New Results on MWT Subgraphs

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Abstract

Let P be a polygon in the plane. We disprove the conjecture that the so-called LMT-skeleton coincides with the intersection of all locally minimal triangulations, $LMT(P)$, even for convex polygons P. We introduce an improved LMT-skeleton algorithm which, for any simple polygon P, exactly computes $LMT(P)$, and thus a larger subgraph of the minimum-weight triangulation $MWT(P)$. The algorithm achieves the same in the general point set case provided the connectedness of the improved LMT-skeleton, which is given in allmost all practical instances.

We further observe that the β-skeleton of P is a subset of $MWT(P)$ for all values $\beta > \sqrt{\frac{4}{3}}$ provided P is convex or near-convex. This gives evidence for the tightness of this bound in the general point set case.

Keywords: Computational geometry, minimum-weight triangulation, LMT-skeleton, β-skeleton

1 Introduction

A *triangulation* of a set S of n points in the Euclidean plane is a maximal set of non-crossing line segments (called *edges*) which have both endpoints in S. A triangulation of S that minimizes the sum of edge lengths is called a *minimum-weight triangulation*, $MWT(S)$, of S. Despite of the simplicity of this concept, the structural and computational properties of an $MWT(S)$ are not well understood. For example, it is not known whether there exists an algorithm computing an $MWT(S)$ in time polynomial in n. For a catalogue of properties of optimal triangulations, and minimum-weight triangulations in particular, the reader may consult the recent survey paper by Aurenhammer and Xu [1].

To gain more insight into the structure of an $MWT(S)$, considerable efforts have been put into the study of its subgraphs. Gilbert [9] pointed out that the shortest edge defined by S always belongs to $MWT(S)$. Another simple observation is that *unavoidable edges*, which are

edges not being crossed by any other edge defined by S, have to appear in any triangulation of S and thus are in any $MWT(S)$. Trivially, all edges of the convex hull of S are unavoidable. The number of unavoidable edges does not exceed $2n-2$, see Xu [13], and usually is very small as most of these edges do occur on the convex hull.

Only in recent years several less trivial (and more rich) subgraphs of $MWT(S)$ have been identified. Among them, the most prominent ones are the *LMT-skeleton* and the *β-skeleton* of S, respectively. In this note, we will remark on both structures and improve on known results.

The following notation is used throughout. S denotes a set of n points in the Euclidean plane. For two points p and q in S, let pq be the (straight line) edge connecting them. When appropriate, an edge will also be considered just as a pair of points. The *length* of an edge pq is the Euclidean distance between p and q, denoted by $|pq|$. The *weight* of a set of edges is the sum of their lengths. Two distinct edges are said to *cross* when they intersect in their interiors.

2 The LMT-skeleton

A triangulation T of S is called *locally minimal* if every 4-sided polygon drawn by T and not containing points from S is optimally triangulated. (That is, every convex quadrilateral contains the shorter one of its two diagonals.) Let $LMT(S)$ denote the intersection of all locally minimal triangulations for S. Then $LMT(S)$ is a subgraph of every $MWT(S)$, as such a triangulation of course is locally minimal, too. Whereas it is not known how to compute $LMT(S)$ in polynomial time, a large subgraph of $LMT(S)$, the so-called *LMT-skeleton* of S, can be computed by a simple and cute method, proposed in Belleville et al. [3] and in Dickerson and Montague [7]. The definition of the LMT-skeleton is procedural and can be stated as follows.

Consider some edge set $E \subset S \times S$. An edge $e \in E$ is called *redundant in E* if e is no edge of the convex hull of S and there is no point-empty quadrilateral formed by E that has e as its shortest diagonal. Note that a redundant edge cannot appear in any locally minimal triangulation which is subset of E. Edge e is called *unavoidable in E* if no other edge in E crosses e. Unavoidable edges have to appear in every triangulation which is subset of E. The LMT-skeleton algorithm puts $E = S \times S$ and proceeds in several rounds. Each round identifies all edges redundant in the current set E, and then eliminates them from the set. When no more edge in the reduced set E can be classified as redundant, the algorithm includes all edges that are unavoidable in E into the LMT-skeleton, and then stops. It is clear that the produced LMT-skeleton is a subset of $LMT(S)$. The number of rounds (but not the LMT-skeleton) depends on the ordering in which the edges are examined.

