# On the Crossing Number of Complete Graphs (Extended Abstract)

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## ABSTRACT
We determine $\overline{cr}(K_{11}) = 102$ and $\overline{cr}(K_{12}) = 153$. Moreover, we give improved upper and lower bounds on the asymptotic saving of crossings when drawing $K_n$ optimally.

## Categories and Subject Descriptors
G.2.1 [**Mathematics of Computing**]: Discrete Mathematics—*Combinatorics, Counting problems*; F.2.2 [**Nonnumerical Algorithms**]: Geometric problems and computations—*computations on discrete structures*

## Keywords
Rectilinear crossing number, complete graph, order type

## 1. INTRODUCTION
The *crossing number* of a graph $G$ is the least number of edge crossings that can be attained when drawing $G$ in the plane. *Drawing* may be interpreted in several ways leading to different concepts of crossing number; Pach and Tóth [13] discuss this issue in detail. Crossing number problems have a quite long history in the graph theory community (see, e.g., Tutte [19] and Erdös and Guy [6]) and have more recently been of interest to computer scientists (see Leighton [12] and Garey and Johnson [7]). For an overview paper, see Pach and Tóth [14].

In the present work, we are concerned with the *geometric* version of the crossing number problem, where the edges of the underlying graph are required to be straight line segments in the plane. Following Harary and Hill [10], we will use the notation *rectilinear crossing number*, $\overline{cr}(G)$, of a graph $G$. The vertices of $G$ are assumed to be in general position, meaning that no three vertices are allowed to be collinear in the drawing. Our specific interest is that of finding $\overline{cr}(K_n)$, the rectilinear crossing number of the complete graph on $n$ vertices.

Determining $\overline{cr}(K_n)$ is commonly agreed to be a difficult task. In fact, the asymptotic value, as $n$ tends to infinity, is unknown for *any* interpretation of the crossing number of $K_n$ considered in the literature; see Richter and Thomassen [15]. From an algorithmic point of view, deciding whether $\overline{cr}(G) \leq k$ for a given graph $G$ and parameter $k$ is NP-hard, as has been proved in Bienstock [3]. Only for very small $n$ the exact values of $\overline{cr}(K_n)$ are known. Whereas the instances $n \leq 9$ have been settled quite a time ago in Erdös and Guy [6], no progress has been made until in 2001 two groups of researchers (Brodsky et al. [4] and the authors [1], respectively) independently found $\overline{cr}(K_{10}) = 62$. In [4] the goal was reached by a purely combinatorial argument, while in [1] the result came as a byproduct of the exhaustive enumeration of all combinatorially inequivalent point sets (so-called *order types*) of size 10.

In fact, the order type data base in [1] gives some more information about $K_n$. For instance, beside the drawing constructed in [4] there is one (and only one) inequivalent drawing of $K_{10}$ which also achieves $\overline{cr}(K_{10}) = 62$. (The interested reader may consult Aichholzer and Krasser [2] for a collection of new results on the crossing properties of small geometric graphs.) For completeness, we include the following table from there.

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| $\overline{cr}(K_n)$ | 0 | 1 | 3 | 9 | 19 | 36 | 62 |
| $I_n$ | 1 | 1 | 1 | 3 | 2 | 10 | 2 |

**Table 1: $\overline{cr}(K_n)$ is attained by exactly $I_n$ drawings**

## 2. NEW RESULTS
Our results mentioned above came from a simple observation: point sets of the same order type yield equivalent drawings of the complete graph. To go further, we tailored the exhaustive search approach in [1] to generate inequivalent drawings of $K_n$ for larger $n$, and, luckily, were surprisingly successful.

