Towards Compatible Triangulations

Oswin Aichholzer*
Franz Aurenhammer
Hannes Krasser†
Institute for Theoretical Computer Science
Graz University of Technology, Graz, Austria
{oaich,auren,hkrasser}@igi.tu-graz.ac.at

Ferran Hurtado‡
Departament de Matematica Aplicada II
Universitat Politecnica de Catalunya, Barcelona, Spain
hurtado@ma2.upc.es

Abstract

We state the following conjecture: any two planar n-point sets (that agree on the number of convex hull points) can be triangulated in a compatible manner, i.e., such that the resulting two planar graphs are isomorphic. The conjecture is proved true for point sets with at most three interior points. We further exhibit a class of point sets which can be triangulated compatibly with any other set (that satisfies the obvious size and hull restrictions). Finally, we prove that adding a small number of Steiner points (the number of interior points minus two) always allows for compatible triangulations.

1 Introduction

Can any two planar point sets (that agree on the number of points and extreme points) be triangulated in the same manner? This intuitive question is the topic of the present paper. Apart from the theoretical interest in this basic problem, questions of this kind arise in various areas of application, including image analysis, morphing, and cartography.

To define the problem more carefully, let S_1 and S_2 be two finite sets of points in the Euclidean plane. Two triangulations T_1 of S_1 and T_2 of S_2 are called *compatible* if the face lattices formed by their triangles, edges, and

*Research supported by APART [Austrian Programme for Advanced Research and Technology] of the Austrian Acedemy of Sciences, and the EU Improving Human Potential Programme HPRI-1999-CT-00071

†Research supported by the FWF [Austrian Fonds zur Förderung der Wissenschaftlichen Forschung]

‡Research supported by Proyectos DGES-MEC PB98-0933 and Gen. Cat. 1999SGR00356

vertices (points) are isomorphic. Compatible triangulations obviously exhibit isomorphic graphs of edges. The converse is not true, in general, as an edge isomorphism between T_1 and T_2 need not preserve the external face. This led us to prefer the notion of *compatible triangulations* to *isomorphic triangulations* which sometimes has been used in the literature.

The problem of triangulating two given point sets compatibly comes in two flavors, namely where the bijection between the points of S_1 and S_2 is either fixed or variable. The case of fixed correspondence is a known problem and has been studied in Saalfeld [7]. He pointed out that compatible triangulations do not always exist and proposed several heuristic approaches for their construction. The challenging problem left open is to determine the complexity of the related decision problem, which so far is neither known to be NP-complete nor to be polynomially solvable.

The problem becomes easier if S_1 and S_2 represent the cyclic order of the vertices of two simple n-gons. Given a bijection between the polygon vertices (there are only n cyclic shifts to choose from) the existence can be decided in time $O(n^2)$, and compatible triangulability of the polygons can always be achieved by adding $O(n^2)$ extra points (so-called Steiner points) in either polygon; see Aronov et al. [2]. Improvements which take into account the number of reflex vertices [5] or the quality of the angles of the produced triangles [3] exist.

The present work is concerned with the problem of finding compatible triangulations *without* fixed point correspondence. To our knowledge this problem has not been studied before. The main question is, of course, under which propositions compatible triangulations do exist. It is clear that in both sets S_1 and S_2 the number n of points must be the same, as has to be the number h of points on their convex hulls $CH(S_1)$ and $CH(S_2)$, by the well-known formula $e = 3n - h - 3$ on the number of edges of a triangulation. Moreover, convex hull points necessarily map to convex hull points, and their cyclic order is preserved. This follows from the easily proven fact below: if two triangulations are compatible then any two corresponding triangles ijk and $i'j'k'$ must have the same orientation, either clockwise or counter-clockwise. (We ignore the symmetric case where the orientation is reversed for each matching pair of triangles.) Intuitively speaking, this means that the underlying point sets S_1 and S_2 'locally' exhibit the same *order type*, as defined in Goodman and Pollack [4]. It is interesting to note that the last two properties need *not* hold if only graph isomorphism rather than compatibility of the triangulations is required.

