On the Crossing Number of Complete Graphs

O. Aichholzer*, Graz, and **F. Aurenhammer**, Graz,
and **H. Krasser**†, Graz

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Abstract

Let $\overline{cr}(G)$ denote the rectilinear crossing number of a graph G. We determine $\overline{cr}(K_{11}) = 102$ and $\overline{cr}(K_{12}) = 153$. Despite the remarkable hunt for crossing numbers of the complete graph K_n – initiated by R. Guy in the 1960s – these quantities have been unknown for $n > 10$ to date. Our solution mainly relies on a tailor-made method for enumerating all inequivalent sets of points (order types) of size 11.

Based on these findings, we establish a new upper bound on $\overline{cr}(K_n)$ for general n. The bound stems from a novel construction of drawings of K_n with few crossings.

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1 Introduction

The *crossing number* of a graph G is the least number of edge crossings attained by a drawing of G in the plane. *Drawing* may be interpreted in several ways leading to different concepts of crossing number; Pach and Tóth [19] discuss this issue in detail. Crossing number problems have a quite long history in the graph theory community (see, e.g., Tutte [25] and Erdös and Guy [10]) and have more recently been of interest to computer scientists (see

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Leighton [17] and Garey and Johnson [11]). We refrain from an attempt to survey the relevant literature in its generality, and point to a recent overview paper by Pach and Tóth [20] instead.

In the present paper, we are concerned with the geometric version of the crossing number problem, where the edges of the underlying graph are required to be straight line segments in the plane. Following Harary and Hill [14], we will use the notation *rectilinear crossing number*, $\overline{cr}(G)$, of a graph G. The vertices of G are assumed to be in general position, meaning that no three vertices are allowed to be collinear in the drawing. Our specific interest is that of finding $\overline{cr}(K_n)$, the rectilinear crossing number of the complete graph K_n on n vertices. The results of this paper appeared as a conference version in [3].

Determining $\overline{cr}(K_n)$ is commonly agreed to be a difficult task. In fact, the exact asymptotic behavior, as n tends to infinity, is unknown for *any* interpretation of the crossing number of K_n considered in the literature; see Richter and Thomassen [21]. From an algorithmic point of view, deciding whether $\overline{cr}(G) \leq k$ for a given graph G and parameter k is NP-hard, as has been proved in Bienstock [7]. Only for very small n the exact values of $\overline{cr}(K_n)$ are known. Whereas the instances $n \leq 9$ have been settled quite a time ago in Erdös and Guy [10], no progress has been made until in 2001 two groups of researchers (Brodsky et al. [8] and the authors [2], respectively) independently found $\overline{cr}(K_{10}) = 62$. In [8] the goal was reached by a purely combinatorial argument, while in [2] the result came as a byproduct of the exhaustive enumeration of all combinatorially inequivalent point sets (so-called *order types*) of size 10. For completeness, we include Table 1 from [4].

n	4	5	6	7	8	9	10
$\overline{cr}(K_n)$	0	1	3	9	19	36	62
I_n	1	1	1	3	2	10	2

Table 1: $\overline{cr}(K_n)$ arises in exactly I_n inequivalent drawings

Considerable effort has also been put into the study of the asymptotic behavior of $\overline{cr}(K_n)$. Let $\overline{\nu}(n) = \overline{cr}(K_n)/\binom{n}{4}$ be – loosely speaking – the rate of improvement when drawing K_n optimally. Note that $\overline{\nu}(n)$ is bounded from above by one. Adopting the notation in Scheinerman and Wilf [22], we define $\overline{\nu}^* = \lim_{n\to\infty} \overline{\nu}(n)$. The quantity $\overline{\nu}^*$ is of separate interest, by a connection to Sylvester's four point problem drawn in [22]. Namely, $\overline{\nu}^*$ is the infimum, over all open sets R with finite area, of the probability that four randomly chosen points in R are in convex position.

Lower bounds on $\overline{\nu}^*$ have been improved several times [9, 26], and most significantly by Ábrego and Fernández-Merchant [1] and Lovász et al. [18] who independently proved $\overline{\nu}^* > \frac{3}{8}$ by finding a lower bound on the number of k-sets. In fact, a refinement to 0.37501 is given in [18]. Improving on these techniques, Balogh and Salazar [6] obtained the current best lower bound of 0.37533. The best known asymptotic *upper* bound has been given by Brodsky et al. [9] and reads $\overline{\nu}^* < 0.3838$.

