On Flips in Polyhedral Surfaces*

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Abstract

Let V be a finite point set in 3-space, and let $\mathcal{S}(V)$ be the set of triangulated polyhedral surfaces homeomorphic to a sphere and with vertex set V. Let abc and cbd be two adjacent triangles belonging to a surface $S \in \mathcal{S}(V)$; the *flip* of the edge bc would replace these two triangles by the triangles abd and adc. The flip operation is only considered when it does not produce a self–intersecting surface. In this paper we show that given two surfaces $S_1, S_2 \in \mathcal{S}(V)$, it is possible that there is no sequence of flips transforming S_1 into S_2, even in the case that V consists of points in convex position.

1 Introduction

When two adjacent triangles abc and cbd in a triangulation of a plane point set form a convex quadrilateral, to *flip* the edge bc consists of removing bc and placing ad instead. This fundamental operation was first introduced by Lawson [8], originally called an *exchange*. A special case is the *Delaunay flip* defined as follows: Let $\alpha = (\alpha_1, \ldots, \alpha_6)$ be the angles from the triangles abc and cbd in nondecreasing order, and analogously $\beta = (\beta_1, \ldots, \beta_6)$ for the triangles abd and adc. If α is lexicographically smaller than β then the Delaunay flip

*Part of this work was carried out while the first and second author visited the Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya

†Contact author. Research supported by APART [Austrian Programme for Advanced Research and Technology] of the Austrian Academy of Sciences, and the EU Improving Human Potential Programme HPRI-1999-CT-00071

‡Research in part supported by the NWO (STW) (Dutch Organisation for Scientific Research)

§Research supported by MEC-DGES-SEUID PB98-0933 and Gen. Cat. 1999SGR00356

replaces bc by ad. Observe that the general flip is reversible while the Delaunay flip works only in one direction.

It is a well–known fact, see e.g. [7], that starting with any triangulation of a point set it is possible to reach its Delaunay triangulation by simply performing Delaunay flips while it is possible. This clearly implies that for any two triangulations T_1 and T_2 there is a sequence of general flips transforming T_1 into T_2, a fact already shown in [8].

The edge flip is an example of a *local transformation*, i. e. a way to produce a new object in a class from previously generated objects, by means of some small change. Local transformations have been used to efficiently enumerate all objects in a class, for example with the reverse search scheme due to Avis and Fukuda [4], or to obtain an optimal object, possibly by combining the transformation with some strategy, like simulated annealing, to escape local optima. A fundamental issue is whether the class of objects is *connected* with respect to the transformation, that is, whether every object is reachable from any starting object in the class after some finite sequence of transformations. In this work we consider an important class of objects, the polyhedral surfaces, under the most natural transformation, the edge flip which is both, a local and constant size operation.

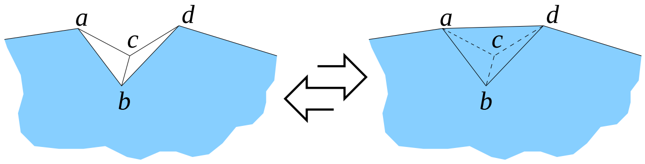


Figure 1: From left to right: flip at the reflex edge bc. From right to left: flip at the convex edge ad.

Let V be a finite point set in 3-space, and let $\mathcal{S}(V)$ be the set of triangulated polyhedral surfaces homeomorphic to a sphere and with vertex set V. Let abc and cbd be two adjacent triangles belonging to a surface $S \in \mathcal{S}(V)$; the *flip* of the edge bc replaces these two triangles by the new triangles abd and adc such that the surface remains to be homeomorphic to a sphere in 3-space (Figure 1). That is, the flip operation is only considered when it does not produce a self–intersecting surface.