We will assume throughout that all point sets considered are in general position, meaning that no three points within a set lie on a common straight line. There are simple examples of point sets not in general position that do not admit compatible triangulations even if the propositions mentioned above are all met; see e.g. Krasser [6].

Throughout, let $|CH(S)|$ denote the number of extreme points in a point set S. Our main conjecture can be formulated in the following way.

Conjecture 1 *(Main Conjecture on Compatible Triangulations)*
Let S_1 and S_2 be two sets of points without fixed correspondence. Then compatible triangulations for S_1 and S_2 exist if these sets fulfill the following properties:

- $|S_1| = |S_2|$,
- $|CH(S_1)| = |CH(S_2)|$, *and*
- S_1 *and* S_2 *are in general position.*

We also consider an extended version of the compatible triangulation problem, where the correspondence between the points *on the convex hulls* (but not between the interior points) is prescribed. The status of the extended problem is still unresolved, too.

Conjecture 2 *(Extended Conjecture on Compatible Triangulations)*
Conjecture 1 still holds if the correspondence between the extreme points of S_1 and S_2 is given by a cyclically shifted labeling.

The motivation for stating the extended conjecture stems from the attempt to prove Conjecture 1 by induction, which naturally leads to assuming the stronger version as a hypothesis.

In this paper, first steps towards proving the conjectures above are undertaken. In fact, obtaining an affirmative answer would be a deep theoretical result, showing that all planar point sets of the same size and hull size are 'topologically equivalent' in this sense. The result, if true, gains in importance in view of the huge number of inequivalent order types for n points; see Aichholzer et al. [1] for the exact numbers of order types for $n \leq 10$.

Section 2 exhibits a family of point sets each of whose members allows for a compatible triangulation with any other point set (within the evident size and hull restrictions). In Section 3 we prove Conjecture 2 provided the number of interior points in either set is three or less. These restricted cases already reflect part of the intrinsic complexity of the problem: the orientation of the three interior points need to be reversed to allow for a solution in some cases. Section 4 shows that compatible triangulability can always be achieved by adding at most $k-2$ Steiner points, where k denotes the number of interior points. This contrasts the afore-mentioned situation for simple polygons [2], where a quadratic number of Steiner points is necessary in the worst case, and is a result of practical relevance. Finally, Section 5 concludes the paper with some related remarks.

2 A family of universal sets

A point set U is called *weak universal* if for every point set S with $|S| = |U|$ and $|CH(S)| = |CH(U)|$ there is a triangulation of S compatible with a triangulation of U. The set U is called *universal* if, in addition, the correspondence between the extreme points of S and U may be prescribed by a cyclic shift. Conjecture 2, if true, states that all point sets are universal. Here we prove that certain point sets are indeed universal.

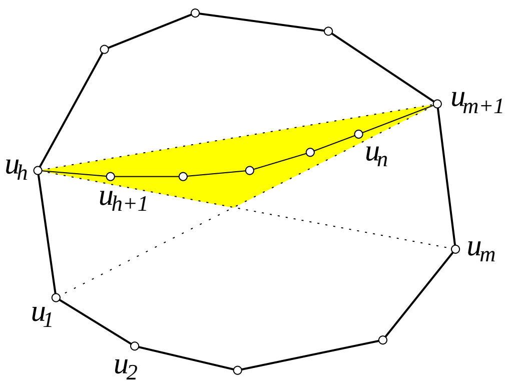

Figure 1: The point set U.

Let U be a point set with extreme points $u_1, \ldots, u_h$ (as they appear counter-clockwise on the convex hull) and internal points $u_{h+1}, \ldots, u_n$ such that, for some $1 \leq m \leq h-2$,

- the subset $U \setminus \{u_1, \ldots, u_m\}$ is in convex position, and
- the internal set $\{u_{h+1}, \ldots, u_n\}$ lies to the left of both $u_h u_m$ and $u_1 u_{m+1}$; see Figure 1.

