# 3-Colorability of Pseudo-Triangulations*

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## Abstract

Deciding 3-colorability for general plane graphs is known to be an NP-complete problem. However, for certain classes of plane graphs, like triangulations, polynomial time algorithms exist. We consider the family of pseudo-triangulations (a generalization of triangulations) and prove NP-completeness for this class. The complexity status does not change if the maximum face-degree is bounded to four, or pointed pseudo-triangulations with maximum face degree five are treated. As a complementary result, we show that for pointed pseudo-triangulations with maximum face-degree four, a 3-coloring always exists and can be found in linear time.

## 1 Introduction

The *chromatic number* of a graph is the smallest number of colors needed to color its vertices so that no two adjacent vertices share the same color. Graphs with chromatic number 3 are said to be *(vertex) 3-colorable*. Determining the chromatic number of a graph is known to be a computationally hard problem. Interestingly, deciding 3-colorability of a *plane* graph is still NP-complete [9]. For the class of triangulations, though, 3-colorability can be decided in linear time; it is necessary and sufficient that every interior (i.e., non-extreme) vertex has even degree. Alternatively, we can use the following constructive approach: Start with the three different colors of a single triangle. Then the color of the third vertex of each edge-adjacent triangle is determined. This process is iterated until either a contradiction occurs (an already colored vertex is forced to have a different color) or a proper coloring is obtained.

Also for some other types of graphs the decision problem can be solved efficiently. Beside (obvious) graph classes like paths, cycles, trees, and quadrangulations, the class of maximal outerplanar graphs (or, equivalently, triangulations of polygons, or of point sets in convex position) is also 3-colorable. Ellingham et al. [3], and in a different formulation Diks, Kowalik, and Kurowski [2], give a characterization of planar graphs with isolated non-triangular faces that are 3-colorable. Moreover, 3-colorability is linear-time decidable for general locally connected graphs [5]. See [8] for a survey on 3-colorability.

In the present work we consider the class of pseudo-triangulations, which generalize triangulations in several aspects. In fact, as we shall see, this class is rich enough to lead to a wide spectrum of coloring results. We show that deciding 3-colorability for pseudo-triangulations is NP-complete. In fact, any plane geometric graph can be reduced, with respect to 3-coloring, to a (pointed) pseudo-triangulation. For the special case of pointed pseudo-triangulations with constant maximum face-degree, the problem remains NP-complete if the degree bound is at least five. As a complementary result, we prove that for pointed pseudo-triangulations with maximum face-degree four, a 3-coloring always exists and can be found in linear time. Some intermediate results for a varying number of pointed vertices are given as well.

We assume that point sets that serve as vertex sets for geometric graphs are in general position, that is, no three points lie on a common straight line. For a point set $S$, let $n = |S|$, and denote with $|CH(S)|$ the number of extreme points of $S$.

## 2 (Pointed) Pseudo-Triangulations

Pseudo-triangulations are a versatile generalization of the well-known concept of (geometric) triangulations [7]. Instead of triangles, their faces are *pseudo-triangles*, that is, simple polygons with exactly three convex vertices. In a geometric straight-line graph $G$, a vertex $v$ is called *pointed* if there exists a line through $v$ such that all edges of $G$ incident to $v$ lie on one side. The *rank* of a pseudo-triangulation is its number of non-pointed vertices; see [1] for further details. Pseudo-triangulations with rank zero are called pointed. These structures are of particular interest, because they are planar Laman graphs, and are minimally rigid [10].

**Lemma 1** *Any plane geometric graph $G(S)$ on $S$ can be extended to a pseudo-triangulation, $T(S')$, such that:*

- $S \subseteq S'$ *and* $|S'| = \Theta(n)$

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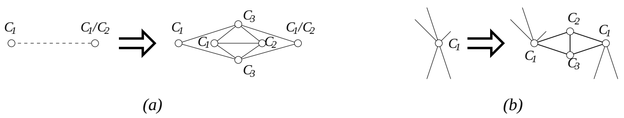


Figure 1: (a) Transforming a plane graph into a pseudo-triangulation. (b) Transforming a plane graph into a pointed plane graph. Note that these gadgets are arbitrarily flat. Colors $C_1$ to $C_3$ indicate possible color configurations.

