

An Improved Lower Bound on the Number of Triangulations

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Abstract

Upper and lower bounds for the number of geometric graphs of specific types on a given set of points in the plane have been intensively studied in recent years. For most classes of geometric graphs it is now known that point sets in convex position minimize their number. However, it is still unclear which point sets minimize the number of geometric triangulations; the so-called double circles are conjectured to be the minimizing sets. In this paper we prove that any set of n points in general position in the plane has at least $\Omega(2.631^n)$ geometric triangulations. Our result improves the previously best general lower bound of $\Omega(2.43^n)$ and also covers the previously best lower bound of $\Omega(2.63^n)$ for a fixed number of extreme points. We achieve our bound by showing and combining several new results, which are of independent interest:

- (1) Adding a point on the second convex layer of a given point set (of 7 or more points) at least doubles the number of triangulations.
- (2) Generalized configurations of points that minimize the number of triangulations have at most $\lfloor n/2 \rfloor$ points on their convex hull.
- (3) We provide tight lower bounds for the number of triangulations of point sets with up to 15 points. These bounds further support the double circle conjecture.

1 Introduction

Bounding the number of geometric graphs of a specific type on a given set of n points in the plane is a very active research topic in the field of combinatorial geometry. The first results [5] on bounding the number of plane Hamiltonian cycles were obtained already in the 1970s. Of particular interest are bounds on $T(n)$, the largest number of triangulations any set of n points can have, since bounds on $T(n)$ can be used to derive bounds on other graph classes (see, for example, [13, 18, 21]). The current bounds stem from Dumitrescu et al. [8], who constructed a set of n points with $\Omega(8.65^n)$ triangulations, and Sharir and Sheffer [20] who proved $T(n) < 30^n$.

Bounds on the minimum number of geometric graphs on a given point set have received comparably less attention, since for many classes of geometric graphs it is known that point sets in convex position minimize their number, see, for example, [2]. However, points in convex position do *not* minimize the number of triangulations. Specifically, points in convex position have C_{n-2} triangulations (where $C_m \in \Theta(4^m/m^{3/2})$ is the m th Catalan number) and the so-called *double circles* have only $\sqrt{12}^{n-\Theta(\log n)}$ triangulations [19]. A *double circle* of size $n = 2h$ has h extreme points, and each edge ab on the convex hull boundary is assigned a different inner point q_{ab} s.t. there is no segment between two points of the double circle crossing the segments $q_{ab}a$ or $q_{ab}b$.



Aichholzer, Hurtado, and Noy [3] proved the first non-trivial asymptotic lower bound on $t(n)$, the smallest number of triangulations any set of n points can have, namely $t(n) \in \Omega(2.33^n)$. They conjectured that $t(n)$ equals the number of triangulations of the double circle. This conjecture is further supported by the results in this paper. The best current bounds on $t(n)$ are as follows: For h extreme points and i inner points, McCabe and Seidel [15] proved $t(n) \in \Omega(2.72^h \cdot 2.2^i)$ and $t(n) \in \Omega(2.63^i)$ when h is a constant. For general point sets, Sharir, Sheffer, and Welzl [22] proved $t(n) \in \Omega(2.4317^n)$.

In this paper we prove the following theorem:

► **Theorem 1.** *Let $t(n)$ be the smallest number of triangulations any set of n points in general position in the plane can have. Then $t(n) \in \Omega(2.631035^n)$ for $n \geq 35064$.*

Our proof uses two basic ingredients: (1) a structural theorem by Aichholzer et al. [3], which provides the mechanism to obtain asymptotic bounds from a large induction base, and (2) tight lower bounds for the number of triangulations of point sets with up to 15 points. These tight lower bounds, together with a combination of known general bounds (Section 5), feed into the structural theorem which then proves Theorem 1 (Section 6).

Computing tight lower bounds on the minimum number of triangulations for $12 \leq n \leq 15$ by brute-force is impossible with current technology. We hence prove two new structural results which we believe to be of independent interest:

- (1) Adding a point on the second convex layer of a given point set (of 7 or more points) at least doubles the number of triangulations (Section 2).
- (2) Generalized configurations of points that minimize the number of triangulations have at most $\lfloor n/2 \rfloor$ points on their convex hull (Section 3).

These results allow us to significantly reduce the number of point configurations to consider and to parallelize computations (Section 4). We could hence complete the automated part of our proof in roughly two months of computation time using up to 128 CPUs (which amounts to several hundred thousand CPU hours). After giving some necessary definitions and background, we describe our proof and computation strategy in more detail below.

Definitions, notation, and background. Let S be a finite point set in general position in the plane, that is, no three points of S lie on a line. A *triangulation* of S is a maximal plane graph on S . We denote the set of all triangulations of S with $\mathcal{T}(S)$ and the number of triangulations of S with $\text{tr}(S) = |\mathcal{T}(S)|$.

The infinite family of sets of n points in general position in the plane can be partitioned into a finite number of equivalence classes by their order types. Two point sets have the same *order type* if there exists a bijection between them such that all point triples have the same orientation [11], that is, clockwise, counterclockwise, or collinear. Point sets with the same order type share many important properties. For example, they contain the same points on the convex hull boundary and the same pairs of edges cross. In particular, the number of triangulations is the same for all point sets of the same order type.

There is a finite number of order types with n points, so it is possible to enumerate them (see the database of all point set order types for up to 11 points [1, 4]). Points in the plane can be mapped to arrangements of lines, whose relative position also determines the orientation of each triple, and hence the order type. Line arrangements can be generalized to *pseudo-line arrangements*, that is, x -monotone curves that intersect pairwise exactly once in a proper crossing. We call the equivalence classes obtained from triple orientations in pseudo-line arrangements *abstract order types*. Abstract order types can be realized by *generalized configurations of points* [9], which are point sets where every pair of points is connected by

exactly one pseudo-line. A point triple (a, b, c) is oriented clockwise iff point c is in the half-plane to the right of the directed pseudo-line through a and b . Many concepts involving point sets, like triangulations, can easily be abstracted to generalized configurations of points. An abstract order type is *realizable* if it is the order type of a point set. All abstract order types with up to eight points are realizable [10]. See [12] for further details.

Proof and computation strategy. The model of pseudo-line arrangements is the crucial tool for exhaustive enumeration of (abstract) order types in [1], and the generation of larger sets fulfilling certain properties [4]. A given representation of a pseudo-line arrangement is augmented by an additional pseudo-line in all possible ways. We use a similar strategy to compute tight lower bounds on the number of triangulations of point sets with up to 15 points: Suppose we know that, when adding a point to a point set (a pseudo-line to a pseudo-line arrangement), the number of triangulations increases by at least a factor α . To verify whether there exists an abstract order type S with n points with $\text{tr}(S) \leq b$, we need to only extend those sets of size $n - 1$ that have at most b/α triangulations (and for these, extend only sets of size $n - 2$ with at most b/α^2 triangulations and so on). The basic idea is to select all order types of size n_0 ($n_0 = 8$ in our case) that have at most b/α^{n-n_0} triangulations, and extend them to order types of size n , provided that the number of triangulations of the intermediate order types are within the bounds.