We are going to prove that U is a universal set. The proof of the result requires a preliminary lemma.

Lemma 1 *Let P be a convex polygon with vertices $v_1, \ldots, v_h$ in counter-clockwise order, and let I be an arbitrary point set interior to P. For any choice of indices $1 \leq i, j \leq h$ with $i \leq j-2$ there exists a simple path π of edges such that*

(a) path π connects v_i, I, and v_j (in this order) and these are the only vertices of π;

(b) the simple polygon bounded by π and $v_i v_{i+1} \ldots v_j$ can be triangulated without using an edge that connects two vertices of π.

Proof. By induction on $d = j - i \geq 2$, the distance between v_i and v_j on P's boundary. For $d = 2$ the result is trivial: sort $I \cup \{v_i, v_j\}$ radially around v_{i+1}, to produce a path star-shaped as seen from v_{i+1}, and triangulate by connecting all points in I to this vertex.

Let now $d > 2$. If the set I is totally to the right of $v_{i+2} v_j$ then connect v_i and v_{i+2} and slice off P the triangle $v_i v_{i+1} v_{i+2}$ which cannot contain any point in I. Apply induction to the remaining convex polygon; the boundary distance between v_i and v_j is now $d-1$.

If, on the other hand, there are points of I to the left of $v_{i+2}v_j$ then let p be the one maximizing the angle $v_i v_{i+1} p$; see Figure 2. Connect p to v_{i+1} and v_{i+2} and observe that the resulting triangle encloses no points of I. Now consider the convex polygon $Q = pv_{i+2} \dots v_j$ (shaded in Figure 2). Let I' and I'' be the subsets of I exterior and interior to Q, respectively. By construction, no point of I' lies to the right of $v_{i+1}p$. So, as done for the case $d = 2$ above, construct the path π' for $I' \cup \{v_i, p\}$ that is star-shaped as seen from v_{i+1}, and build the corresponding triangulation. Finally, apply induction to the polygon Q whose boundary distance between p and v_j is $d-1$. This yields a simple path π'' for $I'' \cup \{p, v_j\}$ whose concatenation with π' obviously is still simple and satisfies the statement of the lemma. ◻

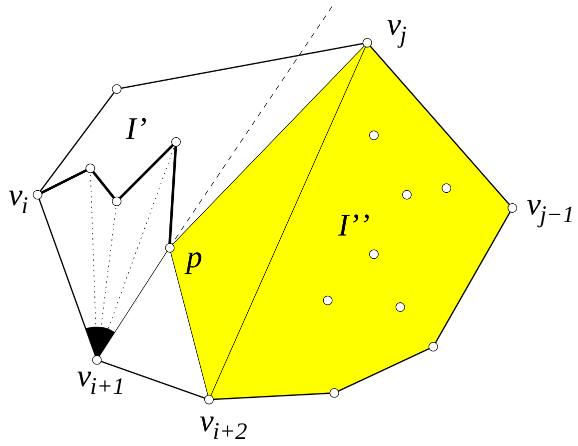

Figure 2: Constructing the triangulation $T(\pi)$.

Theorem 1 *Let U be a set as described above. Then U is a universal set.*

PROOF. Let S be any point set with $|S| = |U| = n$ and with extreme points $v_1, \dots, v_h$ in counter-clockwise order. We construct for $CH(S)$ the path π from v_h to v_{m+1}, and the associated triangulation $T(\pi)$ on π's right-hand side, whose existence is given by Lemma 1. (Note that the index m is given from U and that the counter-clockwise boundary distance between v_h and v_{m+1} is at least two.) The crucial property is that $T(\pi)$ can be reproduced on the simple polygon $u_h \dots u_n u_{m+1} u_m \dots u_1$: only edges between non-consecutive vertices of the path $u_h \dots u_n u_{m+1}$ lie outside this polygon, but those correspond to edges outruled by the definition of $T(\pi)$. It remains to be observed that any triangulation of $CH(S)$ on π's left-hand side can be put on top of the convex polygon $u_{m+1} u_{m+2} \dots u_n$, making compatible the global triangulations. ◻