2 Overview of results

Encouraged by the achievements in [2, 4], we tailored the exhaustive search approach to generate inequivalent drawings of K_n for larger n. We report on these modifications and the obtained spectrum of drawings in Section 3. Section 4 offers a novel construction for drawings with few crossings, for general n. The present section states the obtained results.

Our first result completely resolves the situation for problem size $n = 11$.

Theorem 1 *The rectilinear crossing number of K_{11} is 102, and this value is attained by exactly 374 inequivalent drawings. Each drawing shows exactly 3 extreme vertices.*

For $n \geq 12$ our results are numerous though less complete.

By carefully constructing drawings with low numbers of crossings, in Sections 3 and 4, we obtain upper bounds superior (except for $n = 13$) to all previous ones. For $n \leq 20$ these bounds are listed in lines 2 and 5 of Table 2. Lines 3 and 6 comment on the number of inequivalent drawings that exist for the respective number of crossings. Some drawings for $n \leq 81$ are selected in Table 3; see Section 3.

Earlier experiments, by Thorpe and Harris [24], were based on randomized search and achieved drawings of K_{12} and K_{13} with 155 and 229 crossings, respectively. Their latter result thus competes with our best examples for $n = 13$.

All drawings reported in Tables 1, 2, and 3 possess exactly three extreme vertices. This adds evidence to the commonly believed conjecture that the convex hull of every minimal rectilinear drawing of K_n is a triangle.

Theorem 1 now directly leads to another result. We apply a simple counting argument, first used in Guy [13]. Each of the n subgraphs K_{n-1} of K_n contains at least $\overline{cr}(K_{n-1})$ crossings, each being counted $n-4$ times, because the quadruple of vertices determining the crossing appears in all but 4 such subgraphs. This implies the recurrence relation

$$\overline{cr}(K_n) \geq \lceil \frac{n}{n-4} \cdot \overline{cr}(K_{n-1}) \rceil.$$

n	11	12	13	14	15
upper bound	102	153	229	324	447
drawings	374	≥ 1	≥ 4534	≥ 20	≥ 16001

n	16	17	18	19	20
upper bound	603	798	1029	1318	1657
drawings	≥ 36	≥ 37269	≥ 1	≥ 13243	≥ 6

Table 2: Upper bounds on $\overline{cr}(K_n)$

We further exploit that $\overline{cr}(K_n)$ and $\binom{n}{4}$, the maximum number of crossings in a drawing of K_n, are known to have the same parity for n odd; see Erdös and Guy [10]. (Another proof of this interesting property follows from arguments in Subsection 4.1.) Plugging in the value $\overline{cr}(K_{11}) = 102$ yields the specific lower bound $\overline{cr}(K_{12}) \geq 153$. Lower and upper bounds happen to match, which allows us to conclude:

Theorem 2 *The rectilinear crossing number of K_{12} is 153. The only known drawing that attains this value is depicted in Figure 1.*

In fact, this result is not incidental. The drawing in Figure 1 is highly symmetric such that each subgraph K_{11} is drawn optimally, which is sufficient for global optimality. This situation cannot be achieved in general, not even for smaller n where it is met only for K_4 and K_6. [1]

Concerning the asymptotic upper bound, Brodsky et al. [9] offer a recursive construction that yields $\overline{\nu}^* < 0.3838$, which is superior to previous results in Singer [23] and in Jensen [16]. We improve the upper bound on $\overline{\nu}^*$ further, not only by exploiting our larger starting examples, but with a novel method for constructing drawings of K_n with few crossings. The method is proven optimal within its scope and is described in Section 4. The following new bound is obtained.

Theorem 3 $\overline{\nu}^* < 0.380739$.

3 Generating drawings

This section describes our approach for generating inequivalent drawings of K_n for $n \geq 11$. We start with reviewing some basic notions on point configurations.

[1] By combining our techniques with a novel method for order type extension (which is of interest in its own right), Aichholzer and Krasser [5] very recently showed that the upper bounds in line 2 of Table 2 are tight up to $n = 17$; see the web page [28] for details.