We are facing various challenges with this strategy due to the vast number of abstract order types. First of all, we need the factor α to be as large as possible. The work by Sharir, Sheffer, and Welzl [22] on vertices with degree 3 in random triangulations implies that every point set contains at least one point which one can remove to reduce the number of triangulations by $1/2$ (see Section 5 for details). However, their result does *not* show which point to remove (or add in our strategy). An abstract order type of size n can be obtained from n different (parent) order types of size $n - 1$. If we do not know which point increases the number of triangulations by a factor $\alpha \geq 2$, we have to extend each order type in all possible ways. Such extensions are computationally infeasible, since we can expect an order type to be created close to $n!/n_0!$ times. However, if we know *which* point gives a factor of at least α , we can identify a *unique* parent order type and compute its number of triangulations only if this order type of size n is to be further extended. In Section 2 we prove that adding a point on the second convex layer of the order type increases the number of triangulations by a factor of at least 2. This structural result gives us the necessary control over the extensions and allows us to check the number of triangulations of each *relevant* order type of size n_0 to n only once. We can hence distribute the work load among different independent processes, each handling a fixed disjoint set of order types.

In principle it is not sufficient to extend order types by interior points only. However, in Section 3 we prove that we need to consider only abstract order types with at most $\lfloor n/2 \rfloor$ points on the convex hull. Since $\lfloor 15/2 \rfloor = 8$ we can start our extension from $n_0 = 8$ and extend by adding interior points only. For $n = 8$ all abstract order types are realizable. Hence our improved bound actually applies to all abstract order types (and not only to point sets). Finally, since all abstract order types with the minimum number of triangulations for $12 \leq n \leq 15$ are realizable as point sets, our results support the double circle conjecture.

2 A factor 2 for points on the second convex layer

Let S be a set of $n \geq 7$ points in general position in the plane. We denote the convex hull of S by $\text{CH}(S)$ and the extremal points of S , that is, the points of S which lie on the boundary of $\text{CH}(S)$, with $\text{extr}(S)$. We say that a point $p \notin S$, with $(S \cup \{p\})$ in general position, is

interior-extremal for S if $p \in \text{extr}((S \cup \{p\}) \setminus \text{CH}(S \cup \{p\}))$. That is, p is an extremal point of the second convex layer of the extended set $(S \cup \{p\})$.

Let $T \in \mathcal{T}(S)$ be an arbitrary triangulation of S . We call a simple polygon whose boundary is formed by edges from T and which does not contain vertices of S in its interior an *empty polygon* of T . For a point $p \notin S$ let P be an empty polygon of T formed by k edges such that p lies in the interior of P , and P is star shaped with respect to p . Then we call P a *k-star* of T for p (see Figure 1). An *edge-flip* (or flip) in a triangulation T is the operation of first removing an inner edge e from T such that the two triangles incident to e form a convex quadrilateral Q , and then inserting the opposite diagonal of Q into T .

The following technical lemma is proven in the appendix.

► **Lemma 2.** *Let $p \notin S$ be an interior-extremal point for S , T a triangulation of S , and P a convex k -star of T for p with $k < |S|$. Let $S' = S \cup \{p\}$ and let T' be the triangulation of S' obtained from T by removing all edges in the interior of P and adding all edges connecting p to vertices of P . Then at least one edge of the boundary of P can be flipped in T' .*

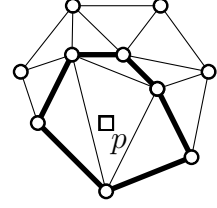


Figure 1 A convex 6-star for p .

2.1 Certificates for triangulations

We now argue that for any given interior-extremal point $p \notin S$, any triangulation $T \in \mathcal{T}(S)$ contains a star for p of a certain type (see below) which we call the certificate of T . We will argue in an algorithmic way, flipping edges of T to p in order to increase the degree of p . However, note that the certificate refers to the original triangulation T . See Figures 2 and 3 for an illustration of all certificates.

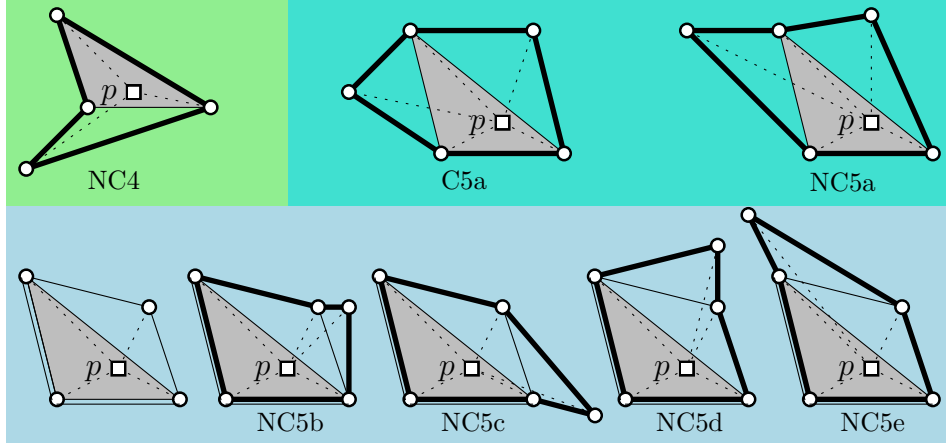
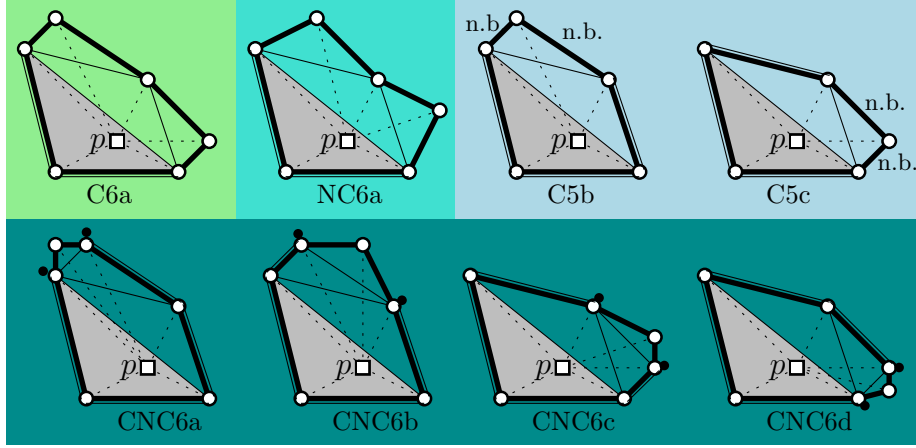


Figure 2 Certificates. Blocked edges are drawn with a double line.

First add p to S and connect p with three edges to the corners of the triangle Δ of T in which p lies to obtain a triangulation T' of $S \cup \{p\}$. By Lemma 2 we can flip at least one edge of Δ , so that p has degree 4. If there exists one flip such that the resulting 4-star is non-convex, then this is the certificate of T (Figure 2: NC4). Otherwise, if at least two of the edges of Δ can be flipped (both flips are in a convex 4-star), we get a (convex or non-convex) 5-star which we take as the certificate of T (Figure 2: C5a and NC5a).

If both previous situations do not exist, then we flip the unique flippable edge to obtain a convex 4-star (c.f. Figure 2, lower left), where the two edges of the boundary near p are *blocked*. We call an edge on the boundary of a star for p blocked, if this edge cannot be



■ **Figure 3** Certificates. Blocked edges are drawn with a double line, non-blocked edges are marked with n.b. The vertices of Certificates CNC which are marked with a black dot can be convex or non-convex.

flipped to p in the underlying triangulation. Note that p can be placed anywhere in the shaded triangle.

