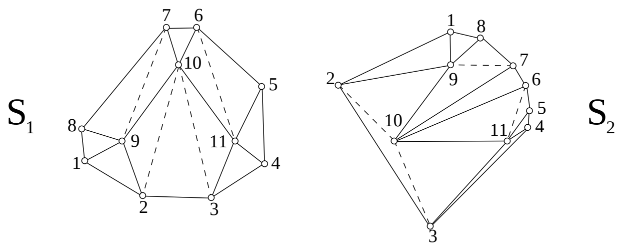

Figure 2: Compatible triangulations with three interior points.

3 Remarks

Unfortunately, the constructions above do not seem to generalize to the case of more interior points. Proving our conjectures for general point sets would be a deep result, showing that all planar point sets of the same size and hull are 'topologically equivalent' in this sense. We tested the conjectures for small point set sizes, utilizing a database of combinatorially inequivalent point sets (i.e., order types); see [2]. If two points sets exhibit the same order type then every triangulation of one set also leads to a compatible triangulation for the other. By checking all different pairs of order types we could verify the conjectures for up to 8 points. Although our database provides all order types of size up to 10, exhaustive tests turned out to be time consuming and are still incomplete for size 9. In fact, determining the time complexity of computing compatible triangulations for two given n-point sets is an open problem.

References

[1] O.Aichholzer, F.Aurenhammer, F.Hurtado, *Some universal sets.* Manuscript, 1999.

[2] O.Aichholzer, F.Aurenhammer, H.Krasser, *Enumerating order types for small point sets with applications.* Submitted.

[3] B.Aronov, R.Seidel, D.Souvaine, *On compatible triangulations of simple polygons.* Computational Geometry: Theory and Applications 3 (1993), 27-35.

[4] H.Krasser, *Kompatible Triangulierungen ebener Punktmengen.* Master Thesis, Institute for Theoretical Computer Science, Graz University of Technology, Graz, Austria, 1999.

[5] A.Saalfeld, *Joint triangulations and triangulation maps.* Proc. 3rd Ann. ACM Sympos. Computational Geometry, Waterloo, Canada, 1987, 195-204.