The fact that the LMT-skeleton of S tends to be a connected graph even for large point sets S comes as a surprise, and for the first time allows for a rapid construction of $MWT(S)$ for practical purposes. Several variants of the LMT-skeleton have been considered recently, see [3, 7, 4, 2], but the question whether these skeletons coincide with $LMT(S)$ has remained open. In this note we give a counterexample. We further propose a new variant, the so-called *improved LMT-skeleton*, and show that this structure is identical to $LMT(S)$ when restricted to simple polygons. As a consequence, $LMT(S)$ for arbitrary point sets S coincides with the improved LMT-skeleton (and thus can be constructed in polynomial time) provided the

connectedness of this structure, which is given in almost all practical instances. In this sense, the improved LMT-skeleton exploits this global subgraph approach to its utmost generality.

Let us first exhibit an example where the LMT-algorithm fails to produce all edges of $LMT(S)$. In the convex and 7-sided polygon in Figure 1 only one edge (the diagonal p_3p_6) is classified as redundant. Thus none of the remaining diagonals is unavoidable, that is, the LMT-skeleton algorithm leaves the polygon's interior empty. On the other hand, the polygon allows only for a single locally minimal triangulation, which therefore coincides with $LMT(S)$ (and $MWT(S)$).

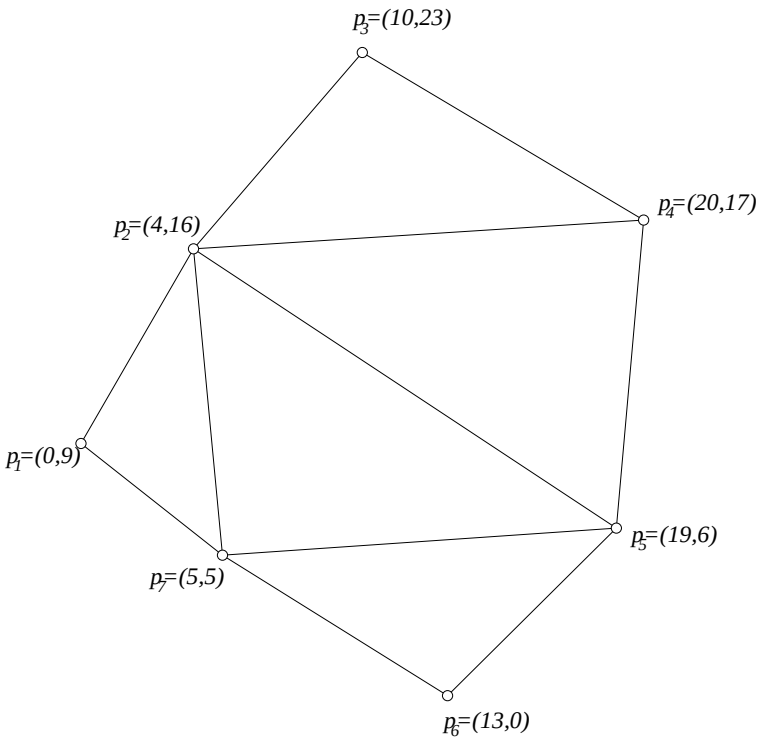

Figure 1: Convex 7-gon with a single locally minimal triangulation but empty LMT-skeleton.

3 The improved LMT-skeleton

The reason why the LMT-algorithm does not produce $LMT(S)$ in general is buried in the definition of a redundant edge. An edge e may not be classified as redundant because of being shortest in some quadrilateral Q, but for some edge f of Q, no quadrilateral witnessing f's non-redundancy need to share a triangle with Q. In this case, e also cannot appear in any locally minimal triangulation. We therefore strengthen the notation of redundancy. To be non-redundant in some set E of edges, e must either be a convex hull edge or be shortest in some point-empty quadrilateral Q from E, and for each edge f of Q that is no convex hull edge, there must exist some point-empty quadrilateral from E that has f as shortest diagonal *and*

that shares a triangle with Q. The set of edges produced by the resulting modified algorithm is a superset of the original LMT-skeleton but still is a subset of $LMT(S)$. We call this set the *improved LMT-skeleton* of S, or $skel^+(S)$ for short.

We proceed to prove the following result on the improved LMT-skeleton for simple polygons. To adapt the concepts above — defined for arbitrary point sets S — to the polygon case it suffices to replace 'convex hull edge' by 'polygon boundary edge' in the definition of redundancy, and to initialize the set E so as to contain all boundary edges and diagonals of the polygon.

