On Flips in Polyhedral Surfaces*

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Abstract

Let V be a finite point set in 3D-space, and let $\mathcal{S}(V)$ be the set of triangulated polyhedral surfaces homeomorphic to a sphere and with vertex set V. Let abc and cbd be two adjacent triangles belonging to a surface $S \in \mathcal{S}(V)$; the *flip* of the edge bc would replace these two triangles by the triangles abd and adc. The flip operation is only considered when it does not produce a self–intersecting surface. In this paper we show that given two surfaces $S_1, S_2 \in \mathcal{S}(V)$, it is possible that there is no sequence of flips transforming S_1 into S_2, even in the case that V consists of points in convex position.

1 Introduction

When two adjacent triangles abc and cbd in a triangulation of a plane point set form a convex quadrilateral, to *flip* the edge bc consists of removing bc and placing ad instead. A special case is the *Delaunay flip* which increases the smallest angle of the triangles involved. It is a well–known fact, see e.g. [5], that starting with any triangulation of a point set it is possible to reach its Delaunay triangulation by simply performing Delaunay flips while it is possible. This clearly implies that for any two triangulations T_1 and T_2 there is a sequence of general flips transforming T_1 into T_2.

The edge flip is an example of a *local transformation*, i. e. a way to produce a new object in a class from previously generated objects, by means of some small change. Local transformations have been used to efficiently enumerate all objects in a class, for example with the reverse search scheme due to Avis and Fukuda [4], or to obtain an optimal object, possibly by combining the transformation with some strategy, like simulated annealing, to escape local optima. A fundamental issue is whether the class of objects is *connected* with

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respect to the transformation, that is, whether every object is reachable from any starting object in the class after some finite sequence of transformations. In this work we consider an important class of objects, the polyhedral surfaces, under the most natural transformation.

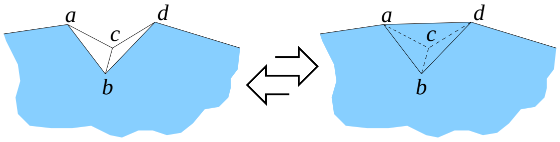

Figure 1: From left to right: flip at the reflex edge bc. From right to left: flip at the convex edge ad.

Let V be a finite point set in 3D-space, and let $\mathcal{S}(V)$ be the set of triangulated polyhedral surfaces homeomorphic to a sphere and with vertex set V. Let abc and cbd be two adjacent triangles belonging to a surface $S \in \mathcal{S}(V)$; the *flip* of the edge bc replaces these two triangles by the new triangles abd and adc (Figure 1). Note that the flip operation is only considered when it does not produce a self–intersecting surface. The systematic exploration via flip of $\mathcal{S}(V)$ has been suggested in [1, 2] for surface reconstruction from samples, in order to obtain the polyhedral surface that optimizes criteria such as mean curvature or total absolute curvature. Nevertheless, we show in this paper that given two surfaces $S_1, S_2 \in \mathcal{S}(V)$, there might be no sequence of flips transforming S_1 into S_2, even in the case that V consists of points in convex position. Hence a new strategy for systematic generation is required, an issue explored in the work [3].

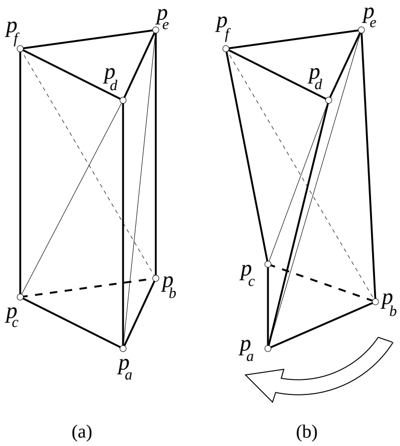

Figure 2: Schönhardt's polyhedron (right)

Nicely, and maybe not surprisingly, our examples are based on Schönhardt's famous 6-vertex polyhedron that cannot be partitioned into tetrahedra without adding extra vertices. To obtain Schönhardt's polyhedron take a triangular prism with diagonals across the rectangular faces. The prism is then twisted in such a way that the polyhedron gets concave at these diagonals as indicated in Figure 2(a) and (b).