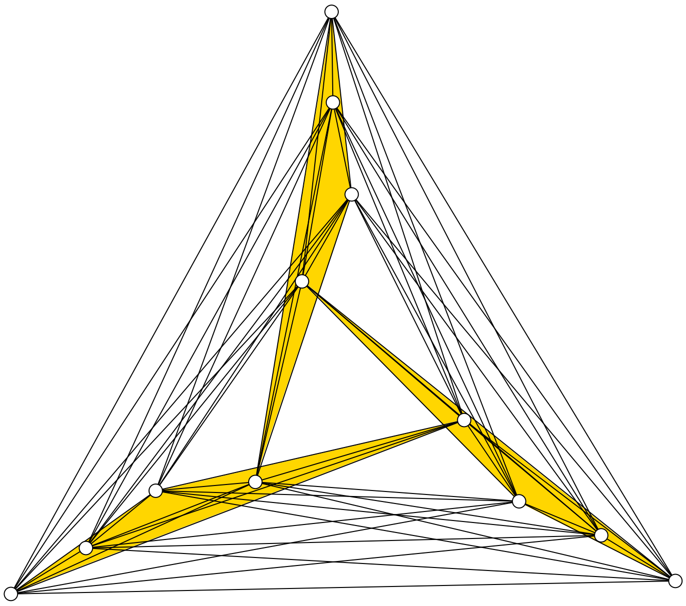

Figure 1: Minimal drawing of K_{12} in Theorem 2

3.1 Order types and λ-matrices

The *order type* of a set of n points (in general position in the plane) is a mapping that assigns to each ordered triple of points an orientation, either clockwise or counter-clockwise; see Goodman and Pollack [12]. Two n-point sets S_1 and S_2 are called *(combinatorially) equivalent* if they have the same order types (up to global reversal of orientation), that is, if there exists a bijection $\varphi : S_1 \to S_2$ such that every point triple a, b, c in S_1 agrees in orientation with the triple $\varphi(a), \varphi(b), \varphi(c)$ in S_2.

Equivalent point sets exhibit identical crossing properties in the following sense. Any two line segments ab and cd spanned by S_1 cross if and only if the corresponding two line segments $\varphi(a)\varphi(b)$ and $\varphi(c)\varphi(d)$ in S_2 do so. In particular, whether or not a drawing of K_4 with vertices a, b, c, d contains a crossing, can be read off the orientations of the four point triples in the quadruple $\{a, b, c, d\}$; a crossing occurs if and only if the number of iso-oriented triples is even. We will call two drawings of K_n *equivalent* if and only if they live on equivalent n-point sets.

We are left with the problem of enumerating all different order types of given size n. This task has been recently accomplished for values $n \leq 10$ by the authors [2]. In a nutshell, the approach proceeds in two steps (for fixed size n).

Combinatorial step: Generate a candidate list of so-called *abstract order* *types* which is guaranteed to contain all order types.

Geometrical step: Find point coordinates for each of the abstract order types in the list which are *realizable*, i.e., which are in fact order types.

To our advantage, the combinatorial step is comparatively easy, whereas the geometrical step is an intricate problem. The general problem of deciding the existence of a realizing point set is NP-hard. An exponential algorithm is known which, however, turned out to be too slow already for $n = 9$. The interested reader is referred to [2] for a detailed discussion of this material. As we shall see, the geometrical step needs only partially be carried out for our present purpose.

Following are the details on the combinatorial step. An *abstract* order type is a mapping that assigns to each ordered triple $\{i, j, k\} \subset \{1, \ldots, n\}$ a clockwise or counter-clockwise orientation. An abstract order type does not necessarily have a realizing point set, because the geometry of a point set imposes restrictions on the choice of orientations. Still, whether two (abstract) line segments ij and kl cross can be decided by checking orientations in the quadruple $\{i, j, k, l\}$. As has been shown in Goodman and Pollack [12], each abstract order type can be encoded by a λ*-matrix*. In this matrix, each entry $\lambda(i, j)$ gives the number of ordered triples i, j, k which are oriented counter-clockwise. Moreover, a specific set of abstract order types which contains all the (realizable) order types, is representable by so-called *wiring diagrams*. The latter are combinatorial abstractions of straight line arrangements and correspond to certain abstract order types by duality; see [12].