Theorem 1 *For any simple polygon P, $skel^+(P)$ coincides with $LMT(P)$.*

Proof. Consider some edge $e \in LMT(P)$. We assume $e \notin skel^+(P)$ and show that this leads to a contradiction.

Let E be the subset of edges that remains after repeatedly eliminating redundant edges (diagonals of P) with the improved LMT-algorithm. By definition, $skel^+(P)$ then consists of all edges that are unavoidable in E. Hence, by our assumption of $e \notin skel^+(P)$, edge e is not unavoidable in E. That is, there is another edge $f \in E$ that crosses e. As being contained in E, edge f cannot be redundant. Also, as crossing e, edge f is no boundary edge of P. So there must exist some quadrilateral Q formed by E that has f as its shortest diagonal. We include f into an initially empty set T of edges and note that — by construction with the improved LMT-algorithm — for each edge g of Q that is no boundary edge of P, there must be some quadrilateral Q' which has g as its shortest diagonal and which shares a triangle with Q. We add each such edge g to the set T, too, and repeat this process for the edges of Q' and so on, until no more edges can be added to T.

In fact, constructing T triangulates the polygon P. Each edge of T is a diagonal of some quadrilateral and of P as well, so quadrilaterals sharing a triangle are encountered in a tree-like fashion. Triangulation T, on the one hand, is locally minimal and, on the other, does not include edge e. But this contradicts the assumption of $e \in LMT(P)$. □

The proof above may fail for the case of general point sets S. We do not know whether the construction of T is guaranteed to lead to a triangulation of S in this case. Edges crossing each other might be included, though we did not succeed to give an example. Still, the following observation can be made for general point sets.

Corollary 1 *Let S be a finite and planar point set, and assume that $skel^+(S)$ forms a connected graph. Then $skel^+(S) = LMT(S)$.*

Proof. If $skel^+(S)$ forms a connected graph then it subdivides the convex hull of S into simple polygons. Each polygon boundary edge belongs to $LMT(S)$ because of $skel^+(S) \subset LMT(S)$, and in the interior of each polygon we have the identity of $skel^+(S)$ and $LMT(S)$ by Theorem 1. □

4 The β-skeleton

Let p and q be two distinct points in S and let $\beta > 1$. Following Kirkpatrick and Radke [12], the edge pq is included in the *β-skeleton* $\beta(S)$ of S if the two circles of diameter $\beta \cdot |pq|$ and passing through both p and q do not enclose any point in S. Interestingly, $\beta(S)$ is a subgraph of every $MWT(S)$ provided β is large enough. The original bound $\beta \geq \sqrt{2}$ in Keil [11] has been improved later in Cheng and Xu [5] to $\beta > 1.1768$. The largest value for which a (simple, four-point) counterexample is available is $\sqrt{\frac{4}{3}}$. To close this gap is an open problem. We observe below that this can be achieved for the β-skeleton of convex polygons, and a certain class of star-shaped polygons. The result is easy to obtain, but strengthens the conjecture that the lower bound $\sqrt{\frac{4}{3}}$ is tight for arbitrary point sets S.

Lemma 1 *(Keil [11]) Let pq be an edge of the β-skeleton $\beta(S)$ for $\beta > \sqrt{\frac{4}{3}}$, and let x and y be two points in S such that the line segment xy intersects the edge pq. Then $|xy| > \max\{|px|, |py|, |qx|, |qy|, |pq|\}$.*

In fact, only the weaker version of Lemma 1 for $\beta > \sqrt{2}$ is proved in [11] but, as mentioned there, the stronger version above still holds. The interested reader may check this by replacing triangle angles of $\frac{\pi}{4}$ by $\frac{\pi}{3}$ in the original proof.