Again by Lemma 2 we can flip at least one of the non-blocked edges of this 4-star. If there exists one flip such that the resulting 5-star is non-convex, then we take this as the certificate of T (Figure 2: NC5b to NC5e). Otherwise, if two edges of the 4-star can be flipped (both flips are in a convex 5-star), we get a (convex or non-convex - in the latter case it is important that we do not have collinear points) 6-star which we take as the certificate of T (Figure 3: C6a and NC6a).

In the remaining case, three edges of the unique convex 4-star are blocked. We make the unique possible flip (exists by Lemma 2), obtaining a convex 5-star, where at least three edges are blocked (those, which have already been blocked for the 4-star). If the remaining two edges are non-blocked, then this is our certificate for T (Figure 3: C5b and C5c). Otherwise, we again make the unique flip (exists by Lemma 2), and obtain a convex or non-convex 6-star, where at least four edges are blocked. This will be the certificate of T (Figure 3: CNC6a to CNC6d).

The above argumentation shows that any triangulation T contains a certificate.

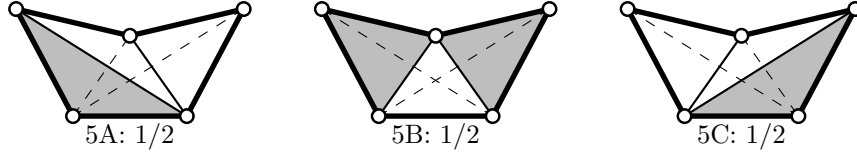
► **Lemma 3.** *Let $p \notin S$ be an interior-extremal point for S and T a triangulation of S . Then T contains a certificate w.r.t. p out of the above list.*

Note that the triangulation T might have more than one certificate, as the above analysis just shows that every triangulation contains a certificate, but does not imply that this is unique. However, it will become clear in the next sections that we can choose any existing certificate for T without causing problems.

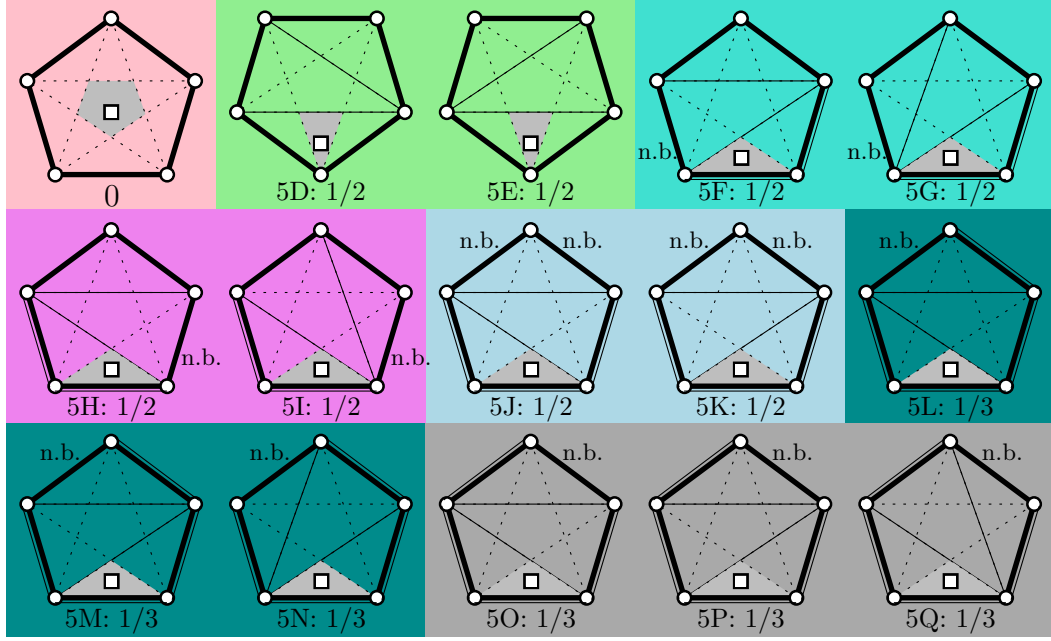
2.2 Distributing weights

Let $p \notin S$ be an interior-extremal point for S and let $S' = S \cup \{p\}$. We will consider all triangulations of S' and distribute weight at most 1 to triangulations of S in a specific way (described below). Initially, all triangulations of S have weight zero.

Let T' be a triangulation of S' and let d be the degree of p in T' . Further, let P be the d -star of p in T' and $G(T')$ be the set of triangulations of S which are obtained from T' by



■ **Figure 4** Non-convex 5-star weight distributing



■ **Figure 5** Convex 5-star weight distributing

deleting p and all its incident edges, and then retriangulating the interior of P in all possible ways. We make a case analysis over the degree d of p in T' .

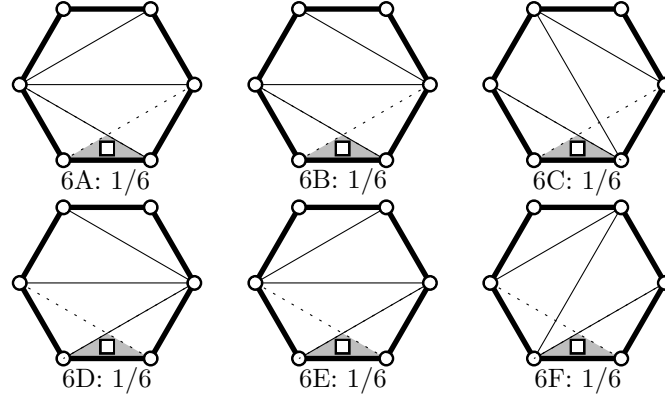
$d = 3$. T' adds weight 1 to the unique triangulation in $G(T')$.

$d = 4$. If P is non-convex, then assign weight 1 to the unique triangulation in $G(T')$. Otherwise assign weight $\frac{1}{2}$ to both triangulations in $G(T')$.

$d = 5$. If P is non-convex, then it can be triangulated in at most 3 different ways. If there are at most two triangulations then we distribute weight $\frac{1}{2}$ to each. If there are three triangulations of P then P has one reflex vertex and 4 diagonals lie in the interior of P . The weight is distributed as indicated in Figure 4, where only those triangulations get weight $\frac{1}{2}$ each, where p lies in the shaded area. Note that in this way a total weight of at most 1 is distributed.

If P is convex, then there are three different positions where p can lie within P ; c.f. Figure 5. If p lies in the central area we do not distribute any weight, and if p lies in the area near a corner c of P we distribute weight $\frac{1}{2}$ to the two triangulations where c is not incident to a diagonal (Cases 5D and 5E). Otherwise, p lies near an edge e , i.e., in the intersection of two neighbored ears. Depending on the blocked versus non-blocked pattern of the edges of P near e , we distinguish 5 disjoint cases as shown in Figure 5. In the first three cases we distribute weight $\frac{1}{2}$ to two triangulations each, and in the remaining 2 cases weight $\frac{1}{3}$ to three triangulations each.

We remark that we distribute weights only for cases depicted in Figures 4 and 5.



■ **Figure 6** Convex or non-convex 6-star weight distributing

$d = 6$. Similar as before we only distribute weight if p lies near an edge e , i.e., in the intersection of two neighbored ears. Moreover, the vertices of e must be convex, while we do not make any assumption about the other vertices. We give weight $\frac{1}{6}$ to 6 different triangulations as indicated in Figure 6. Note that if P has reflex vertices then some of these triangulations do not exist and we simply waste the weight assigned to them.

$d \geq 7$. We do not assign any weight for T' .

Observe that in all cases we assign at most weight 1 for T' , and that there are several cases where we assign weight less than one for T' . We denote with W the total sum of all distributed weights and have the following lemma.

