Transforming Spanning Trees and Pseudo-Triangulations

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Abstract

Let T_S be the set of all crossing-free straight line spanning trees of a planar n-point set S. Consider the graph $\mathcal{T}_S$ where two members T and T' of T_S are adjacent if T intersects T' only in points of S or in common edges. We prove that the diameter of $\mathcal{T}_S$ is $O(\log k)$, where k denotes the number of convex layers of S. Based on this result, we show that the flip graph $\mathcal{P}_S$ of pseudo-triangulations of S (where two pseudo-triangulations are adjacent if they differ in exactly one edge – either by replacement or by removal) has a diameter of $O(n \log k)$. This sharpens a known $O(n \log n)$ bound. Let $\widehat{\mathcal{P}}_S$ be the induced subgraph of pointed pseudo-triangulations of $\mathcal{P}_S$. We present an example showing that the distance between two nodes in $\widehat{\mathcal{P}}_S$ is strictly larger than the distance between the corresponding nodes in $\mathcal{P}_S$.

1 Introduction

Let S be a set of n points in general position in the plane (no three points of S are on a common line). We denote by T_S the set of all crossing-free straight line spanning trees of S. Several authors investigated the question of whether, and how fast, two members of T_S can be transformed into each other by means of predefined rules. Avis and Fukuda [5] considered the graph with node set T_S where two spanning trees are adjacent if they have all but one edge in common (i.e., differ by a single edge move). They showed that the diameter of this graph is at most $2n - 4$. The impact of several more involved transformations, including length-reducing edge moves and so-called edge slides, has been studied in Aichholzer, Aurenhammer, and Hurtado [1]. Recently, Aichholzer and Reinhardt [4] proved that the edge slide distance between two trees in T_S is $O(n^2)$.

Define the graph $\mathcal{T}_S = (T_S, A)$ whose set of arcs A consists of pairs of trees in T_S that intersect each other only in points of S or in common edges. In other words, A contains all pairs (T, T') such that no edge of T crosses any edge of T'. The arcs of $\mathcal{T}_S$ correspond to rather powerful transformations, namely, the replacement of a tree by some 'compatible' tree. Not surprisingly, a bound of $O(\log n)$ on the diameter of $\mathcal{T}_S$ is easily obtained. A core result in [1] states that $\mathcal{T}_S$ contains a path of size $O(\log n)$ from every member of T_S to the *minimum* spanning tree of S such that tree lengths decrease along this path.

In this note we prove that the diameter of $\mathcal{T}_S$ is $O(\log k)$, where k is the number of convex layers $L_1, \ldots, L_k$ of S. That is, L_1 is the boundary of the convex hull of S, and L_i is defined recursively as the boundary of the convex hull of $S \setminus \bigcup_{j<i} L_j$, for $2 \le i \le k$ and $k = \min\{i \mid L_{i+1} = \emptyset\}$. We do not know whether this bound is asymptotically tight. In particular, we do not have any example where the diameter of $\mathcal{T}_S$ is not a constant.

Interestingly, the diameter of $\mathcal{T}_S$ is related to flip distances in pseudo-triangulations. A *pseudo-triangle* is a planar polygon with exactly three interior angles less than π. A pseudo-triangulation of S is a partition of the convex hull of S into pseudo-triangles whose vertex set is S. The flip graph of pseudo-triangulations, $\mathcal{P}_S$, has as its set of nodes all possible pseudo-triangulations of S. Two pseudo-triangulations are connected in $\mathcal{P}_S$ by an arc if they differ in exactly in one edge, either by replacement or removal. In other words, each arc of $\mathcal{P}_S$ corresponds to an exchanging or a removing edge flip. Aichholzer, Aurenhammer, and Krasser [3] (see also [2]) proved that the diameter of $\mathcal{P}_S$ is $O(n \log n)$. Let $\widehat{\mathcal{P}}_S$ be the induced subgraph of elements of $\mathcal{P}_S$ having exactly $2n - 3$ edges (the minimum number of edges a pseudo-triangulation of S can have). The elements of $\widehat{\mathcal{P}}_S$ are called minimum, or pointed, pseudo-triangulations; each vertex of such a pseudo-triangulation is *pointed*, that is, all its incident edges lie in an angle less than π. Bereg [6] showed that the diameter of $\widehat{\mathcal{P}}_S$ is still bounded by $O(n \log n)$, and in [2] it is proved that if the diameter of $\widehat{\mathcal{P}}_S$ is $O(n)$ then the same is true for $\mathcal{P}_S$. On the other hand, the flip distance for triangulations (i.e. pseudo-triangulations having the maximum number of edges) is known to be $\Theta(n^2)$ in the worst case. In [8], Hurtado, Noy, and Urrutia refined the bound for triangulations to $O(nk)$.