- *$G(S)$ is 3-colorable if and only if $T(S')$ is 3-colorable*
- *the rank of $T(S')$ equals the number of non-pointed vertices in $G(S)$*

**Proof.** From [7, Theorem 2.6] it follows that, by adding a linear number of edges, any plane geometric graph $G(S)$ can be augmented in polynomial time to a pseudo-triangulation $T(S)$ without changing the pointedness of the underlying vertices. Instead of adding single edges, we use gadgets like in Figure 1(a) to connect two vertices in order to obtain a pseudo-triangulation $T(S')$.

Observe that $|S'| = \Theta(n)$ holds, as one gadget adds only a constant number of points. Also, the gadgets can be added in a way such that they do not change the pointedness of the involved vertices. (Under the general position assumption, the gadgets can be made sufficiently narrow.) As the additional vertices introduced with each gadget are all pointed, it follows that the number of non-pointed vertices remains unchanged.

Finally, adding the gadgets does not add additional coloring restrictions. The connected vertices might be colored arbitrarily (identically or differently), and still the added vertices of the gadget are 3-colorable. Thus $G(S)$ is 3-colorable if and only if $T(S')$ is 3-colorable. □

As planar graph 3-colorability is known to be NP-complete [9], the previous lemma already leads to the following NP-completeness result.

**Theorem 2** *Deciding whether a pseudo-triangulation is 3-colorable is NP-complete.*

**Proof.** By Lemma 1, we can obtain a pseudo-triangulation $T$ from each plane graph $G$ such that $G$ is 3-colorable if and only if $T$ is 3-colorable. As the transformation can be done in polynomial time, and only a linear number of edges and vertices are added, the claimed NP-completeness result follows. □

Pointed pseudo-triangulations are an important subclass of pseudo-triangulations. They minimize the number of edges over all pseudo-triangulations and thus, in some way, also the number of color restrictions. Nevertheless, we will show that even for this restricted class, 3-colorability is NP-complete. To this end, we prove that pointed planar graph 3-colorability is NP-complete, from which NP-completeness of pointed pseudo-triangulation 3-colorability follows.

**Lemma 3** *Deciding whether a pointed plane geometric graph is 3-colorable is NP-complete.*

**Proof.** We show how to transform a given plane straight-line graph $G$ into a pointed plane straight-line graph $G'$, such that $G$ is 3-colorable if and only if $G'$ is 3-colorable. W.l.o.g., assume that there are no horizontal edges in the given embedding of $G$, as otherwise we slightly rotate the plane. Now every non-pointed vertex $v$ of $G$ is replaced by two duplicates $v_L$ and $v_R$ of $v$. The two copies are placed sufficiently close to the left ($v_L$) and to the right ($v_R$) of $v$, respectively. All edges incident to $v$ from above are now incident to $v_L$, and all edges incident to $v$ from below are moved to $v_R$. In addition, $v_L$ and $v_R$ are connected by a small construction consisting of five edges, as shown in Figure 1(b).

By the general position assumption, the resulting graph $G'$ is plane. Only a linear number of additional vertices has been added, and all the vertices are now pointed. Moreover, the gadget connecting $v_L$ and $v_R$ ensures that in a proper 3-coloring of $G'$ both vertices have to get the same color. Thus $G$ is 3-colorable if and only if $G'$ is 3-colorable. NP-completeness follows as the transformation can be done in polynomial time. □

Combining Lemma 1 with Lemma 3 gives the following theorem.

**Theorem 4** *Deciding whether a pointed pseudo-triangulation is 3-colorable is NP-complete.*

The last result gives rise to an interesting question. On the one hand, pointed pseudo-triangulations have rank 0 and, as shown in Theorem 4, it is NP-complete to decide their 3-colorability. On the other hand, triangulations have maximum rank $r_{max} = n - |CH(S)|$, i.e., all interior vertices are non-pointed, and, as already mentioned in the introduction, 3-colorability can be decided in linear time. So it is natural to ask for which rank the change from 'easy' to 'intractable' happens. With the next two theorems we make a first step towards answering this question. (In the following, several proofs are omitted due to space constraints.)