► **Lemma 4.** *For $|S| > 6$ we have $W < |\mathcal{T}'(S')|$.*

The strict inequality comes from the fact that there is at least one triangulation $T' \in \mathcal{T}'(S')$ where p has degree 7 or more, for which we did not distribute any weight. Actually, for $|S| \geq 6$ it holds that $W \leq |\mathcal{T}'(S')| - (|S| - 6)$, as for any $7 \leq d \leq |S|$ we can construct a triangulation of $\mathcal{T}'(S')$ where p has degree d . One way to see this is that there exist triangulations of S' where p has degree 3 and $|S|$, respectively, that the flip-graph of triangulations is connected, and that an edge flip does not change the degree of any vertex by more than one.

2.3 Combining the results

We now show that any certificate listed in Section 2.1 collects weight at least two. Note that we always get weight 1 from the triangle ($d = 3$). So we only have to argue that in addition we get at least weight 1.

NC4. We get weight 1 from the non-convex 4-gon ($d = 4$).

C5a, NC5a. We get two times weight $\frac{1}{2}$ for each convex 4-gon ($d = 4$).

In the following, we also always get weight $\frac{1}{2}$ for the convex 4-gon ($d = 4$) from the first unique flip. Hence it remains to reason that in addition we get at least weight $\frac{1}{2}$.

NC5b, NC5c, NC5d, NC5e. We get weight $\frac{1}{2}$ from cases 5A, 5B, 5C, and 5B, respectively.

For the remaining certificates, p always lies in a convex 5-gon in the shaded ear-area. If p lies in the 'central wedge' then we get weight $\frac{1}{2}$ from at least one of the two cases 5D or 5E. Otherwise we know that p lies near an edge, where for all remaining certificates there

are two possibilities: p either lies in the wedge close to the bottom edge (as indicated in the figures of the certificates), or in the wedge near the lower left edge. In the following cases we will always consider both possibilities in the just mentioned order (separated by ‘or’ and, if necessary, grouped by squared brackets).

C5b. Weight $\frac{1}{2}$ from case 5J or 5G.

C5c. Weight $\frac{1}{2}$ from case 5I or 5K.

C6a, NC6a. Weight $\frac{1}{2}$ from case 5H (use only the top flip) or 5F (use only the right flip).

CNC6a. [Weight $\frac{1}{3}$ from case 5L plus weight $\frac{1}{6}$ from case 6A] or weight $\frac{1}{2}$ from case 5G.

CNC6b. [Weight $\frac{1}{3}$ from case 5O plus weight $\frac{1}{6}$ from case 6B] or [weight $\frac{1}{3}$ from case 5N plus weight $\frac{1}{6}$ from case 6F].

CNC6c. [Weight $\frac{1}{3}$ from case 5Q plus weight $\frac{1}{6}$ from case 6C] or [weight $\frac{1}{3}$ from case 5M plus weight $\frac{1}{6}$ from case 6E].

CNC6d. Weight $\frac{1}{2}$ from case 5I or [weight $\frac{1}{3}$ from case 5P plus weight $\frac{1}{6}$ from case 6D].

As every triangulation of S gets assigned weight at least two, we get Lemma 5, which together with Lemma 4 implies Lemma 6.

► **Lemma 5.** *For $|S| > 6$ it holds that $2|\mathcal{T}(S)| \leq W$.*

► **Lemma 6.** *For $|S| > 6$ it holds that $2|\mathcal{T}(S)| < |\mathcal{T}'(S')|$.*

Note that for all triangulations in the set $\mathcal{T}'(S')$ we constructed the point p has degree at most 6. From the discussion at the end of Section 2.2 it follows that for $|S| \geq 6$ we can also write $|\mathcal{T}'(S')| \geq 2|\mathcal{T}(S)| + (|S| - 6)$. Hence we obtain Theorem 7, which in combination with the number of triangulations of convex point sets implies Corollary 8.

► **Theorem 7.** *Let $t(i, h) := \min_{S, |S|=i+h} \text{tr}(S)$ denote the minimum number of triangulations that every set of $n = i + h \geq 7$ points, with i inner points and h extreme points, exhibits. Let $t(n) := \min_{n=i+h} t(i, h)$ denote the minimum number of triangulations that every set of $n \geq 7$ points exhibits. The following bounds hold for $t(n)$ and $t(i, h)$.*

$$t(n) \geq 2t(n-1) + (n-7) \tag{1}$$

$$t(i, h) \geq 2t(i-1, h) + (n-7) \tag{2}$$

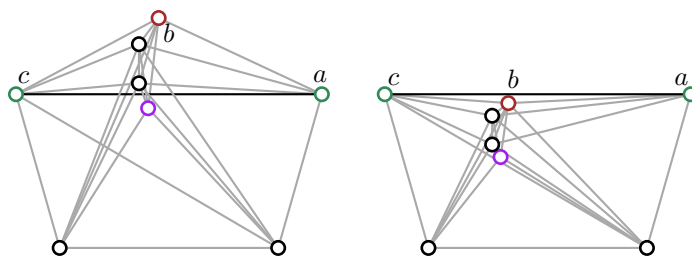
► **Corollary 8.** $t(i, h) = \Omega^*(2^i 4^h)$.

3 Small convex hulls

Here we show that abstract order types which minimize the number of triangulations cannot have more than half of their vertices on the boundary of their convex hulls. We use the following result which is part of the characterization of so-called crossing-minimal point sets [16]. For completeness the proof is given in the appendix.

► **Proposition 9** ([16]). *Let P be an abstract order type with at least four extreme points and three consecutive vertices a , b , and c on the convex hull boundary s.t. no two points inside this triangle are in convex position with a and c . Then the following holds.*

1. *There exists an abstract order type Q and a bijection between P and Q s.t. the images of a and c are consecutive vertices on the convex hull of Q and any point triple in P is oriented as its image in Q (either clockwise or counterclockwise).*
2. *For every crossing-free geometric graph on P , its image on Q is also crossing-free.*



■ **Figure 7** Illustration for Proposition 9. All point triples in the two point sets are the same except that all points are to the left of ac in the right set.

See Figure 7. Intuitively, we need a set Q and a mapping from the triangulations on Q to the triangulations on P that keeps the image crossing-free.

► **Lemma 10.** *Let P and Q be defined as in Proposition 9. Then P has strictly more triangulations than Q .*

Proof. We give an injective mapping of the triangulations of Q to the triangulations of P . Recall that a geometric graph is a triangulation of a point set with h extreme points if it is crossing-free and has $3n - 3 - h$ edges. The analogous holds for graphs on abstract order types. Let h be the number of extreme points of P . Let T be an arbitrary triangulation of Q , which has $3n - 3 - (h - 1)$ edges. As ac is on the boundary of the convex hull, it is an edge of T . We remove ac from T and draw the corresponding graph T' on P , as indicated by the bijection between P and Q . As T' has $3n - 3 - h$ edges, it remains to show that T' is crossing-free.

Suppose that T' on P contains a crossing. Then, the only crossings may occur between two edges containing both a and c . However, T' does not contain the edge ac . Hence, the only crossings could be between an edge as and an edge ct for some points s and t , and at least one of s and t is inside the triangle abc in P (as otherwise the orientation of all point triples in $\{a, c, s, t\}$ would be the same as in Q , implying that there is no such crossing). However, they cannot be both inside the triangle abc due to the characterization of P (as $\{a, c, s, t\}$ would be in convex position). But this means that as and ct cannot cross. Hence, T' is crossing-free.