3.2 Strategy and results

The observations above led us to the following strategy for the combinatorial step for $n = 11$. Utilizing wiring diagrams, we generated a list of abstract order types in λ-matrix representation. For the sake of time efficiency, we generated all matrices in *natural labelling*: index 1 is extreme, and indices 2 through n are sorted clockwise around 1. (These geometric notions still make sense in the abstract setting; for example, a line segment ij is *extreme* if and only if $\lambda(i, j) \in \{0, n - 2\}$.) Natural labelling suggests itself from wiring diagrams; see [2] for more details. As there are several naturally labelled λ-matrices for a fixed order type (namely the number of extreme indices times two, for re-orientation), only the *(lexicographically) minimal* matrix was taken into account for each order type.

Our computations for $n = 11$ resulted in the tremendous number of 18 410 581 880 matrices, thereof 2 343 203 071 being minimal and thus corresponding

drawing	D_{27}	D_{36}	D_{39}	D_{40}	D_{81}
crossings	6186	21191	29737	33093	618616

Table 3: Numbers of crossings in selected drawings

to pairwise inequivalent abstract order types. Clearly, matrix generation and minimality check had to be done on-line to avoid storage problems. For each minimal matrix, we calculated the number of crossings in the (abstract) K_{11} for the respective order type. The minimum number, 102, was obtained by 374 abstract order types with exactly 3 extreme indices each. Reliability of the calculations is given by their completely combinatorial nature.[2]

At this late point, geometry came into play. We tried to find realizing point sets for the 374 abstract order types above, utilizing methods from [2]. Either strategy (point insertion using the existing order type data base for $n = 10$, and simulated annealing starting from scratch, respectively) was successful in realizing all order types in question. In particular, integer coordinate representations (16-bit) have been obtained for the corresponding 11-point sets. The point sets have been re-checked by re-calculating their λ-matrices: the sets are pairwise inequivalent and yield drawings of K_{11} with exactly 102 crossings. This proves the correctness of our geometric computations.

Once having available all existing minimal drawings of K_{11}, constructing drawings of K_n for $n \geq 12$ with few crossings suggests itself. Following suit, for each of the corresponding 11-point sets we considered the arrangement of straight lines through each pair of points and placed a 12^{th} point in an arrangement cell in all possible ways. This provided us with a large collection of 12-point sets, the 'best' of which were used in turn to generate a collection of 13-point sets in the same manner, and so on. We continued through $n = 45$ with this strategy, examining hundred-thousands of sets for each value of n.

It should be noted that this insertion strategy – when applied to *all* inequivalent $(n-1)$-point sets – in general fails to produce all inequivalent n-point sets. In particular the question in [8], of whether a minimal drawing of K_{n+1} is always attainable from one of the minimal drawings of K_n, remains unsettled.

For $n \leq 20$, the obtained results are displayed in Table 2. Particularly pleasing is the discovery of a drawing of K_{12} with 153 crossings, see Figure 1, which we could prove minimal by examining lower bounds. Selected drawings for larger n are given in Table 3, for later use in Section 4.

[2] These calculations took about one month on a 1GHz PIII processor under Linux.

4 Upper bound constructions

Arbitrarily large drawings with few crossings are typically constructed by taking the best drawing D known so far, and recursively replacing each of its vertices by a copy of D. We intend to show that better results are achieved by a different, somewhat counter-intuitive construction. Replacement of vertices is done only *once*, with a fixed configuration of points in *convex* position. Despite the fact that convex configurations locally *maximize* the number of crossings, this strategy pays off in the global sense as we shall see.

4.1 Lenses and halvings

Consider some drawing D_n of K_n for n *even*, and let Ω be a sufficiently large disk containing D_n. For a given point configuration C, let $\sigma(C)$ denote the narrowest strip that – within Ω – contains all the straight lines that pass through pairs of points in C.