**Theorem 5** *For all constants $c \geq 1$ and any $r \leq r_{max} - \Theta(\sqrt[c]{n})$ it is NP-complete to decide whether a pseudo-triangulation of rank $r$ is 3-colorable.*

**Theorem 6** *Whether a pseudo-triangulation $T(S)$ of rank $r \geq r_{max} - \Theta(\log n)$ is properly 3-colorable can be decided in polynomial time.*

## 3 Constant Maximum Face-Degree

In this section, we consider pseudo-triangulations with constant maximum face-degree, that is, pseudo-triangulations where each interior face is a pseudo-triangle with at most a (small) constant number of vertices. The following two statements will allow us to transform any pseudo-triangulation with high maximal face-degree into one with smaller maximal face-degree while keeping rank and colorability properties; cf. Figure 2.

**Lemma 7** *Any pseudo-triangle with $k > 5$ vertices can be subdivided into two pseudo-triangles of sizes strictly less than $k$, by adding an interior vertex of degree two, such that the pointedness of the involved vertices persists.*

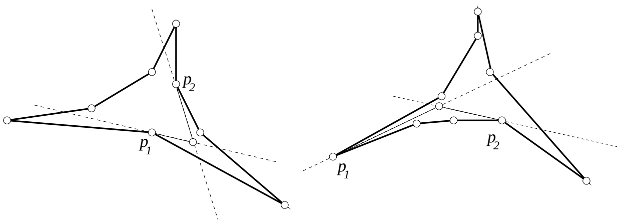


Figure 2: Subdividing large pseudo-triangles.

**Corollary 8** *Every pseudo-triangulation $T(S)$ with maximum face-degree $k > 5$ can be transformed into a pseudo-triangulation $T'(S')$ with maximum face-degree five in polynomial time such that:*

- *$S \subseteq S'$ and $|S'| = \Theta(n)$*
- *$T(S)$ is 3-colorable if and only if $T'(S')$ is 3-colorable*
- *the rank of $T'(S')$ equals the rank of $T(S)$*

### 3.1 Face-Degree $\leq 5$ Pseudo-Triangulations

By combining Corollary 8 with Theorem 4, we obtain a result for pointed pseudo-triangulations with maximum face-degree five.

**Corollary 9** *Deciding whether a pointed pseudo-triangulation with maximum face-degree five is 3-colorable is NP-complete.*

A more general statement (that includes the previous corollary) is the following.

**Theorem 10** *For all constants $c \geq 1$ and any rank $r \leq r_{max} - \Theta(\sqrt[c]{n})$ it is NP-complete to decide whether a pseudo-triangulation of rank $r$ and with maximum face-degree five is 3-colorable.*

### 3.2 Face-Degree $\leq 4$ Pseudo-Triangulations

Pseudo-triangles of size larger than five can always be subdivided as described in Lemma 7. This result cannot be extended to smaller pseudo-triangles. In fact, the situation changes completely if we bound the face-degree of a pseudo-triangulation by four.

**Theorem 11** *Pointed pseudo-triangulations with maximum face-degree four are 3-colorable.*

**Proof.** To prove the theorem, we use the concept of combinatorial (pointed) pseudo-triangulations [6]. These are combinatorial embeddings of (pointed) pseudo-triangulations, where the edges need not be straight lines and pointedness is not a geometric property anymore. Instead, each pointed vertex has a mark in one incident face, namely the one where it is pointed to, and for each face all but three vertices (the corners) have marks in this face. The only exception is the outer face, where all (at least three) incident vertices have their mark in. Note that a given combinatorial (pointed) pseudo-triangulation can be embedded such that every angle with a mark is larger than $\pi$ and all other angles are smaller than $\pi$ [4, Section 5.2]. Thus, with respect to 3-colorability, combinatorial pointed pseudo-triangulations are equivalent to geometric pointed pseudo-triangulations.

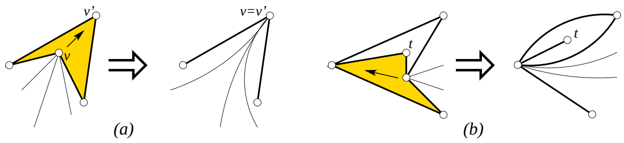


Figure 3: Move operation to collapse a pseudo-triangle (a), and a degenerate case (b).