Since there exists a triangulation of P that contains the edge ac , there are strictly more triangulations of P than of Q . ◀

► **Corollary 11.** *An abstract order type minimizing the number of triangulations is crossing-minimal (in terms of [16]).*

As noted in [16], there cannot be large crossing-minimal point sets with more than half of the points being extreme. Suppose there is an abstract order type with four or more extreme points that minimizes the number of triangulations. Consider the triangles that are formed by all triples of points that are consecutive on the convex hull boundary. By Lemma 10, each such triangle must contain (at least) two points. Any point can only be in at most two such triangles. So for h points on the convex hull boundary, we need at least h interior points to have two points inside each such triangle.

► **Theorem 12.** *Let P be an abstract order type that minimizes the number of triangulations with $|P| \geq 6$. Then P has at most $\lfloor |P|/2 \rfloor$ extreme points.*

The double circle is an example with $n/2$ extreme points where the local improvement due to Lemma 10 is indeed no longer possible.

n	upper bound nr. of triangulations	number of generated abstract order types	all realizable order types	thereof respecting the triangul. bound
8	534	-	3 315	3 315
9	1 069	158 829	158 817	158 817
10	2 139	13 645 342	14 309 547	13 635 816
11	4 278	437 639 146	2 334 512 907	436 539 440
12	8 556	570 219 167	unknown	unknown
13	17 112	32 884 369	unknown	unknown
14	34 224	101 218	unknown	unknown
15	68 448	1	unknown	1

■ **Table 1** Abstract extension from $n = 8$ to $n = 15$.

4 Tight lower bounds for up to 15 points

Exact values for the minimum numbers of triangulations have been known for sets of at most 11 points [3]. To compute those values also for sets with $12 \leq n \leq 15$ points, we start with the exhaustive data base of all order types of cardinality 8 and extend all sets in the abstract setting (that is, without realizing them as point sets). All arguments used in Sections 2 and 3 hold also in the abstract setting, since we used only arguments which use sidedness properties of the underlying (abstract) order type (and no metric properties). By Theorem 12, and because $\lfloor 15/2 \rfloor = 8$, it is sufficient to extend sets by adding interior points only. Moreover, choosing the right insertion order, by Lemma 6 we can further reduce the number of order types that have to be extended: For $n = 12, \dots, 15$, the (in general conjectured triangulation minimizing) double circles have 2236, 7147, 20979, 68448 triangulations, respectively. To show that, for example, for $n = 13$ the minimal number of triangulations is in fact 7147, it is sufficient to extend all (abstract) order types for $n = 12$ with at most $\lfloor 7147/2 \rfloor = 3573$ triangulations. To do so, we need all order types for $n = 11$ with at most $\lfloor 3573/2 \rfloor = 1786$ triangulations, and so on. Table 1 shows the maximum number of triangulations to be considered for each cardinality from 8 to 15. Note that starting at 8 is necessary as for $n \geq 9$ there exist non-realizable abstract order types and the exhaustive list of order types is limited to the realizable ones. For $n = 9$ there exist 158 830 abstract order types, thus all of them except the one with 9 points in convex position were generated.

We want to extend only those order types that have *at most* a certain number of triangulations. To do so we need a counting algorithm that is run for *each* candidate order type to accept or reject order types for further extensions. Previous sets of experiments comparing different counting algorithms [6, 7] yield three potential candidates described in [6, 7, 17]. However [7] is harder to translate to an abstract setting. After implementing and testing the other two, we decided to use the algorithm by Ray and Seidel [17].

To “translate a counting algorithm to the abstract setting” we use only one primitive, namely **orientation**. That is, given any three points, we answer in $O(1)$ time whether the triple turns left (1) or right (-1). Recall that we do not have collinear triples. Our input is the Λ -matrix [14] of the input order type, that is, a 3-dimensional matrix where the entry $\Lambda[i][j][k]$ contains either 1 or -1 for $i \neq j \neq k$. Using Λ we can pre-compute for *every* directed edge $\vec{e} = \overrightarrow{ab} \neq \overleftarrow{ba} = \overleftarrow{e}$ of the input order type S , the set of *empty* triangles Δ_{abc} such that $\Lambda[a][b][c] = 1$. That is, *every* such vertex $c \in \Delta_{abc} \subset S$ lies to the left of the oriented line supporting $\vec{e} = \overrightarrow{ab}$. The set of points c is saved in a data structure Δ so that we can retrieve it in $O(1)$ time.

The algorithm by Ray and Seidel is a divide-and-conquer algorithm that uses memoiza-

tion for efficiency. The algorithm considers a *closed* polygon whose vertices are points of the input order type S , and which might contain other points of S . We call such a polygon, with its contained points, a *pointgon* P . The task is to count all different triangulations of P . The algorithm chooses an edge ab of the boundary of P . Afterwards, all empty triangles \triangle_{abc} fully contained in P are enumerated. Observe that (1) the set of triangles \triangle_{abc} is pairwise-crossing, (2) no two such triangles can appear in the same triangulation of P , and (3) every triangulation of P contains exactly one of those triangles. Hence the number of triangulations $\text{tr}(P)$ of P can then be expressed as $\text{tr}(P) = \sum_{\triangle_{abc}} \text{tr}(P \mid \triangle_{abc})$, where $\text{tr}(P \mid \triangle_{abc})$ is the number of triangulations of P containing the triangle \triangle_{abc} .

If for the empty triangle \triangle_{abc} , the point c is a vertex of P , then the algorithm divides P into two sub-pointgons P_{ac}, P_{cb} and proceeds recursively on each one. For such a splitting triangle \triangle_{abc} , the number of triangulations is $\text{tr}(P \mid \triangle_{abc}) = \text{tr}(P_{ac}) \cdot \text{tr}(P_{cb})$. If c is fully contained inside P , the algorithm continues counting with the pointgon $P_c = P \setminus \triangle_{abc}$ (P_c is a pointgon where the edge ab is replaced by the two edges ac, cb). Thus $\text{tr}(P \mid \triangle_{abc}) = \text{tr}(P_c)$. Observe that in both cases the resulting sub-polygon(s) are smaller than P (less triangles in any of their triangulations than in P) and hence the algorithm eventually terminates. To compute the number of triangulations of S , the convex hull of S (with its interior points) is used as initial pointgon P .

5 A huge induction base

From the abstract extension described in the last section we obtain the *exact values* of $t(n)$ for $3 \leq n \leq 15$, see Table 5 in the appendix. Moreover, in a similar manner we could also produce *exact values* of $t(i, h)$ for $3 \leq i + h \leq 16$, see Table 2.

$n \setminus h$	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	-	-	-	-	-	-	-	-	-	-	-	-	-
4	1	2	-	-	-	-	-	-	-	-	-	-	-	-
5	2	3	5	-	-	-	-	-	-	-	-	-	-	-
6	4	6	9	14	-	-	-	-	-	-	-	-	-	-
7	11	13	19	28	42	-	-	-	-	-	-	-	-	-
8	32	30	43	62	90	132	-	-	-	-	-	-	-	-
9	96	89	102	145	207	297	429	-	-	-	-	-	-	-
10	305	272	250	352	497	704	1001	1430	-	-	-	-	-	-
11	991	849	776	878	1230	1727	2431	3432	4862	-	-	-	-	-
12	3297	2694	2453	2236	3114	4344	6071	8502	11934	16796	-	-	-	-
13	11204	9022	7865	7147	8025	11139	15483	21554	30056	41990	58786	-	-	-
14	?	?	?	23128	20979	29004	40143	55626	77180	107236	149226	208012	-	-
15	?	?	?	?	68448	76473	105477	145620	201246	278426	385662	534888	742900	-
16	?	?	?	?	?	203748	280221	385698	531318	732564	1010990	1396652	1931540	2674440

Table 2 Minimum number of triangulations for n points with h extreme points. Overall *tight* lower bounds for fixed $3 \leq n \leq 15$ are selected in gray, see also Table 5 in the appendix.