The vertices of D_n are now replaced by point configurations $C_1, \ldots, C_n$, in a way such that no C_i is split by an edge of D_n. We say that a replacement satisfies the *halving property* if, in addition, for each C_i the strip $\sigma(C_i)$ avoids all configurations but C_i and halves them into groups of sizes $\frac{n}{2}$ and $\frac{n}{2} - 1$, respectively. Replacements of this kind exist for configurations C_i of arbitrary order types, by the following well-known geometric fact: for each point p among $n = 2k$ points in general position, there exists a straight line through p that has k and $k - 1$ points on its sides. (We remark that separations into sets of different sizes neither need to exist, nor are of use to us in this context.) Thus, appropriate affine transformations of the C_i's will always guarantee the halving property.

Define a *lens* as a configuration L of points on a (small portion of a) semi-circle; see Figure 2. The orientation of the strip $\sigma(L)$ defines a *curved* side and a *straight* side of L in an obvious sense. We replace the vertices of D_n by lenses of equal size c such that (1) the resulting drawing $D_{c \cdot n}$ of $K_{c \cdot n}$ fulfills the halving property, and (2) for each lens, the majority of the remaining vertices of $D_{c \cdot n}$ lies on its curved side.

The following is a count of the crossings in $D_{c \cdot n}$, which we categorize into five types as done similarly in [23, 9]. See Figure 2 again.

Type I The first type is between edges *internal* to the lenses. As lenses are convex and their number is n, there exist $n \cdot \binom{c}{4}$ such crossings.

Type II Crossing between an edge internal to a lens L and some *external* edge. By construction, the external edge has to start in L. Each triple x, y, z of points in L (ordered on its semi-circle) defines an edge xz internal to L

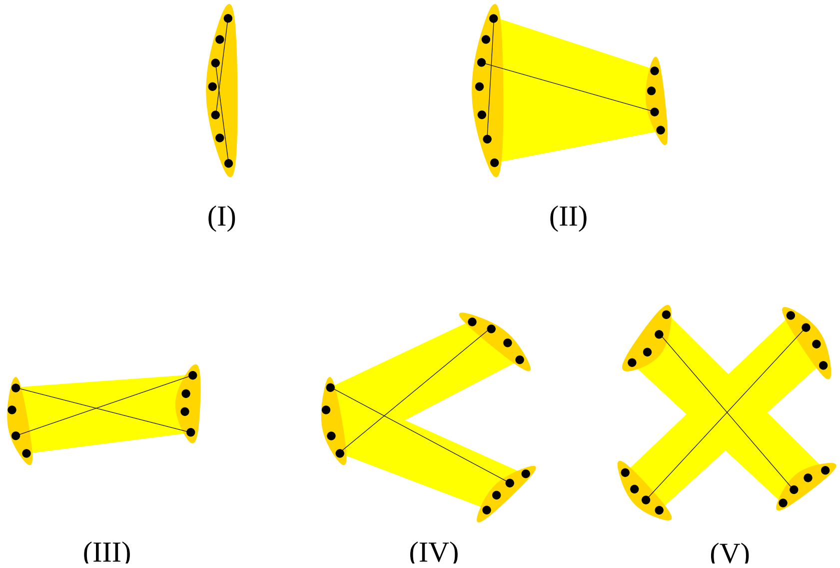

Figure 2: Crossings defined by lenses

that crosses all edges yp whose second endpoint p lies on the straight side of L. (Note that p cannot lie on the curved side of L because of the halving property.) There are $(\frac{n}{2}-1)\cdot c$ choices for p, and n choices for L, which gives $c\cdot\binom{c}{3}\cdot n\cdot(\frac{n}{2}-1)$ crossings of this type.

The remaining types concern external edges only.

Type III This type refers to crossings between edges connecting *two* given lenses L and L'. There are $\binom{c}{2}$ choices of two endpoints in L, and of two endpoints in L', respectively, and each quadruple yields one crossing. Furthermore, there are $\binom{n}{2}$ pairs L, L' to choose from. This amounts to $\binom{c}{2}^2\cdot\binom{n}{2}$ crossings.

Type IV Crossings between edges connecting *three* given lenses L, L', and L''. Let L be the (unique) lens that contains two endpoints of the edge pair defining such a crossing. Then there are $\binom{c}{2}$ possibilities for choosing these two endpoints in L, and c choices for the other endpoint of either edge in L' and L'', respectively. Taking into account that L' and L'' have to lie on the same side of the strip $\sigma(L)$ to allow for crossings of this type at all, we have $n\cdot[\binom{\frac{n}{2}}{2}+\binom{\frac{n}{2}-1}{2}]$ valid choices for the triple L, L', L''. So we obtain $\binom{c}{2}\cdot c^2\cdot n\cdot[\binom{\frac{n}{2}}{2}+\binom{\frac{n}{2}-1}{2}]$ crossings of this type.