3 Few interior points

Below we prove Conjecture 2 for point sets with at most three interior points. Even in these seemingly simple situations, the illusion of a quick proof is destroyed by the fact that the orientation of the three interior points might have to be reversed to achieve compatibility. We first focus on the following class of point sets.

Lemma 2 *Let S be a point set containing an extreme point p such that $S \setminus \{p\}$ is in convex position. Then Conjecture 2 is true.*

Proof. The set S fulfills the requirements for the set U in Section 2 for $m = 1$ and thus is a universal set by Theorem 1. ◻

Theorem 2 *Let S_1 and S_2 be two n-point sets each of which contains $k \leq 3$ interior points. Then Conjecture 2 is true.*

Proof. The cases $k = 0$, where S_1 and S_2 are in convex position, and $k = 1$, where star-like compatible triangulations always exist, are trivial.

The case $k = 2$ is still easy. It is always possible to find two extreme points of S_1 labeled a and b such that (1) a and b lie on different sides of the line through the two interior points of S_1, and (2) the (extreme) points of S_2 labeled a and b lie on different sides of the line through the two interior points of S_2. (The non-existence of such a pair a, b would imply that one of S_1 and S_2 totally lies on one side of the respective line.) We build two triangles for S_1 by joining its interior points x and y and connecting a and b to both of them. The remaining extreme points can be connected to either x or y accordingly. Doing the same for S_2 results in compatible triangulations.

The case $k = 3$ is more intriguing. Let us refer to the triangle formed by the three interior points of S_1 (or S_2) as the *interior triangle* of S_1 (or S_2). We distinguish between two situations, the former being similar to the case $k = 2$.

(a) S_1 as well as S_2 contains extreme points with labels a, b, c each of which allows for a triangle adjacent to the respective interior triangle such that no overlap of these four triangles occurs. In this case, we integrate these triangles for both sets. For extreme points with labels intermediate to a, b, and c, edges can be drawn now to a unique interior point, in a compatible manner and much like the case $k = 2$.

(b) Otherwise. The solution we are going to construct will not contain the interior triangle. (In fact, no such solution exists.) Let xyz be this triangle for S_1. Let D_x be the subset of S_1 not on x's side of the line through y and z. Define subsets D_y and D_z analogously; see Figure 3(a). Since labels as above cannot be found, the subsets of S_2 that correspond to D_x, D_y, and D_z have to look as in Figure 3(b), where two of them are totally contained in the same trilateral cell. Without loss of generality, let the third subset correspond to D_z. We wish to redefine D_x, D_y, and D_z so as to be disjoint sets, by introducing 'breakpoint' labels a, b, and c such that D_x ranges from $a + 1$ to b, D_y ranges

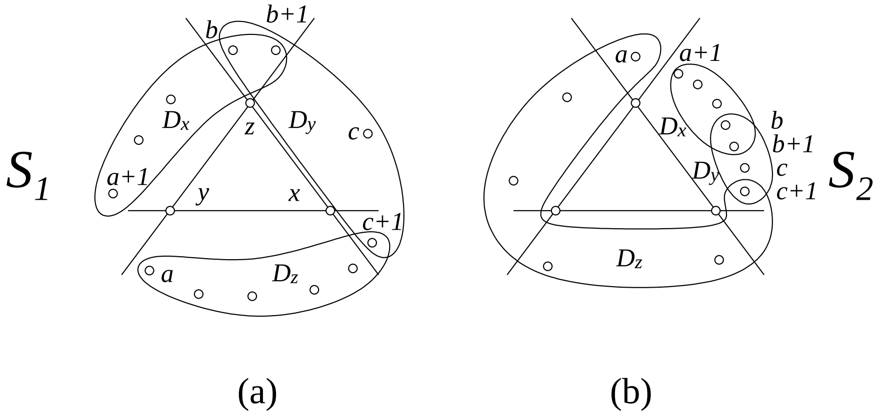

Figure 3: Two subsets move into the same cell.

