φ-Self-Approaching Curves (extended abstract)

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Abstract

We consider all planar oriented curves that have the following property. For each point B on the curve, the rest of the curve lies inside a wedge of angle φ with apex in B; here $\varphi < \pi$ is fixed. This property restrains the curve's meandering. We provide an upper bound for the length of such a curve, divided by the distance between its endpoints, and prove this bound to be tight. A main step is in proving that the curve's length cannot exceed the perimeter of its convex hull, divided by $1 + \cos\varphi$.

Keywords. Self-approaching curves, convex hull, detour, arc length.

1 Introduction

Let f be an oriented curve in the plane running from A to Z, and let φ be an angle in $[0, \pi)$. Suppose that, for every point B on f, the curve segment from B to Z is contained in a wedge of angle φ with apex in B. Then the curve f is called *φ-self-approaching*. The name refers to the fact that, at least for $\varphi \leq \pi/2$, the curve always gets closer to each point on its remaining part. The quantity

$$\frac{\text{length}(f)}{AZ}$$

is called the *detour* of f, where AZ denotes the euclidean distance from A to Z.

At the 1995 Dagstuhl Seminar on Computational Geometry, R. Seidel [1] posed the following open problems. Is there a constant, $c(\varphi)$, so that the detour of every φ-self-approaching curve is at most $c(\varphi)$? If so, how small can one prove $c(\varphi)$ to be?

Both questions are answered by our analysis. We provide, for each φ in $[0, \pi)$, a constant $c(\varphi)$ with the above property, and prove it minimal by constructing a φ-self-approaching curve such that this factor $c(\varphi)$ is achieved.

Self-approaching curves are interesting for different reasons. If a *mobile robot* wants to get to the kernel of an unknown star-shaped polygon and continuously follows the

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angular bisector of the innermost left and right reflex vertices that are visible, the resulting path is self-approaching for $\varphi = \pi/2$; see Icking and Klein [3]. Since the minimum value of $c(\pi/2)$ is known to be ≈ 5.3331, as shown in Icking et al. [4], one immediately obtains an upper bound for the competitive ratio of the robot's strategy.

Furthermore such properties of curves are interesting in their own right. For example, in the book by Croft et al. [2] on open problems in geometry, *curves with increasing chords* are defined by the property that for every four consecutive points A, B, C, D on the curve, B is closer to C than A to D. The open problem of how to bound the length of such curves divided by the distance between its endpoints has been solved by Rote [5]; he showed that the tight bound equals $\frac{2}{3}\pi$. There is a nice connection to the curves studied in this paper. Namely, a curve has increasing chords if and only if it is $\pi/2$-self-approaching in both directions.

2 The tight upper bound for the detour of φ-self-approaching curves

Theorem 1 *The length of a φ-self-approaching curve is not greater than $c(\varphi)$ times the distance of its endpoints, where*

$$c(\varphi) := \begin{cases} 1 & \text{for } \varphi = 0, \\ \max\limits_{\beta \in [0..\frac{\pi}{2}]} \dfrac{2\beta + \pi + 2}{\sqrt{5 - 4\cos\beta}} & \text{for } \varphi = \pi/2, \\ \max\limits_{\beta \in [0..\varphi]} \dfrac{(1 + e^{\pi \cot\varphi}) e^{\beta \cot\varphi} - \frac{2}{1+\cos\varphi}}{\cos\varphi \sqrt{((1 + e^{\pi\cot\varphi}) e^{\beta\cot\varphi} - \cos\beta)^2 + \sin^2\beta}} \quad (*) & \text{otherwise.} \end{cases}$$

The bound $c(\varphi)$ is tight.

Theorem 1 is proved by the following steps: First we show that for three consecutive points B, C, and D on a φ-self-approaching curve f the inequality

$$CD \leq BD - \cos\varphi \cdot \text{length}(f \text{ from } B \text{ to } C)$$

holds. From this we conclude that the length of the curve cannot exceed the perimeter of its convex hull, divided by $1 + \cos\varphi$. This fact is the main tool in our analysis. It allows us to derive an upper bound for the curve's length by circumscribing it with a simple, closed convex curve whose length can be easily determined. Finally, we demonstrate that the resulting bound is tight, by constructing φ-self-approaching curves for which the upper bounds are achieved.

Figure 1 demonstrates how such a curve with maximal detour looks like, for $\varphi = 2$. It consists of three pieces of logarithmic spirals from A to B, to C, and to D, followed by a sequence of many cycloids inside a rectangular area. The angle β maximizes $(*)$ in the definition of $c(\varphi)$. The optimal detour of $c(\varphi)$ is only attained in the limiting case where the height of the rectangle tends to 0.

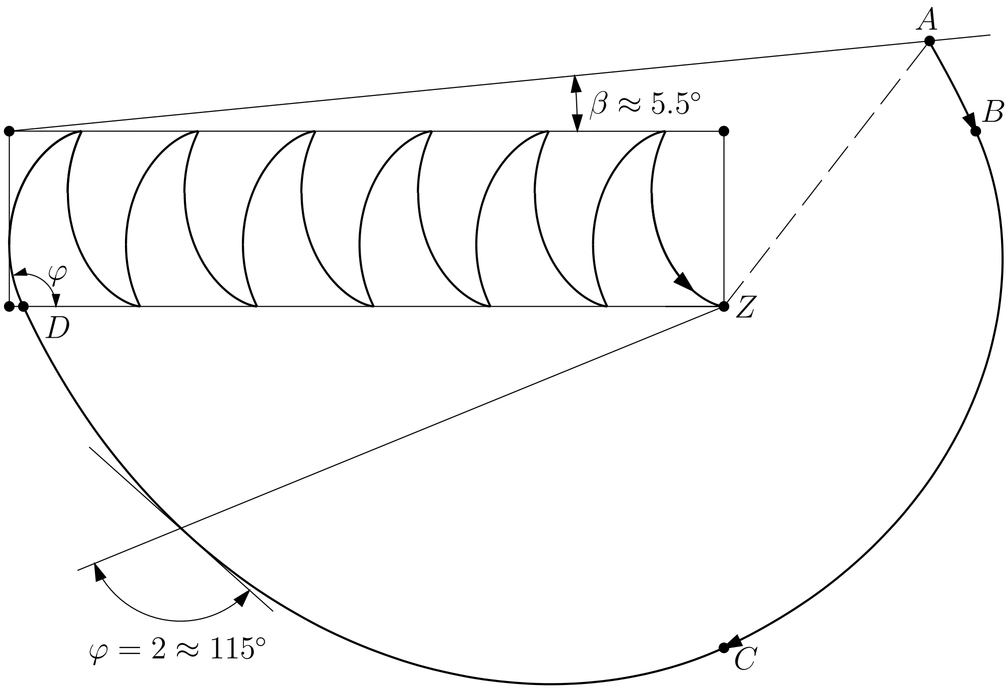

Figure 1: The construction of the φ-self-approaching curve with maximal detour for $\varphi = 2$.

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