Type V Crossings between edges connecting *four* given lenses. Let $\gamma(D_n)$ be the number of crossings in the drawing D_n. We have c possibilities to choose an endpoint in one lens, and $\gamma(D_n)$ valid choices for lens quadruples.

The number of crossings of type V thus is $c^4 \cdot \gamma(D_n)$.

Adding up crossings of types I to V and replacing $N = cn$ we get that the total number of crossings is

$$\left(\frac{24\gamma(D_n) + 3n^3 - 7n^2 + 6n}{n^4}\right)\binom{N}{4} + O(N^3).$$

Several comments are in order. Observe first that crossing types are sorted by increasing share in $\gamma(D_{c\cdot n})$, except for very small values. Next, types III, IV, and V are *invariant*, in the sense that the *order type* of the substituted configuration C does not affect the number of such crossings – as long as the halving property is satisfied. For type III and type V this is not surprising, in view of their correspondence to the edges and the crossings, respectively, in the original drawing D_n.

Type II crossings are minimized, and type I crossings maximized, by lenses. Finally, configurations C of size c that lead to minimal drawings of K_c (i.e., with $\overline{cr}(K_c)$ crossings) obviously minimize type I. Interestingly, in a trade-off between types I and II, the lenses win over all other configurations.

Lemma 1 *Let L and C be a lens and a configuration of different order type, respectively, of size c. Then L causes fewer crossings in $D_{c\cdot n}$ than C, when only types I and II are counted.*

Proof Fix a bijection $\varphi : L \to C$. Then there exists a point triple x, y, z in L which disagrees in orientation with the triple $\varphi(x), \varphi(y), \varphi(z)$. The latter triple gives rise to $\frac{n}{2} \cdot c$ type II crossings for C, whereas the former triple is responsible for $(\frac{n}{2} - 1) \cdot c$ type II crossings for L, which is c crossings less.

On the other hand, observe that reorientation of *any* triple T in a set of c points can create at most $c - 3$ crossings, as the number of quadruples (potential crossings) that contain T is $c - 3$.

To sum up, L causes at least $3 \cdot t$ crossings less than C if the order types disagree in t triples. □

From Lemma 1 and the foregoing discussion, we conclude the following general result.

Theorem 4 *Given some drawing D_n of K_n for n even, let D_N be a drawing of K_N, for $N > n$, obtained by substituting arbitrary point configurations for the vertices of D_n such that the halving property is fulfilled. Then $\gamma(D_N)$ is minimized if each substitute is a lens (of proper orientation).*

Noteworthy more is hidden in the proof of Lemma 1. If a single triple is reoriented in an *abstract* order type A of size 4, then the status of A switches from crossing (even number of iso-oriented triples) to non-crossing (odd number of iso-oriented triples), or vice versa. Consequently, in any abstract order type of size $c \geq 4$, the number of crossings *created or eliminated* when reorienting one of its triples is exactly $c-3$. So, if c is odd, crossings change by an even number. The set of abstract order types is obviously connected under triple reorientation, which now implies the parity property of K_c for odd c, mentioned (and used) in Section 2.

4.2 The improvement

The constructions in Singer [23], and in Brodsky et al. [9] (which led to drawings with the fewest crossings to date) follow the *recursive* replacement scheme mentioned in Subsection 4.1. Before quantifying the impact of lenses, let us briefly comment on our experiments using the recursive scheme. The order type generation method in Section 3 provides us with plenty of improved drawings to start the recursion. The drawing D_{39} in Table 3 turned out to fit this purpose best. Without further effort (which has been necessary in [9], who additionally split configurations) we obtained an upper bound of 0.380891 for the quantity $\overline{\nu}^*$ in Section 2, thus improving over all previous results. The details are omitted.

As for our results when using lenses, we examined various promising candidates for a start drawing D_n, the winner being the drawing D_{36} in Table 3. For $c \to \infty$ the bound $\overline{\nu}^* < 0.380858$ was obtained, which is already slightly better than the bound we got from recursive replacement in the drawing D_{39}.