The entries of Table 2 marked with a question mark are entries that do currently not seem to be computable within a reasonable amount of time (approx. six to twelve months of computation using a cluster of several CPUs). To extend Tables 2 and 5 with lower bounds (not with *exact* values) for $t(n)$ and $t(i, h)$, for larger values of $n = i + h$, we use the inequalities provided by Theorem 7 along with two other inequalities from previous work of Sharir, Sheffer, and Welzl [22], and Aichholzer, Hurtado, and Noy [3], respectively.

Sharir, Sheffer, and Welzl [22] derived lower bounds for the number of triangulations by considering the expected number of vertices with degree three in a triangulation. Let \hat{v}_3 be the *expected* number of inner vertices with degree three for a triangulation of a set S of $n = i + h$ points, with i the number of inner points and h the number of extreme points.

Let $f(i, h)$ be an upper bound for \hat{v}_3 . Then there exists an inner vertex $v \in S$ for which it holds that, in a (random) triangulation, $P(\deg(v) = 3) \leq f(i, h)/i$ (the probability that v has degree 3). From that we get $\text{tr}(S \setminus \{v\}) = \text{tr}(S) \cdot P(\deg(v) = 3)$, as any triangulation

of $S \setminus \{v\}$ corresponds to a triangulation of S where v has degree 3 (just remove v and its three incident edges). This implies $\text{tr}(S) \geq \frac{i}{f(i,h)} \text{tr}(S \setminus \{v\})$, which in turn implies:

$$t(i, h) \geq \frac{i}{f(i, h)} t(i-1, h) \quad (3)$$

The following two upper bounds for \hat{v}_3 were shown by Sharir et al. [22]: (1) $\hat{v}_3 \leq \frac{i}{2}$ for $n \geq 7$ (Lemma 2.2 in [22]) and (2) $\hat{v}_3 \leq \frac{2i+h/2}{5}$ for $n \geq 6$ (Lemma 2.3 in [22]). Combining them with (3) we obtain:

$$t(i, h) \geq 2t(i-1, h) \quad \text{for } n \geq 7 \quad (4)$$

$$t(i, h) \geq \frac{5i}{2i+h/2} t(i-1, h) \quad \text{for } n \geq 6 \quad (5)$$

Observe that our lower bound (2) from Theorem 7 is better than bound (4) of Sharir et al. by the rather small additive factor of $n-7$. Our bound (2) is, however, considerably stronger in the sense that we know it holds for an *interior-extremal* point while bound (4) holds for *some* inner point of S , not necessarily interior-extremal. This additional property of our bound made it feasible to extend our computations to 15 points.

Aichholzer, Hurtado, and Noy [3] proved the following lower bound for $t(i, h)$:

$$t(i, h) \geq \left\lceil \left(h \cdot \sum_{EXT} + i \cdot \sum_{INT} \right) / (6n - 4h - 6) \right\rceil \quad \text{for } n = i + h \geq 4, \quad (6)$$

where \sum_{INT} is bounded by

$$\sum_{INT} \geq t\left(\frac{n}{2} + 1\right)^2 + 2 \cdot \sum_{k=1}^{\frac{n}{2}-1} \min_j \left\{ t\left(\frac{n}{2} + 1 + j\right) \cdot t\left(\frac{n}{2} + 1 - j\right) \mid 0 \leq j \leq \min\left\{k, \frac{n}{2} - 2\right\} \right\}$$

for $n \geq 4$ even¹, and by

$$\sum_{INT} \geq 2 \cdot \sum_{k=0}^{\frac{n-3}{2}} \min_j \left\{ t\left(\frac{n+3}{2} + j\right) \cdot t\left(\frac{n+1}{2} - j\right) \mid 0 \leq j \leq \min\left\{k, \frac{n-5}{2}\right\} \right\}$$

for $n \geq 5$ odd. For all $n \geq 4$, \sum_{EXT} is bounded by $\sum_{EXT} \geq \sum_{k=3}^{n-1} t(k) \cdot t(n+2-k)$.

Observe that (6) makes use of the general minimum number of triangulations $t(n)$. To see that this gives indeed a recursive inequality observe that \sum_{EXT} and \sum_{INT} denote expressions that involve values of $t(k)$ only for $k < n$.

Combining the bounds. We want to extend Tables 2 and 5 (in the appendix) with lower bounds for $t(n)$ and $t(i, h)$ for higher values of $n = i + h$. For this we consider lower bounds (2), (5), and (6), and for each new entry $t(i, h)$ of Table 2 we keep the largest value found among the three inequalities. For the missing entries of Table 2 with $n \leq 15$ we additionally test the largest bound found against the general tight lower bound found at $t(n)$ and we keep the largest of the two. As a reference, the missing entries (marked with a question mark) of Table 2 are filled as in Table 3. Observe, for example, that for $n = 15$ entries for $h \leq 6$ match that of entry for $h = 7$. This is because the general tight lower bound for $n = 15$ is larger than the bounds we obtain using the three aforementioned inequalities. For $n \geq 16$, entry $t(n)$ of Table 5 is set after the corresponding entries $t(i, h)$ of Table 2 have been computed. More precisely (and by definition of $t(n)$): $t(n) = \min_{n=i+h} t(i, h)$.

¹ Note that in [3] there was a typo in that formula, as the sum was wrongly taken up to $n/2 + 1$. However, in the text right after the formula this was stated correctly and it was also used correctly for the computations given in [3].

$n \setminus h$	3	4	5	6	7	8
3	1	-	-	-	-	-
4	1	2	-	-	-	-
5	2	3	5	-	-	-
6	4	6	9	14	-	-
7	11	13	19	28	42	-
8	32	30	43	62	90	132
9	96	89	102	145	207	297
10	305	272	250	352	497	704
11	991	849	776	878	1230	1727
12	3297	2694	2453	2236	3114	4344
13	11204	9022	7865	7147	8025	11139
14	≥ 26223	≥ 20979	≥ 20979	23128	20979	29004
15	≥ 68448	≥ 68448	≥ 68448	≥ 68448	68448	76473
16	≥ 161787	≥ 157957	≥ 153659	≥ 148800	≥ 143264	203748

■ **Table 3** Completing missing entries of Table 2. Overall lower bounds for fixed $3 \leq n \leq 16$ are selected in gray. Entries for $h \geq 9$ are omitted for readability.

6 Improved Asymptotics

Aichholzer et al. [3] proved a structural theorem which provides the mechanism to obtain asymptotic bounds from a large induction base such as the one we provide in Tables 2 and 5 (in the appendix). For simplicity and completeness we summarize their results (mainly Theorem 2 and Corollary 1 of [3]) in the following theorem.

► **Theorem 13** ([3]). *Let $a \geq 3$ be an integer. If for every n with*

$$\left\{ \begin{array}{l} a \leq n \leq 2a - 2 \\ a \leq n \leq 2a + 2 \\ a \leq n \leq 2a^2 - 11a + 21 \\ a \leq n \leq 2a^2 - 13a + 29 \end{array} \right\} \text{ the bound } t(n) \geq \frac{1}{k} \cdot \tau^{n-2} \text{ with } \left\{ \begin{array}{l} k = 1 \\ k = 2 \\ k = 3 \\ k = 4 \end{array} \right\} \text{ holds,}$$

then the bound $t(n) \geq \frac{1}{k\tau^2} \cdot \tau^n = \Omega(\tau^n)$ holds for all $n \geq a$.