In [9] a procedure to construct triangulated polyhedra of minimal area is given. The flip operation was used as a basic step to locally improve the polyhedron in the vicinity. An incremental algorithm for 3D triangulation is presented in [5]. There the authors use the flip operation (called swap) to make the triangulated polyhedral surface more “smooth”. The systematic exploration via flip of $\mathcal{S}(V)$ has been suggested in [1, 2] for surface reconstruction from samples, in order to obtain the polyhedral surface that optimizes criteria such as mean curvature or total absolute curvature. Nevertheless, we show in this paper that given two surfaces $S_1, S_2 \in \mathcal{S}(V)$, there might be no sequence of flips transforming S_1 into S_2, even in the case that V consists of points in convex position. Hence a new strategy for systematic generation is required, an issue explored in the work [3].

In fact we might have expected more. Let W be a plane point set; we lift every point $p = (x, y) \in W$ to the point $p^* = (x, y, x^2 + y^2)$ on the unit paraboloid, and obtain a lifted set W^*. Given a triangulation T of W we can lift every triangle pqr in T to the corresponding triangle, obtaining in this way a *polyhedral terrain*. If abc and cbd are two adjacent triangles in T such that a Delaunay flip is possible at bc, it is well known that in this case b^*c^* is a reflex edge of the terrain (looking from below) and that the Delaunay flip corresponds to sticking a tetrahedron $a^*b^*c^*d^*$ to the terrain. The new terrain has lost the reflex edge b^*c^* and gained the convex edge a^*d^*; the Delaunay triangulation of W is reached when the terrain becomes the lower convex hull of W^* [6].

Now one would expect that given a polyhedral triangulated surface S such that its vertex set V is in convex position, it would be possible to use flips only at reflex edges for transforming S into the convex hull of V. This is not the case as we prove next as a warming up construction, which is simpler than the surface in Section 2 we use for our stronger result.

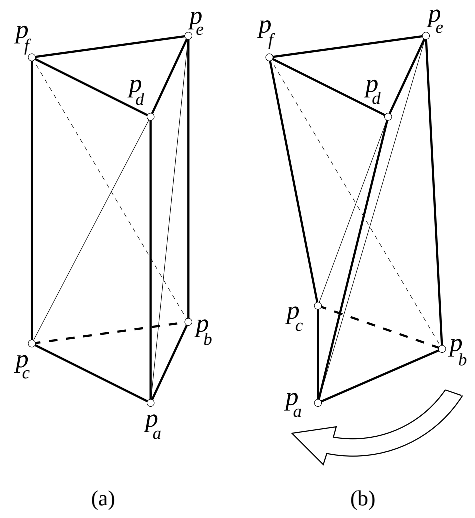


Figure 2: Schönhardt's polyhedron (right)

Nicely, and maybe not surprisingly, our examples are based on Schönhardt's famous 6-vertex polyhedron that cannot be partitioned into tetrahedra without adding extra vertices. To obtain Schönhardt's polyhedron take a triangular prism with diagonals across the rectangular faces. The prism is then twisted in such a way that the polyhedron gets concave at these diagonals as indicated in Figure 2(a) and (b).

Now let's carve out of a cube a hole like Schönhardt's polyhedron without the triangular top and bottom lids, and slightly open along the edge $p_a p_d$ (Figure 3). The vertices of the cube can be carefully perturbed in such a way that the vertex set of the resulting polyhedron Q is in convex position. Observe that the only reflex edges in Q are p_3p_7 and p_2p_6. A flip at p_3p_7 would replace triangles $p_3p_7p_8$ and $p_3p_7p_2$ by triangles $p_2p_7p_8$ and $p_3p_8p_2$, giving a self–intersecting surface. The situation is similar at the edge p_2p_6, hence no flip is possible at the reflex edges of Q, which proves the claim. It can be seen that starting with flips at suitable convex edges, it is possible to transform Q into its convex hull. Our example in the next section shows that this is not the case for all surfaces.