For both graphs $\mathcal{P}_S$ and $\widehat{\mathcal{P}}_S$ it is an open problem to determine tight asymptotic bounds on the diameter. We prove that the diameter of $\mathcal{P}_S$ is $O(n \log k)$, using our result for the graph $\mathcal{T}_S$. In particular, if the diameter of $\mathcal{T}_S$ is constant then the diameter of $\mathcal{P}_S$ is $\Theta(n)$. We also demonstrate that the distance between certain nodes in $\widehat{\mathcal{P}}_S$ is strictly larger than the distance between the correspond-

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ing nodes in $\mathcal{P}_S$. A more comprehensive study of the diameters of $\mathcal{T}_S$ and $\mathcal{P}_S$ can be found in Huemer [7].

2 An upper bound on the diameter of $\mathcal{T}_S$

We show that the diameter of $\mathcal{T}_S$ can be related to the number k of convex layers of the point set S.

Observation 1 *Let Δ be a triangulation of S. Let x be a point of S which lies on some layer L_i of S, for $i \geq 2$. Then Δ contains an edge xy that does not cross L_i and such that y lies on a layer L_j with $j < i$.*

Proof. If such an edge does not exist then x must be a pointed vertex of Δ. But a triangulation does not contain pointed vertices, except on layer L_1. $\square$

Theorem 1 *Let a point set S be given whose number of convex layers is k. The diameter of $\mathcal{T}_S$ is $O(\log k)$.*

Proof. Define a *layer tree* to be a non-crossing spanning tree of S which contains all but one edge of each layer L_i of S and which connects consecutive layers (by single edges); Figure 1 gives an example. We show below that any given spanning tree $T \in T_S$ can be transformed into some layer tree using $O(\log k)$ transformations, as defined by the arcs of $\mathcal{T}_S$. This implies the theorem, because two layer trees clearly can be made to coincide by applying at most two transformations.

Consider some triangulation Δ that contains T. Due to Observation 1, for every point $x \in S \setminus L_1$ there is an edge in Δ to some layer with lower index. We select one such edge per point in $S \setminus L_1$, and in addition, all edges but one of L_1. The selected edges constitute a new spanning tree, T', of S. As T' and T live in the same triangulation, a single transformation is capable of replacing T by T'.

For a point $x \in S$, let $g_k(x)$ be the shortest path from x to a point on L_k such that $g_k(x)$ does not cross T'. As no edge of T' crosses any layer twice, $g_k(x)$ visits points on layers with increasing index. On the other hand, by Observation 1, there is a path $g_1(x)$ from x to L_1 that does not cross T' and that visits points on layers with decreasing index. Now, for all points x on layers $L_1, \ldots, L_{k/2}$ take the path $g_1(x)$, and for all points x on layers $L_{k/2}, \ldots, L_k$ take the path $g_k(x)$. The union of all these paths with L_1 is a connected graph, G. By construction, G neither crosses the tree T' nor the layer $L_{k/2}$. We select from G a spanning tree, T'', which contains L_1 minus one edge. T' can be transformed into T'' in one step, and T'' can be transformed into a spanning tree T_1 that contains, in addition, $L_{k/2}$ minus one edge, in one more step.

In summary, after a constant number of transformations we arrive at two independent subproblems of size $k/2$. Therefore, with the same effort, we can transform the tree T_1 into a tree that contains, in addition to L_1 and $L_{k/2}$, from *both* layers $L_{k/4}$ and $L_{3k/4}$ all edges but one. We conclude that $O(\log k)$ tranformations suffice to generate a layer tree for S. $\square$

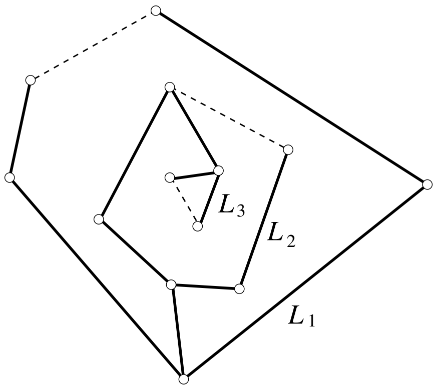

Figure 1: A layer tree for $k = 3$

3 Bounding the diameter of $\mathcal{P}_S$

Next we show that an upper bound on the diameter of $\mathcal{T}_S$ also gives an upper bound on the diameter of the flip graph $\mathcal{P}_S$ of pseudo-triangulations. We make use of a lemma from [3] on flip distances in simple polygons.