To use Theorem 13 we need to ensure a lower bound $t^-(n, k) = \frac{1}{k} \cdot \tau^{n-2}$ within a given interval $a \leq n \leq b(a, k)$. For $k = 3, 4$ the upper end $b(a, k)$ of the interval is quadratic in a , resulting in very large intervals for which $t(n) \geq t^-(n, k)$ has to be checked. Hence, for proving a bound of $t(n) \geq \frac{1}{k} \cdot \tau^{n-2}$ for some chosen τ and all $n \geq a \geq 3$, we have to verify $t(n) \geq t^-(n, k)$ for all $a \leq n \leq b(a, k)$, using Table 5 (which in turn requires Table 2) for large enough values of n , see Section 5. In other words, considering four intervals $a_k \leq k \leq b_k = b(a_k, k)$ define $\tau_{\max} = \max_{1 \leq k \leq 4} \{ \min_{a_k \leq n \leq b_k} (k \cdot t(n))^{\frac{1}{n-2}} \}$ and let k_{\max} be the $1 \leq k \leq 4$ at which τ_{\max} is achieved. Theorem 13 implies that $t(n) \geq \frac{1}{k_{\max}\tau^2} \cdot \tau^n = \Omega(\tau_{\max}^n)$ for all $n \geq a_{k_{\max}}$. Table 4 shows lower bounds on $t(n)$ for $16 \leq n \leq 40$ along with values of $\tau = \tau(n, k)$, for $1 \leq k \leq 4$.

For example, following Theorem 13 we obtain that for $k = 2$ and a base range of $29 \leq n \leq 40$, $\tau(n, 2) \geq 2.3778$, and thus $t(n) \geq \Omega(2.3778^n)$ for $n \geq 19$. To obtain a better lower bound for $t(n)$ we extend Table 4 for larger values of n . More precisely, we extend it up to $n \leq 70131$. Figure 11 in the appendix shows the growth of $\tau(n, 2)$. We thus obtain the following values of $\tau(n, k)$ (exact and truncated up to the shown precision):

$$\begin{aligned} \tau(n, 1) &\geq 2.630983 && \text{for } 35066 \leq n \leq 70130 \\ \tau(n, 2) &\geq 2.631035 && \text{for } 35064 \leq n \leq 70130 \\ \tau(n, 3) &\geq 2.584586 && \text{for } 190 \leq n \leq 70131 \\ \tau(n, 4) &\geq 2.588402 && \text{for } 190 \leq n \leq 69759 \end{aligned}$$

Theorem 13 now implies that $t(n) \geq \Omega(2.631035^n)$ for $n \geq 35064$, which proves Theorem 1.

n	$t(n) \geq$	Eq.	$k = 1$	$k = 2$	$k = 3$	$k = 4$
16	143264	(5)	2.3350	2.4535	2.5256	2.5781
17	304818	(5)	2.3206	2.4303	2.4969	2.5453
18	657451	(5)	2.3100	2.4122	2.4742	2.5190
19	1434439	(5)	2.3022	2.3981	2.4559	2.4978
20	3160629	(5)	2.2966	2.3868	2.4412	2.4805
21	7023620	(5)	2.2926	2.3778	2.4291	2.4662
22	17826487	(6)	2.3043	2.3856	2.4344	2.4697
23	461173087	(6)	2.3171	2.3948	2.4415	2.4752
24	121862959	(6)	2.3309	2.4055	2.4503	2.4825
25	315689696	(6)	2.3417	2.4133	2.4562	2.4871
26	837664309	(6)	2.3539	2.4229	2.4641	2.4939
27	2205647348	(6)	2.3645	2.4309	2.4707	2.4993
28	5849635236	(6)	2.3749	2.4391	2.4774	2.5050
29	14684224691	(5)	2.3798	2.4417	2.4786	2.5052
30	37399997550	(6)	2.3856	2.4454	2.4810	2.5067
31	95073504069	(6)	2.3908	2.4486	2.4831	2.5079
32	241432822521	(6)	2.3956	2.4516	2.4850	2.5089
33	612064398552	(6)	2.4000	2.4543	2.4866	2.5098
34	1561238258557	(6)	2.4046	2.4572	2.4886	2.5110
35	3973209273535	(6)	2.4087	2.4598	2.4902	2.5120
36	9878502893156	(6)	2.4109	2.4606	2.4901	2.5113
37	24308011357985	(6)	2.4124	2.4606	2.4893	2.5098
38	59495132694369	(5)	2.4133	2.4602	2.4881	2.5081
39	145702365782129	(5)	2.4143	2.4599	2.4870	2.5065
40	357019041982701	(5)	2.4152	2.4597	2.4861	2.5050

■ **Table 4** Lower bounds on $t(n)$ and $\tau(n, k)$ for $16 \leq n \leq 40$ and $k = 1, \dots, 4$.

Computational limits. With our current methods a stronger lower bound on $t(n)$ seems not feasible, see Figure 11 in the appendix for example. The current bound required (exact) computations on very large numbers — for both, the number of triangulations (integer arithmetic) and for $\tau(n, k)$ (floating-point arithmetic). For the former we dealt with integers that require close to 100 000 bits of precision. For the latter we performed *exact* floating-point arithmetic with truncation up to 128 bits of precision. Extending Table 4 up to $n \leq 70131$ and computing the respective $\tau(n, k)$, for $1 \leq k \leq 4$, took about 65 hours on a desktop machine with a somewhat recent processor: Intel i7-4770 CPU at 3.40GHz. For arithmetic on big integers we used the well-known GMP library (for C), and for exact floating-point arithmetic we used the also well-known MPFR library (also for C). The data produced for this paper can be downloaded from: <http://www.victoralvarez.net/papers/aahpsv.tgz> (around 500 MB).

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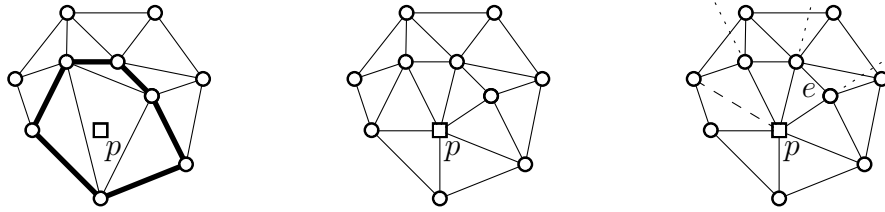
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A Omitted proofs

► **Lemma 2.** *Let $p \notin S$ be an interior-extremal point for S , T a triangulation of S , and P a convex k -star of T for p with $k < |S|$. Let $S' = S \cup \{p\}$ and let T' be the triangulation of S' obtained from T by removing all edges in the interior of P and adding all edges connecting p to vertices of P . Then at least one edge of the boundary of P can be flipped in T' .*

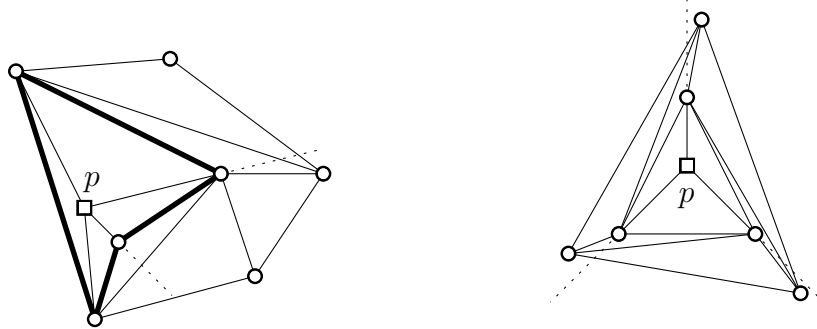


■ **Figure 8** A convex 6-star for p , retriangulating and flipping an edge.