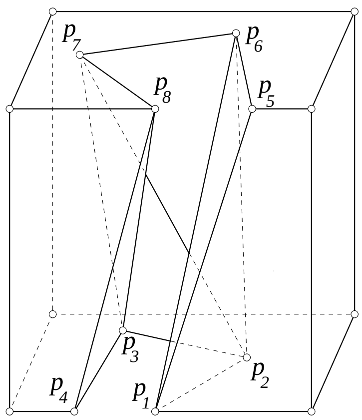


Figure 3: A polyhedron Q in which flips at reflex edges are not sufficient

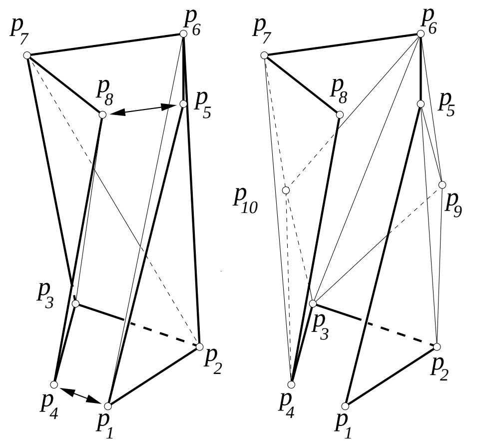
2 Main result

The main contribution of this paper is to establish Theorem 1, which follows from the construction given in the following paragraphs.

Theorem 1 *Given a set V of vertices in 3D, the class $\mathcal{S}(V)$ of triangulated surfaces with vertex set V is not connected by flip, even if V is in general and convex position.*

Intuitive explanation. Figure 4 shows the intuition behind our construction. We construct our polyhedron P out of Schönhardt's polyhedron by wrapping a layer of plasticine around all faces except the top and bottom triangles. We fill up the concave parts to get convex on the outside and cut the plasticine along a vertical edge (e.g. edge $p_a p_d$ in Fig. 2(b)) such that the resulting body gets homeomorphic to a sphere.

Realization. To simplify the description we split the boundary of P into two parts, called inner and outer surface of P. The inner surface is shown in



(a) (b)

Figure 4: Intuitive construction: (a) Inner surface of P , (b) Outer surface of P

Figure 4(a). It is obtained by taking the surface of Schönhardt's polyhedron except the top and bottom triangle and opening it up along the edge p_ap_d. In other words the vertices p_a and p_d are split into p_1, p_4 and p_5, p_8, respectively, and moved apart. Note that we also reverse the in-out direction of the surface.

For the outer surface of P to gain some volume we add two vertices p_9 and p_{10} near the middle of the two edges p_2p_6 and p_3p_7, see Figure 4(b). Their location is chosen in a way that all vertices are in extreme position. Then the outer surface is the part of the convex hull of all vertices, which is bounded by the polygonal cycle $p_1p_5p_6p_7p_8p_4p_3p_2p_1$ (the boundary between inner and outer surface) and includes p_9 and p_{10}. Choosing positions of p_9 and p_{10} appropriately guarantees that the resulting polyhedron P is homeomorphic to a sphere.

Figure 5 gives the edge graph of P. The left figure shows the inner surface (top, red (dark gray)) and outer part (bottom, green (light gray)). Edges common to both surfaces are drawn in black. The four flippable edges (see discussion below) are drawn with thin lines. The right figure shows all edges of P, using the same color-code.

$p_1 =$	(383, -924, 0)	$p_6 =$	(259, 966, 3999)
$p_2 =$	(866, 500, 1)	$p_7 =$	(-966, -259, 4000)
$p_3 =$	(-866, 500, 0)	$p_8 =$	(383, -924, 4000)
$p_4 =$	(-383, -924, 0)	$p_9 =$	(778, 778, 2000)
$p_5 =$	(924, -383, 4000)	$p_{10} =$	(-1100, 0, 2000)