from $b+1$ to c, and D_z ranges from $c+1$ to a. This is always possible unless two of the sets are singletons consisting of the same point p. In this case, however, removal of p leaves the set $S_1 \setminus \{p\}$ in convex position such that compatible triangulations exist by Lemma 2. Otherwise, we assign labels x, y, and z to the interior triangle of S_2 and draw compatible edges as shown in Figure 4. Note that the orientation of xyz has been reversed for S_2. The two shaded polygons for S_1 are convex and thus can be triangulated compatibly with their counterparts for S_2. It remains to partition the still untriangulated area for S_1 into convex parts. Possible reflex angles may occur at x or y. If, as in Figure 4, this happens for x then there must exist a point labeled d in $D_z \setminus D_y$ which makes this angle convex. (If d would not exist then all extreme points of S_2 would lie on the same side of the line through x and z.) The situation for y is similar. This completes the construction of compatible triangulations. ◻

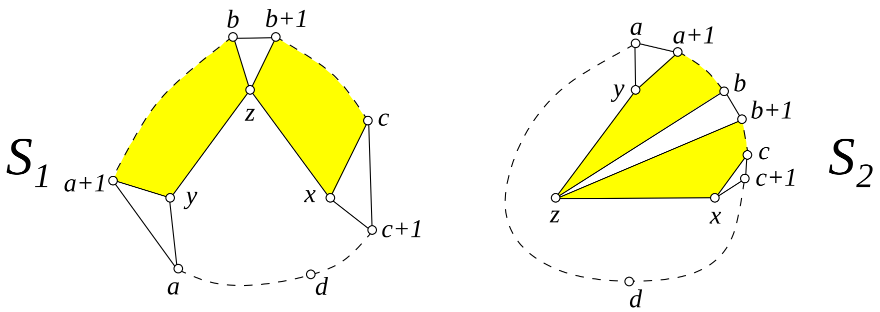

Figure 4: Compatible edge skeletons that yield convex parts.

4 When Steiner points are allowed

Since it is still left open whether Conjectures 1 and 2 are true, the question of whether compatibility can be always forced by adding a reasonably small number of Steiner points suggests itself. We demonstrate below that a linear number suffices. An efficient algorithm for finding compatible triangulations is implicit in the proof.

Let v be an extreme point of the set S. The *ear* $E(v)$ of v is the triangle spanned by the two edges of $CH(S)$ incident to v. An ear is called *empty* if it does not enclose any (interior) point of S.

Theorem 3 *Let S_1 and S_2 be two n-point sets whose $n-k$ extreme points are in fixed correspondence. If $k \geq 2$ then S_1 and S_2 can be triangulated compatibly by introducing at most $k-2$ Steiner points.*

Proof. By induction on k, the number of interior points. Let $\sigma(k)$ be the number of Steiner points needed. We trivially have $\sigma(2) = \sigma(3) = 0$ by Theorem 2, so let us assume $k \geq 4$.

Let $v_1, \ldots, v_h$ be the extreme points of S_1, and $w_1, \ldots, w_h$ the corresponding extreme points of S_2, in counter-clockwise order. We may assume that for no index i the ears $E(v_i)$ and $E(w_i)$ are simultaneously empty. Otherwise, compatible edges $v_{i-1}v_{i+1}$ and $w_{i-1}w_{i+1}$ are introduced, and the sets $S_1 \setminus \{v_i\}$ and $S_2 \setminus \{w_i\}$ are considered. It remains to discuss the cases below.