Lemma 2 *Let Q be a simple polygon with m edges. The flip distance between any two triangulations of Q is $O(m)$, if exchanging, removing, and inserting edge flips are allowed.*

Theorem 3 *Let a set S of n points be given. If the diameter of $\mathcal{T}_S$ is d then the diameter of $\mathcal{P}_S$ is $O(nd)$.*

Proof. Every pseudo-triangulation of S can be completed to a triangulation by applying $O(n)$ inserting edge flips. It thus suffices to show that any two triangulations Δ_1 and Δ_2 are connected in $\mathcal{P}_S$ by a sequence of $O(nd)$ flips. Let Δ_1 and Δ_2 contain spanning trees T_1 and T_2 of S, respectively. There is a path of length d in $\mathcal{T}_S$ which connects T_1 and T_2. We show that for consecutive trees T and T' on this path, the distance in $\mathcal{P}_S$ between any triangulation Δ containing T and any triangulation Δ' containing T' is $O(n)$.

Let us 'cut' the triangulation Δ along the edges of T. This gives several triangulated polygons. Note that no edge of T' crosses any edge of such a polygon, because T' and T are adjacent in $\mathcal{T}_S$. By Lemma 2, we can modify the triangulation within each such polygon Q_i so as to contain all the edges of T' within Q_i in $O(m_i)$ flips, if Q_i has m_i edges. Thus Δ can be transformed into a triangulation Δ'' that contains T' in $O(\sum m_i)$ flips. This sum is bounded by $n + 2(n-1)$, counting the edges of L_1 plus two times the edges of T. Similarly, in a second step, we cut Δ'' along T' and transform Δ'' into the desired triangulation Δ' using another $O(n)$ flips. $\square$

Corollary 4 *The diameter of the flip graph $\mathcal{P}_S$ is bounded by $O(n \log k)$, where k is the number of convex layers of S.*

Lemma 2 also holds for pointed pseudo-triangulations [3]. Thus, we are also interested in the graph $\widehat{\mathcal{T}}_S$ of pointed spanning trees of S, where two trees are adjacent if there exists a pointed pseudo-triangulation which contains them both. If we can bound the diameter of $\widehat{\mathcal{T}}_S$ by d then the diameter of $\widehat{\mathcal{P}}_S$ is $O(nd)$, applying the argumentation of Theorem 3. Moreover, the bound $O(nd)$ carries over to the flip graph $\mathcal{P}_S$ by a result in [2].

4 Comparing distances in $\mathcal{P}_S$ and $\widehat{\mathcal{P}}_S$

In a pointed pseudo-triangulation the number of edges is minimum. This suggests that this type of pseudo-triangulation is most flexible as far as adaption by flips is concerned. In other words, one might conjecture that the distance between two nodes of $\mathcal{P}_S$ does not increase when we are required to stay within the subgraph $\widehat{\mathcal{P}}_S$ of $\mathcal{P}_S$. In the following we present an example that refutes this conjecture. Assume $n > 9$ in the following.

The underlying set S of n points is shown in Figures 2 and 3. It consists of three subsets $P = \{p_1, .., p_{n/3}\}$, $Q = \{q_1, .., q_{n/3}\}$, and $R = \{r_1, .., r_{n/3-1}\}$ in convex position, and one interior point m. The last point is chosen to lie to the left of $p_1q_{n/3}$, of $q_1r_{n/3-1}$, and of $r_1p_{n/3}$. Two pointed pseudo-triangulations PT_1 (Figure 2) and PT_2 (Figure 3) are drawn on S.