Table 1: Coordinates of 10 vertices in general, convex position

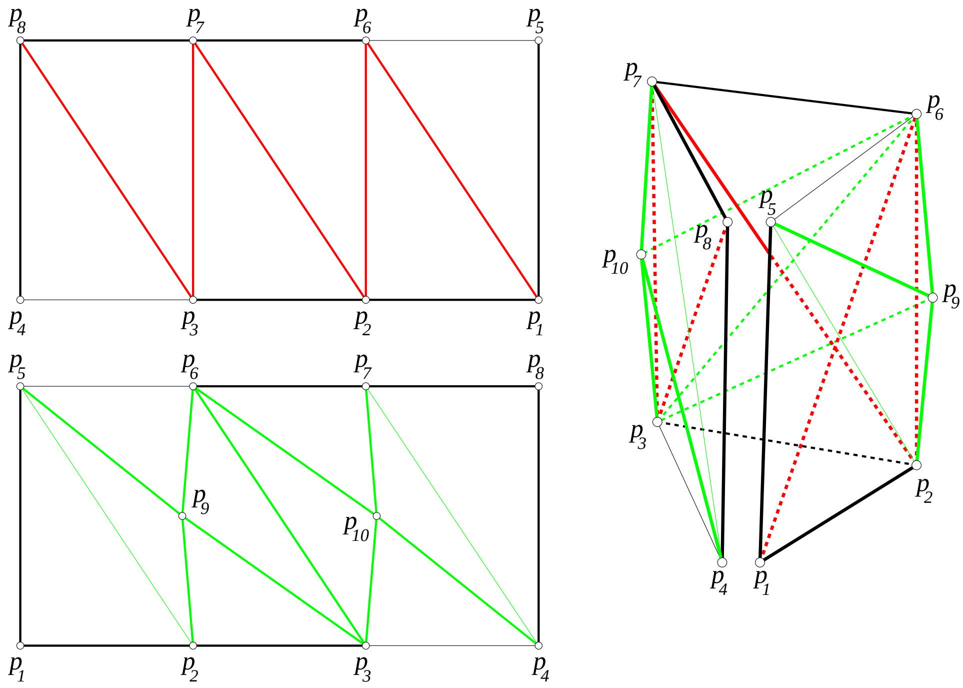


Figure 5: Graph of the surface of P: Inner and outer parts (left) and entire polyhedron (right) with labeled vertices

Let $V = p_1, \ldots, p_{10}$ be the vertex set of P as shown in Figure 5. Table 1 gives one possible realization of the coordinates of V. V is in convex position and its convex hull forms a simplicial polyhedron, i.e. the vertices are in general position (no four are on a common plane). Thus V is in no way degenerated, i.e. any vertex might be moved in any direction by some (small) epsilon. In Table 2 the face list of P is summarized. The surface consists of 16 triangles in general position and 24 edges.

In Figure 6 different views of the resulting polyhedron P are shown. See [10] for an animated (interactive) version and additional information. To en-

$\Delta_1 = (p_1, p_2, p_5)$	$\Delta_5 = (p_3, p_6, p_9)$	$\Delta_9 = (p_4, p_7, p_{10})$	$\Delta_{13} = (p_2, p_3, p_7)$
$\Delta_2 = (p_2, p_5, p_9)$	$\Delta_6 = (p_3, p_6, p_{10})$	$\Delta_{10} = (p_4, p_7, p_8)$	$\Delta_{14} = (p_2, p_6, p_7)$
$\Delta_3 = (p_5, p_6, p_9)$	$\Delta_7 = (p_6, p_7, p_{10})$	$\Delta_{11} = (p_1, p_2, p_6)$	$\Delta_{15} = (p_3, p_7, p_8)$
$\Delta_4 = (p_2, p_3, p_9)$	$\Delta_8 = (p_3, p_4, p_{10})$	$\Delta_{12} = (p_1, p_5, p_6)$	$\Delta_{16} = (p_3, p_4, p_8)$