2 Main result

The main contribution of this paper is to establish Theorem 1, which follows from the construction given below.

Theorem 1 *Given a set V of vertices in 3D, the class $\mathcal{S}(V)$ of triangulated surfaces with vertex set V is not connected by flip, even if V is in general and convex position.*

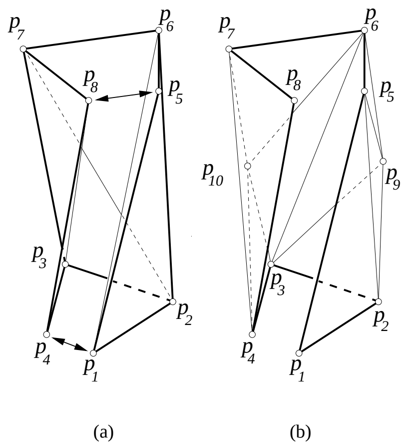

Figure 3: Intuitive construction: (a) Inner and (b) outer surface of P

Figure 3 shows the intuition behind our construction. We construct our polyhedron P out of Schönhardt's polyhedron by wrapping a layer of plasticine around all faces except the top and bottom triangles. We fill up the concave parts to get convex on the outside and cut the plasticine along a vertical edge (e.g. edge $p_a p_d$ in Fig. 2(b)) such that the resulting body gets homeomorphic to a sphere. For the outer surface of P to gain some volume we add two vertices p_9 and p_{10} near the middle of the two edges $p_2 p_6$ and $p_3 p_7$, see Figure 3(b). Please refer to the full paper for more details.

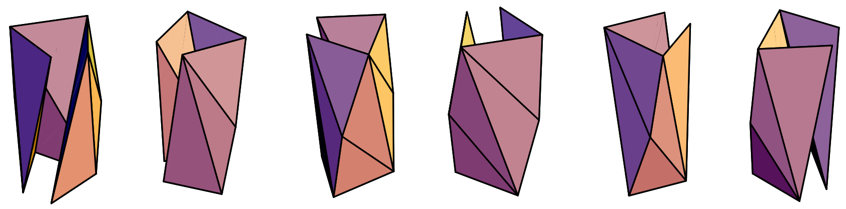

Figure 4: Different views of the polyhedron P

Let $V = p_1, \ldots, p_{10}$ be the vertex set of P. Table 1 gives one possible realization of the coordinates of V. V is in convex position and its convex hull forms a simplicial polyhedron, i.e. the vertices are in general position (no four are on a common plane). In Figure 4 different views of the resulting

$p_1 =$	(383, -924, 0)	$p_6 =$	(259, 966, 3999)
$p_2 =$	(866, 500, 1)	$p_7 =$	(-966, -259, 4000)
$p_3 =$	(-866, 500, 0)	$p_8 =$	(383, -924, 4000)
$p_4 =$	(-383, -924, 0)	$p_9 =$	(778, 778, 2000)
$p_5 =$	(924, -383, 4000)	$p_{10} =$	(-1100, 0, 2000)

Table 1: Coordinates of the 10 vertices of P in general, convex position

polyhedron P are shown. See [6] for an animated (interactive) version and additional information. To enable the interested reader to perform her/his own experiments we also provide a Mathematica© notebook with the definition of the vertices and faces of P.

We were able to show that $\mathcal{S}(V)$ is not connected under the flip operation. This is done by proving that we cannot transform P to the convex hull of V by flips. The crucial step is to show that only four edges of P can be flipped. We thus obtained a subclass of 9 surface triangulations of V and 12 possible flips between them. In particular none of the reflex edges of P can ever be flipped. It follows that we cannot get the convex hull of V by flipping (starting with P), i.e. the class of triangulated surfaces with vertex set V is not connected by flip.

Space limitations preclude the discussion of further details. The interested reader may consult the full version of the paper.

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[6] *Interactive animation and additional information*, `http://www.igi.TUGraz.at/oaich/triangulations/polyhedra.html`