To exploit the concept of lenses to its utmost generality, we use the following observation. Type V contributes the most crossings in an asymptotic sense, namely $\Theta(c^4 n^4)$. To reduce the value of the product c^4, lenses of *different* sizes should be chosen. Optimizing these sizes is not easy, however, even for the particular drawing D_{36}. The number of crossings in the resulting drawing is a degree-4 polynomial P in 35 variables (36 lens sizes with fixed sum). To reach tractability, we fixed certain multiplicative factors between the sizes, and minimized over the resulting simplified polynomial (in 8 variables). The choice of factors was guided by the *share* of each vertex of D_{36} in $\gamma(D_{36})$, that is, the number of crossings of D_{36} caused by the edges incident to that vertex. This gave an additional improvement, the bound $\overline{\nu}^* < 0.380739$ stated in Theorem 3. We refer the reader to the web page [29] for details on the drawing D_{36}. In particular, point coordinates and lens sizes are given there.

We were not able to break this bound so far, neither by changing the shape nor by increasing the size of the start drawing D_n. The intuitive reason is that good drawings tend to balance the share of crossings over all vertices (as is exemplified in Figure 1), whereas our replacement strategy is most effective in unbalanced situations. It might be well possible to gain further improvement by splitting of lenses, e.g., as done in [9] for other configurations.

The superiority of our construction is also apparent from a comparison to the benchmark example K_{81} in [23, 27, 9]. A replacement by lenses of size 3 in the drawing D_{27} (see Table 3) already resulted in an elimination of 4914 crossings, with respect to the best value known before. Even more effect took a replacement in the drawing D_{40} by lenses of size 2, except for one vertex (with minimum share in $\gamma(D_{40})$), which was replaced by a lens of size 3. An additional loss of 384 crossings occured, and two more crossings could be saved by slightly moving the vertices, which finally gave the drawing D_{81} in Table 3.

5 Remarks

Our approach opened the door to various new results on the rectilinear crossing number of K_n. This constitutes one more example where computational experiments, when carried out carefully and combined with theoretical arguments, shed new light into notoriously difficult combinatorial problems.

We are currently building up a data base for $n = 11$ of all *projective* order types, including their realizations if existent. This will enable us to access the (Euclidean) order types of size 11 faster while using affordable disk space. (Our latest calculations show that exactly 2 334 512 907 order types of the latter kind are realizable.) An examination of questions similar to those in the present paper thus will become easier in the future.

There is hope that an inspection of the currently 'best' drawings of K_n for small n might reveal regularities leading to new theoretical insights. A small step in this direction is the observation that, as of yet, all these drawings showed triangular convex hulls. Also, from Tables 1 and 2 we see that, for n odd, the number of inequivalent drawings with small numbers of crossings is quite large, probably because the lack of symmetries. This indicates that drawings attaining $\overline{cr}(K_n)$ might be easier to find for n odd.

Finally, let us remark on a related problem. The *(rectilinear) Hamiltonian cycle problem* asks for the largest number $h(n)$ of crossing-free Hamiltonian cycles a rectilinear drawing of K_n can realize. A conjecture originally stated in Hayward [15] and adopted in Brodsky et al. [9] reads: does there always exist a drawing of K_n which simultaneously attains $\overline{cr}(K_n)$ and $h(n)$? The conjecture has been proved for $n \leq 8$, and disproved for $n = 9$ and 10, in [4]. The present computations for $n = 11$ add to the negative side: the largest number of cycles achieved among the 374 drawings attaining $\overline{cr}(K_{11})$ is 20328, and we constructed a drawing of K_{11} having more cycles.

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Oswin Aichholzer
Graz University of Technology
Institute for Software Technology
Inffeldgasse 16b
Graz
Austria
oaich@ist.tugraz.at

Franz Aurenhammer
Graz University of Technology
Institute for Theoretical Computer Science
Inffeldgasse 16b
Graz
Austria
auren@igi.tugraz.at

Hannes Krasser
Graz University of Technology
Institute for Theoretical Computer Science
Inffeldgasse 16b
Graz
Austria
hkrasser@igi.tugraz.at