In a first step, we generated all combinatorially inequivalent point sets for $n = 11$. In fact, 'only' a list of candidates

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| $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| upper bound | 102 | 153 | 229 | 324 | 447 | 603 | 798 | 1030 | 1318 | 1658 |
| drawings | 374 | $\geq 1$ | $\geq 4272$ | $\geq 22$ | $\geq 2360$ | $\geq 17$ | $\geq 17532$ | $\geq 62$ | $\geq 3069$ | $\geq 54$ |
| lower bound | 102 | 153 | 221 | 310 | 423 | 564 | 738 | 949 | 1204 | 1505 |

**Table 2: New bounds on $\overline{cr}(K_n)$**

was generated, namely all the *abstract order types* that correspond to wiring diagrams; see Goodman and Pollack [8]. For each abstract order type (and there are 2 343 203 071 in the list) we computed the number of (abstract) crossings. As an important fact, both steps can be carried out in a purely combinatorial manner. This allows for tremendous savings in runtime, as well as for reliable computations. Only for those (few) abstract order types which gave the *smallest* number of crossings, we had to perform the more challenging geometrical task of finding point coordinates – if they exist at all.

They do exist, and we could restore them, which gave our first result; it completely resolves the situation for $n = 11$.

THEOREM 1. *The rectilinear crossing number of $K_{11}$ is* 102*, and this value is attained by exactly* 374 *inequivalent drawings. Each drawing shows only* 3 *extreme vertices.*

By carefully constructing larger drawings with low numbers of crossings, we obtained upper bounds superior (except for $n = 13$) to all previous ones. For $n \leq 20$ these bounds are listed in line 2 of Table 2. Line 3 comments on the number of inequivalent drawings that exist for the respective number of crossings. Some drawings for $n \leq 81$ are selected in Table 3. The interested reader is encouraged to visit our web site [20] where integer representations for the realizing point sets are provided.

| drawing | $D_{27}$ | $D_{36}$ | $D_{39}$ | $D_{40}$ | $D_{81}$ |
|---|---|---|---|---|---|
| crossings | 6186 | 21191 | 29737 | 33093 | 618616 |

**Table 3: Selected drawings**

Earlier experiments, by Thorpe and Harris [18], were based on randomized search and achieved drawings of $K_{12}$ and $K_{13}$ with 155 and 229 crossings, respectively. Their latter result thus competes with our best examples for $n = 13$.

All drawings reported in Tables 2 and 3 possess exactly three extreme vertices. This adds evidence to the commonly believed conjecture that the convex hull of every minimal rectilinear drawing of $K_n$ is a triangle.

Concerning lower bounds, Theorem 1 directly leads to an improvement by the following well-known argument, first used in Guy [9]. Each of the $n$ subgraphs $K_{n-1}$ of $K_n$ contains at least $\overline{cr}(K_{n-1})$ crossings, each being counted $n - 4$ times because the 4 vertices determining the crossing appear in all but 4 such subgraphs. This implies the recurrence relation

$$\overline{cr}(K_n) \geq \lceil \frac{n}{n-4} \cdot \overline{cr}(K_{n-1}) \rceil. \qquad (1)$$

We further exploit that $\overline{cr}(K_n)$ and $\binom{n}{4}$, the maximum number of crossings in a $K_n$, are known to have the same parity for $n$ odd; see Erdös and Guy [6].

Plugging in the value $\overline{cr}(K_{11}) = 102$ yields the specific lower bounds listed in line 4 of Table 2 and – when driven to larger $n$ – the general lower bound in Theorem 3. Lower and upper bounds happen to match for $n = 12$, which allows us to break new ground again.