For an interior vertex $v$ we define a merge operation for the pseudo-triangle $\nabla$ to which $v$ is pointed. This operation identifies $v$ with the antipodal vertex $v'$ in $\nabla$, by 'moving' $v$ towards $v'$, see Figure 3(a). In this way $\nabla$ collapses, but the remaining graph is still a valid combinatorial pointed pseudo-triangulation with one vertex, one face, and two edges less.

We iterate this process as long as we have interior vertices. This can be done, as each such vertex is always pointed towards a pseudo-triangle of size four. Whenever there exist interior vertices of degree two, they are merged before other vertices, to avoid degenerate cases as shown in Figure 3(b). Such degeneracies can only happen if vertex $t$ has degree two. Observe that all arguments also hold in the degenerate case, as we still have all relevant properties of combinatorial pointed pseudo-triangulations. However, for simplicity, we prefer to avoid degeneracies.

At the end of all merging steps, no interior vertices are left, and we obtain a (combinatorial) triangulation of a convex point set. Such triangulations are

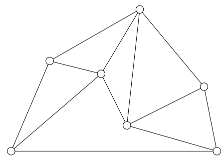

Figure 4: A pseudo-triangulation with rank 1 (one non-pointed vertex) and maximum face-degree four, which can not be 3-colored.

well known to be 3-colorable, and we can assign their colors in linear time.

We finally invert the above process and replicate, in each reversed merge step, the color of the original vertex for the duplicated vertex. This keeps the 3-coloring valid, as these vertices are not connected in the original graph. After all merge steps are undone, the given pointed pseudo-triangulation is 3-colored. □

Note that the above proof also provides a linear time algorithm to find a 3-coloring. The obtained 3-coloring is special in the sense that, for every interior face of size four of the pointed pseudo-triangulation, its reflex vertex has the same color as its antipodal vertex. In fact, up to permutation of the three colors, there is only one coloring with this property. This follows from the facts that (1) a 3-coloring of a triangulation of a convex point set is unique (up to permutation), and (2) the merge steps used in the above proof lead to a unique triangulation of the convex set, independent of the order they are carried out.

Pointed pseudo-triangulations with bounded face-degree four are a special structure concerning 3-colorability. Note that triangulations of convex point sets also fall into that category. Investigating the influence of the rank of a bounded face-degree four pseudo-triangulation on 3-colorability reveals that already a rank of 1 allows pseudo-triangulations which are not properly 3-colorable; see Figure 4. Note that all interior vertices in this example have even degree. So the parity property, which can be used to prove 3-colorability for triangulations, does not carry over to pseudo-triangulations of general rank. In addition, there exist 3-colorable examples with non-pointed interior vertices of odd degree.

In fact, we can prove NP-completeness for a wide range of ranks for maximum face-degree four pseudo-triangulations.

**Theorem 12** *For all constants $c \geq 1$ and any $r$, $\Theta(\sqrt[c]{n}) \leq r \leq r_{max} - \Theta(\sqrt[c]{n})$, it is NP-complete to decide whether a rank $r$ pseudo-triangulation with maximum face-degree four is 3-colorable.*

## 4 Final remarks

To summarize, we have the following results for pseudo-triangulations of maximum face-degree four:

- rank 0 (pointed pseudo-triangulations): always 3-colorable.
- rank $r$, $\Theta(\sqrt[c]{n}) \leq r \leq r_{max} - \Theta(\sqrt[c]{n})$: NP-complete.
- rank $r$, $r_{max} - \Theta(\log n) \leq r \leq r_{max}$: decidable in polynomial time.
- rank $r_{max} = n - |CH(S)|$ (triangulations): decideable in linear time.

For rank $r$ pseudo-triangulations of maximum face-degree five, and rank $r$ pseudo-triangulations without any face-degree bound, 3-colorability is NP-complete as long as $r \leq r_{max} - \Theta(\sqrt[c]{n})$. For both classes, 3-colorability is decidable in polynomial time if $r \geq r_{max} - \Theta(\log n)$. Where precisely do the changes between 'NP-complete' and 'polynomial time decidable' happen? What can be said if a pseudo-triangulation is 'almost pointed' (small constant rank)?

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