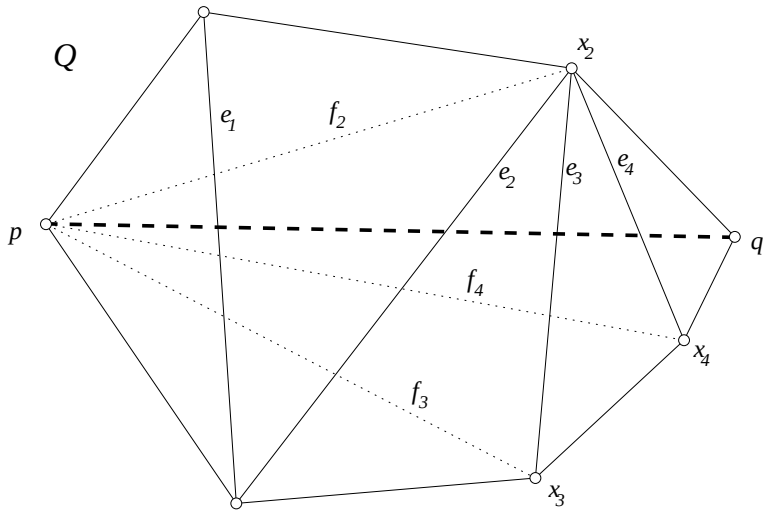

Figure 2: Proof of Theorem 2: pq is not in $MWT(P)$ and thus not in $\beta(P)$.

Theorem 2 *Let P be a convex polygon, let $MWT(P)$ be a minimum-weight triangulation of P, and let $\beta(P)$ be the β-skeleton of P for some $\beta > \sqrt{\frac{4}{3}}$. Then $\beta(P)$ is a subset of $MWT(P)$.*

Proof. We prove the assertion by contradiction. Assume there is an edge pq of $\beta(P)$ which is not in $MWT(P)$. Then edge pq has to intersect some triangles of $MWT(P)$ properly. Let Q be their union, and let $MWT(Q)$ be the restriction of $MWT(P)$ to the convex polygon Q. Each

of the $k \geq 1$ diagonals of $MWT(Q)$ intersects pq. Let $e_1, \ldots, e_k$ denote their total ordering with respect to increasing distance from point p. We now construct a new triangulation of Q which contains pq and has a weight smaller than $MWT(Q)$, giving a contradiction.

Consult Figure 2. First consider e_1, the edge closest to p. Assign e_1 to pq. We have $|pq| < |e_1|$ by Lemma 1. Next, for $i = 2, \ldots, k$, consider e_i, which has one endpoint x_i which is no endpoint of e_{i-1}. Assign e_i to $px_i = f_i$. Again, $|f_i| < |e_i|$ by Lemma 1. So the weight of $\{e_1, \ldots, e_k\}$ exceeds the weight of $\{pq, f_2, \ldots, f_k\}$. It remains to be observed that $\{pq, f_2, \ldots, f_k\}$ also yield a triangulation of Q, as these edges are crossing-free and their number is k. □

Theorem 2 applies to a slightly more general class of polygons, as only visibility from p rather than convexity is required in the proof above. So the theorem is true for all polygons that are star-shaped as seen from the endpoints of their β-skeleton edges, for at least one endpoint per edge.

5 Concluding remarks

We have considered two questions on LMT-skeletons and β-skeletons for restricted cases, contributing evidence to the conjectures that Theorem 1 and Theorem 2 might be true for the general point set case. In particular, the improved LMT-algorithm will indeed construct the intersection of all locally minimal triangulations $LMT(S)$, except for very specially constructed point sets S. This is not the case with previous LMT-algorithms. The advantage carries over to the practically more relevant situation where most edges are removed from the start set $E = S \times S$ by pre-exclusion tests (see Das and Joseph [6] or Drysdale et al. [8]) before an LMT-algorithm is run.

In order to define subsets of an $MWT(S)$ richer than $LMT(S)$, let us consider the following generalization of local minimality. For fixed k, call a triangulation of S *k-minimal* if it is minimum-weight within each of its k-sided and point-empty polygons. Let $LMT_k(S)$ denote the intersection of all k-minimal triangulations of S. Clearly, $LMT_i(S) \subset LMT_j(S)$ for $3 \leq i < j \leq n = |S|$. $LMT_3(S)$ is the set of unavoidable edges, and $LMT_4(S) = LMT(S)$. Interestingly, there exist point sets S (in convex position) where $MWT(S)$ is unique but $LMT_{n-1}(S)$ is empty. We raise the question of constructing $LMT_k(S)$ efficiently for general k. At present, we do not even know of a polynomial-time algorithm for computing 5-minimal triangulations. Popular strategies like edge flipping or greedy edge insertion, which are well known to produce 4-minimal (i.e. locally minimal) triangulations are easily shown to fail.

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