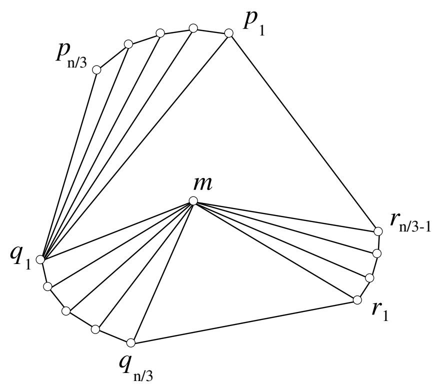

Figure 2: The pseudo-triangulation PT_1

Lemma 5 *The distance between PT_1 and PT_2 in $\widehat{\mathcal{P}}_S$ is at least $n-3$.*

Proof. Every pointed pseudo-triangulation of S has exactly $2n-3$ edges. PT_1 and PT_2 have the $n/3-1$ edges r_im in common, plus the $n-1$ edges of the convex hull of S. Thus $2n/3-1$ edges are different. We will show that to obtain the very first edge of PT_2 that is not in PT_1, at least $n/3-1$ (exchanging) edge flips are required. Then, when the first edge is present, there still remain at least $2n/3-2$ different edges. For each such edge, at least one flip is needed in addition. This gives a lower bound of $n-3$ flips.

PT_1 contains edges mq_i and q_1p_i which must be transformed into edges $p_{n/3}q_i$ and mp_i of PT_2. If the first edge of PT_2 that is created is of type $p_{n/3}q_i$ then this edge crosses $n/3-1$ edges q_1p_i. All these edges must be replaced beforehand. If, otherwise, the first edge of PT_2 that is created is of type mp_i then either all edges mq_i or all edges mr_i must be replaced first, because the point m has to stay pointed. So, in any case, at least $n/3-1$ flips are necessary to obtain the first edge of PT_2. □

Lemma 6 *The distance between PT_1 and PT_2 in $\mathcal{P}_S$ is at most $2n/3$.*

Proof. We construct a sequence of $2n/3$ flips that transforms PT_1 into PT_2. The edge flip that inserts the edge mp_1 into PT_1 is applied first. Now edge q_1p_i is flipped into edge mp_{i+1} by an exchanging flip, for $i = 1, \ldots, n/3-1$. Next, edge mq_i is exchanged by edge $p_{n/3}q_{i+1}$, for $i = 1, \ldots, n/3-1$. Finally, we apply the edge flip that removes $mq_{n/3}$ and gives PT_2. □

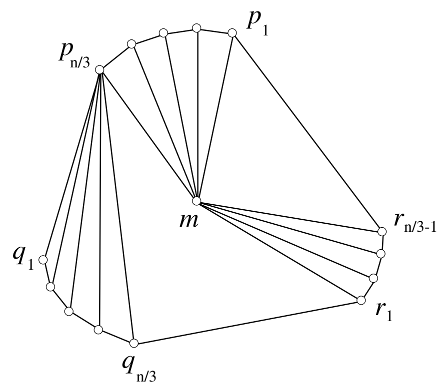

Figure 3: The pseudo-triangulation PT_2

Corollary 7 *There exist pointed pseudo-triangulations PT_1 and PT_2 whose distance in the graph $\widehat{\mathcal{P}}_S$ can only be realized by a flip sequence that affects edges common to PT_1 and PT_2.*

Proof. To see this, let the subset R in Figures 2 and 3 consist of a single point r_1. Then the edge mr_1 is common to PT_1 and PT_2. If flipping mr_1 is not allowed then at least $n-3$ flips are needed to transform PT_1 into PT_2, by the same arguments as in the proof of Lemma 5. Otherwise, we change mr_1 (in PT_1) to the edge mp_1 first. Then we apply the same sequence of $2n/3-2$ exchanging flips as in the proof of Lemma 6. Finally, $mq_{n/3}$ is changed to mr_1 which gives PT_2. This sequence consists of only $2n/3$ flips. □

Note that the reduction in flip distance in Lemma 6 and Corollary 7, respectively, stems from creating the edge mp_1. This edge is outruled in Lemma 5 by the required pointedness of m, and in Corollary 7 by being the result of flipping the common edge mr_1.

5 Conclusion and open problems

We gave a bound on the diameter of the graph $\mathcal{T}_S$ of non-crossing spanning trees and related this result to transforming pseudo-triangulations. The problem of bounding the diameter of $\mathcal{T}_S$ is also of interest on its own. So far we were not able to find two non-crossing spanning trees on the same point set S whose distance in $\mathcal{T}_S$ is more than constant.

Conjecture 1 *The diameter of $\mathcal{T}_S$ is sublogarithmic.*

We also restate the well known problem of determining the diameter of the flip graph $\mathcal{P}_S$ of pseudo-triangulations.

Conjecture 2 *The diameter of $\mathcal{P}_S$ is $\mathbf{o}(n \log n)$.*

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