Table 2: Faces (triangles) of the surface of P

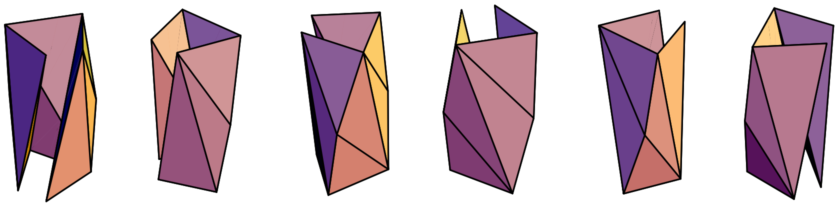


Figure 6: Different views of the polyhedron P

able the interested reader to perform her/his own experiments we also provide a Mathematica© notebook with the definition of the vertices and faces of P.

Flipping edges. We are now ready to show that the class of triangulated surfaces with vertex set V is not connected under the flip operation. We will do this by proving that we cannot transform P to the convex hull of V by flips.

First we show that only four edges of P can be flipped. Recall that there are two different reasons preventing an edge to be flipped. Either the flip causes a proper self–intersection of the surface or it forces an edge to appear twice on the surface (e.g. once for the inner and once for the outer part). Thus the resulting polyhedron would no longer be homeomorphic to a sphere.

Flipping reflex inner edges generates edges intersecting opposite faces of the inner surface (see the red (dark gray) edges in Figure 5). This is a direct consequence of the Schönhardt-type shape of the inner surface. For example flipping the edge p_2p_6 to p_1p_7 generates the two faces $(p_1p_2p_7)$ and $(p_1p_6p_7)$, both intersecting the existing faces $(p_3p_4p_8)$ and $(p_3p_7p_8)$ of P. Another possible flip is p_2p_7 to p_3p_6, forcing the latter to appear twice on the surface. By circularity of the construction similar arguments hold for all inner edges.

For the outer surface consider first the edges on the 'backside' of P. A flip of any of the edges p_2p_3, p_3p_6 and p_6p_7 causes intersection with either $(p_1p_2p_6)$ or $(p_3p_7p_8)$ or both. Flipping p_3p_9 or p_6p_{10} forces again edges to appear twice on the surface. Similar observations show that none of p_2p_9, p_6p_9, p_3p_{10}, p_7p_{10} can be flipped.

The situation on the left and right outer side of P is more involved. A flip of any of the edges p_1p_2, p_1p_5, p_5p_9, (and symmetrically p_7p_8, p_4p_8, p_4p_{10}) results in already existing edges. Thus only four edges remain to be considered and in fact all of them can be flipped. Thus from the 24 edges of the surface of P there are only four possible edge flips, namely p_2p_5 to p_1p_9, p_5p_6 to p_1p_9 and symmetrically p_4p_7 to p_8p_{10}, p_3p_4 to p_8p_{10}.

Obviously the four edges form two independent groups of two edges each. By symmetry we need only to consider one of them, say the edges flippable to p_1p_9. Moreover only one of p_2p_5 or p_5p_6 can be flipped to p_1p_9 at the same time, i.e. we have only to consider two new surfaces. Repeated application of the above arguments shows that for both flips the situation for the non-flippable edges remains unchanged. In other words during any possible combination of the four flips on P no other edge becomes flippable.

We conclude that for the surface of P there are only four possible flips, leading to a total of $3 \times 3 = 9$ different triangulated surfaces of V (including P) 'reachable' from P by flips. Since the two groups of flips are independent this gives a subclass of 9 surface triangulations of V and 12 possible flips between them. In particular none of the inner (red) edges of P can ever be flipped, including the only two reflex edges p_2p_6 and p_3p_7. Since none of the inner edges belongs to the convex hull of V it follows that we cannot get the convex hull of V by flipping when starting with P. That is the class of triangulated surfaces with vertex set V is not connected by flip.

Acknowledgements We would like to thank Wolfgang Moser for implementing the interactive program used in [10].

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