Proof. As p is an interior-extremal point of S , the k -star P has at least one extreme point of S as a vertex. Moreover, as $k < |S|$, there is at least one point of S in the exterior of P . Thus there exist triangles of T' in the exterior of P , implying that not all edges of P can be edges on the boundary of the convex hull of S .

Now assume that none of the edges of P can be flipped in T' . Consider rays emanating from p and going through the vertices of P . For an edge e of P which cannot be flipped and which is not on the convex hull of S , the incident outer triangle must have its third vertex outside the convex wedge formed by the rays passing through the endpoints of e . W.l.o.g. we assume that this vertex is to the right of the right of the two rays, that is, clockwise shifted. Thus, the edge of P cyclically right of e cannot be on the convex hull of S either, and since we assume that it is also not flippable we can repeat the argumentation. But as P has at least one vertex on the convex hull of S , there must be an edge of P for which the third vertex of the outer triangle is not to the right of the wedge formed by the two incident rays, a contradiction. ◀

Note that both conditions, namely that p is an interior-extremal point of S and that P is convex, are crucial, as otherwise the lemma does not hold (see Figure 9).



■ **Figure 9** No flip possible: non-convex 4-star and non-interior-extremal point p .

► **Proposition 9.** *Let P be an abstract order type with at least four extreme points and three consecutive vertices a , b , and c on the convex hull boundary s.t. no two points inside this triangle are in convex position with a and c . Then the following holds.*

1. There exists an abstract order type Q and a bijection between P and Q s.t. the images of a and c are consecutive vertices on the convex hull of Q and any point triple in P is oriented as its image in Q (either clockwise or counterclockwise).
2. For every crossing-free geometric graph on P , its image on Q is also crossing-free.

Proof. See the full version of [16]. We show that Q exists by taking a dual pseudo-line arrangement P^* of P and transforming it to a dual pseudo-line arrangement of Q (consisting of x -monotone curves in the Euclidean plane). See Figure 7 and Figure 10 for an accompanying illustration. W.l.o.g., let a^*c^* be below b^* . For any point p in the triangle abc , the pseudo-line p^* is separating a^*c^* from a^*b^* and b^*c^* . Since any two points p and q inside the triangle are not in convex position with ac , the crossing p^*q^* is not above both a^* and c^* . Hence, we can re-route these pseudo-lines to pass below a^*c^* and again have a pseudo-line arrangement. If we also re-route b^* , b is no longer an extreme point. Since the relative position of no other pseudo-lines and crossings is affected by this re-routing, the other orientations of point triples are maintained. The resulting pseudo-line arrangement is a dual representation of Q .

Suppose there is a graph G' that is crossing-free on P but its image G on Q contains a crossing. Then, the only crossings may occur between two edges containing both a and c . However, the edge ac cannot be crossed in G . Hence, the only crossings could be between an edge as and an edge ct for some points s and t , and at least one of s and t is inside the triangle abc in P . Since these edges do not cross on P , the triple (s, t, a) is oriented in the other direction as the triple (s, t, c) in P (i.e., if P is a point set, the supporting line of s and t separates a and c). This also has to be the case in Q . But this means that as and ct cannot cross. Hence, G is crossing-free. \blacktriangleleft

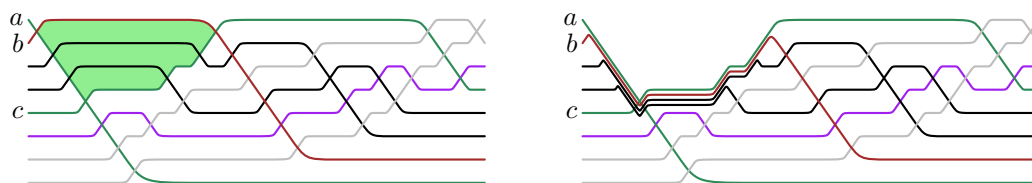
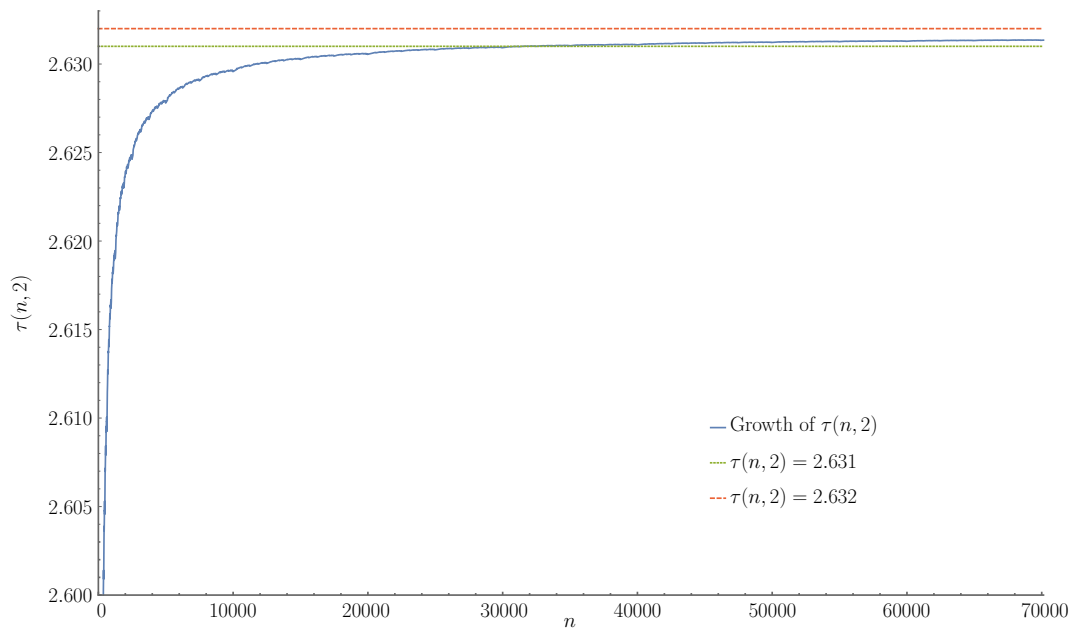


Figure 10 Illustration of the pseudo-line arrangements (dual to the point sets in Figure 7) for Proposition 9.

B Additional tables and figures

n	$t(n)$	n	$t(n)$	n	$t(n)$
3	1	8	30	13	7 147
4	1	9	89	14	20 979
5	2	10	250	15	68 448
6	4	11	776		
7	11	12	2 236		

Table 5 Exact minimum number $t(n)$ of triangulations for $3 \leq n \leq 15$.



■ **Figure 11** $\tau(n, 2)$ is the largest achievable $\tau(n, k)$ for $1 \leq k \leq 4$. Its growth is shown and compared against bounds 2.631 (dotted) and 2.632 (dashed). For clarity, the y -axis of the plot is adjusted to better observe the growth of $\tau(n, 2)$, which exceeds 2.600 for $n \geq 400$.