Case (a) There is an index i such that both ears $E(v_i)$ and $E(w_i)$ are non-empty. Let $p \in S_1$ be the first point hit when rotating the edge $v_{i-1}v_i$ counter-clockwise around v_{i-1}. As $E(v_i)$ contains points, p must be an interior point. So by introducing one Steiner point s slightly to the left of $v_{i-1}p$ and close to edge v_iv_{i+1}, it is possible to create an empty triangle v_ips and three adjacent empty triangles as shown in Figure 5(a). Obviously, the same construction can be repeated for the ear $E(w_i)$ using a suitable Steiner point t, giving four compatible triangles. We now apply induction to the sets $S_1 \cup \{s\} \setminus \{v_i\}$ and $S_2 \cup \{t\} \setminus \{w_i\}$ whose number of interior points is $k-1 \geq 3$. This implies $\sigma(k) \leq \sigma(k-1) + 1 \leq (k-1) - 2 + 1 = k-2$.

Case (b) For each index i, exactly one of the ears $E(v_i)$ and $E(w_i)$ is empty. Clearly S_1 (and S_2) has $h \geq 4$ extreme points in this case. Let us assume $h \geq 5$ first. Then one of the sets, S_1 say, has at least three empty ears. Choose two of them which do not overlap, $E(v_i)$ and $E(v_j)$, for $j \geq i+2$. Consider the convex polygon $CH(S_1) \setminus (E(v_i) \cup E(v_j))$, and some triangulation T_1 thereof which uses only edges which separate $E(v_i)$ and $E(v_j)$. Let T_2 denote the compatible triangulation of the corresponding polygon for S_2. Now sweep T_1, triangle by triangle, with a line ℓ_1 starting at position $v_{i+1}v_{i-1}$ and stopping at position $v_{j-1}v_{j+1}$. When sweeping a triangle $v_av_bv_{b+1}$ we require ℓ_1 to pass through point v_a and to cross the edge v_bv_{b+1}. Sweep T_2 with a line ℓ_2 in a similar manner. Let α_1 be the current number of interior points of S_1 to the left of ℓ_1 (same for α_2). Since $E(v_i)$ and $E(v_j)$ are empty ears we have $\alpha_1 = 0$ in the beginning and $\alpha_1 = k$ at the end. On the other hand, $\alpha_2 \geq 1$ in the beginning and $\alpha_2 \leq k-1$ at the end, because $E(w_i)$ and $E(w_j)$ are both non-empty. As a consequence, there exist indices a and b that yield $\alpha_1 = \alpha_2$ while ℓ_1 sweeps the triangle $v_a v_b v_{b+1}$ and ℓ_2 sweeps the corresponding triangle $w_a w_b w_{b+1}$. Now pick a Steiner point s on (this very position of) ℓ_1 and close to $v_b v_{b+1}$, and put in edges from s to v_b, v_{b+1}, and v_a; see Figure 5(b). Repeat the construction for S_2 which gives a compatible triangle. Finally, apply induction to the respective subsets of S_1 and S_2 on either side of the lines. They have k' and k'' interior points, respectively, for $1 \leq k', k'' \leq k-1$. We may assume $k' \geq 2$ because $k \geq 4$. We get $\sigma(k) \leq \sigma(k') + \sigma(k'') + 1$, with $\sigma(k') \leq k'-2$ and $\sigma(k'') \leq k''-1$ and equality in the last term if $k''=1$. In summary, $\sigma(k) \leq k-2$ again.