THEOREM 2. *The rectilinear crossing number of $K_{12}$ is* 153*. The only known drawing that attains this value is depicted in Figure 1.*

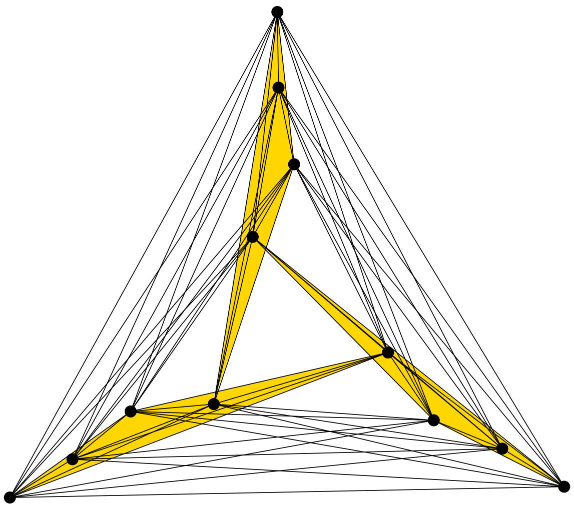

**Figure 1: Minimal drawing of $K_{12}$**

Actually, this result is not incidental. The drawing in Figure 1 is highly symmetric such that each subgraph $K_{11}$ is drawn optimally, which is sufficient for global optimality. This situation cannot be achieved in general, not even for smaller $n$ where it is met only for $K_4$ and $K_6$.

To describe the asymptotic bound, let $\overline{\nu}(n) = \overline{cr}(K_n)/\binom{n}{4}$. Inequality (1) implies that $\overline{\nu}(n)$ is a strictly increasing function and thus exists in the limit. Adopting the notation in [16], we define $\overline{\nu}^* = \lim_{n\to\infty} \overline{\nu}(n)$. By performing calculations through large $n$ we obtained the following value.

THEOREM 3. $\overline{\nu}^* > 0.311507$.

The lower bound for the limit in Theorem 3 has been improved several times. The previously best value has been 0.3001, in Brodsky et al. [5].

The same paper also offers a recursive construction that yields an asymptotic *upper* bound of 0.3838, which is superior to previous results in Singer [17] and in Jensen [11]. Arbitrarily large drawings with few crossings are typically constructed by taking the best drawing $D$ known so far and recursively replacing each of its vertices by $D$. We could show that better results are achieved by a different, somewhat counter-intuitive construction. Replacement of vertices is done only *once*, with a fixed configuration of points in *convex* position. (We omit the details in this extended abstract.)

Despite the fact that convex configurations locally *maximize* the number of crossings, our novel strategy pays off in the global sense. To start with, we used our improved drawings for larger $n$, as given in Table 3. We obtained the following theorem.

THEOREM 4. $\overline{\nu}^* < 0.380739$.

It is worth mentioning that $\overline{\nu}^*$ is a geometric quantity of separate interest, by a connection to Silvester's four point problem, drawn in Scheinerman and Wilf [16].

## 3. REMARKS

Our experimental approach opened the door to various new results on the rectilinear crossing number of $K_n$. This constitutes one more example where computational experiments, when carried out carefully and combined with theoretical arguments, shed new light into notoriously difficult combinatorial problems. The interested reader will find detailed output data of the computations (in particular, point sets in small integer representation) at our web page [20].

A tantalizing question is that for the value of $\overline{cr}(K_{13})$. According to the new bounds in Table 2, $\overline{cr}(K_{13})$ is a member of $\{221, 223, 225, 227, 229\}$. Unfortunately, with current methods it seems out of reach to even perform a computation of all different order types of size 12. Still there is hope that an inspection of the currently 'best' drawings of $K_n$ for small $n$ might reveal regularities that lead to new theoretical insights. A small step in this direction is the observation that, as of yet, all these drawings showed triangular convex hulls. Also, from Tables 1 and 2 we see that, for $n$ odd, the number of inequivalent drawings with small numbers of crossings is quite large, probably because the lack of symmetries. This indicates that drawings attaining $\overline{cr}(K_n)$ might be easier to find for $n$ odd, and leads us to believe that $\overline{cr}(K_{13}) = 229$.

We currently are building up a data base for $n = 11$ of all *projective* order types, including their realizations if existent. This will enable us to access the (Euclidean) order types of size 11 faster while using reasonable sized disk space. An examination of questions similar to those in the present paper then will become easier in the future.

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