The case $h=4$ which has not been covered yet is easier. There is a line ℓ_1 crossing the edges v_1v_2 and v_3v_4 which has exactly 2 interior points of S_1 to its left and $k-2 \geq 2$ such points to its right. There is also a similar line ℓ_2 that crosses the corresponding edges for S_2. Pick two Steiner points on ℓ_1, one close to v_1v_2 and the other close to v_3v_4, and create the two respective triangles, as shown in Figure 5(c). Repeat the construction for ℓ_2 and S_2 which results in two compatible triangles. Now apply induction to the respective subsets of S_1 and S_2 on either side of the lines. This gives $\sigma(k) \leq \sigma(2)+\sigma(k-2)+2 \leq (k-4)+2 = k-2$ which completes the argumentation. ◻

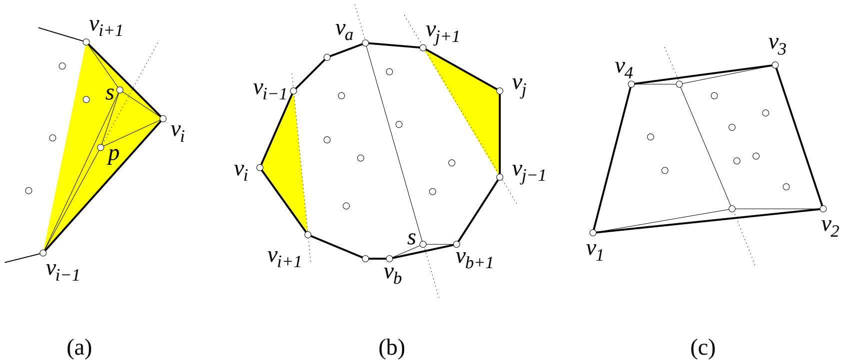

Figure 5: Placing Steiner points.

5 Remarks

It is a challenging open problem to prove our conjectures for general point sets. Unfortunately, the constructions in the proof of Theorem 2 do not seem to generalize easily to the case of more interior points. Approaches to prove Conjecture 2 by induction suffer from the lack of appropriate methods for splitting the problem. Here is a discouraging related fact: given two quadrilaterals, one convex and the other star-shaped, plus one interior point for each, it may not be possible to triangulate them compatibly if the correspondence between their boundary vertices is fixed by a cyclic shift.

We tested Conjecture 2 for small point set sizes, utilizing a database of combinatorially inequivalent point sets (i.e., order types); see [1]. If two point sets exhibit the same order type then every triangulation of one set also leads to a compatible triangulation for the other. By checking all different pairs of order types we could verify the conjecture for up to 8 points. Although our database provides all order types of size up to 10, exhaustive tests turned out to be time consuming and are still incomplete for size 9. In fact, determining the time complexity of computing compatible triangulations, if they exist, for two given n-point sets is an open problem.

Finally, we pose the problem of triangulating compatibly two point sets in *fixed* correspondence when Steiner points are allowed. In view of the applications mentioned in [7], a fast algorithm using a small number of Steiner points would be of practical relevance.

References

[1] O.Aichholzer, F.Aurenhammer, H.Krasser, *Enumerating order types for small point sets with applications.* 17^{th} Ann. ACM Symp. Computational Geometry, Medford, MA, 2001 (to be presented).

[2] B.Aronov, R.Seidel, D.Souvaine, *On compatible triangulations of simple polygons.* Computational Geometry: Theory and Applications 3 (1993), 27-35.

[3] M.Bern, A.Sahai, *Isomorphic triangulations and map conflation.* Personal communication, 2000.

[4] J.E.Goodman, R.Pollack, *Multidimensional sorting.* SIAM J. Computing 12 (1983), 484-507.

[5] E.Kranakis, J.Urrutia, *Isomorphic triangulations with small number of Steiner points.* Int'l J. Computaional Geometry & Applications 9 (1999), 171-180.

[6] H.Krasser, *Kompatible Triangulierungen ebener Punktmengen.* Master Thesis, Institute for Theoretical Computer Science, Graz University of Technology, Graz, Austria, 1999.

[7] A.Saalfeld, *Joint triangulations and triangulation maps.* Proc. 3^{rd} Ann. ACM Sympos. Computational Geometry, Waterloo, Canada, 1987, 195-204.