Geometric achromatic and pseudoachromatic indices

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Abstract

The pseudoachromatic index of a graph is the maximum number of colors that can be assigned to its edges, such that each pair of different colors is incident to a common vertex. If for each vertex its incident edges have different color, then this maximum is known as achromatic index. Both indices have been widely studied. A geometric graph is a graph drawn in the plane such that its vertices are points in general position, and its edges are straight-line segments. In this paper we extend the notion of pseudoachromatic and achromatic indices for geometric graphs, and present results for complete geometric graphs. In particular, we show that for n points in convex position the achromatic index and the pseudoachromatic index of the complete geometric graph are $\lfloor \frac{n^2+n}{4} \rfloor$.

1   Introduction

A *vertex coloring* with k colors of a simple graph G, is a surjective function that assigns to each vertex of G a color from the set $\{1, 2, \ldots, k\}$. A coloring is *proper* if any two adjacent vertices have different color, and it is *complete* if every pair of colors appears on at least one pair of adjacent vertices. The *chromatic number $\chi(G)$ of G* is the smallest number k for which there exists a proper coloring of G using k colors. It is not hard to see that any proper coloring of G with $\chi(G)$ colors is a complete coloring. The *achromatic number* $\alpha(G)$ of G is the biggest number k for which there exists a proper and complete coloring of G using k colors. The *pseudoachromatic number* $\psi(G)$ of G is the

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biggest number k for which there exists a complete coloring of G using k colors. Clearly we have that

$$\chi(G) \leq \alpha(G) \leq \psi(G).$$

The achromatic number was introduced by Harary, Hedetniemi and Prins in 1967 [10]; the pseudoachromatic number was introduced by Gupta in 1969 [9]. Several authors have studied these parameters, and it turns out that the exact determination of the numbers is quite difficult. For details see [3, 6] and the references therein.

The *chromatic index* $\chi_1(G)$, *achromatic index* $\alpha_1(G)$ and *pseudoachromatic index* $\psi_1(G)$ of G, are defined respectively as the chromatic number, achromatic number and pseudoachromatic number of the line graph $L(G)$ of G. Notationally, $\chi_1(G) = \chi(L(G))$, $\alpha_1(G) = \alpha(L(G))$ and $\psi_1(G) = \psi(L(G))$.

A central topic in Graph Theory is to study the behavior of any parameter in complete graphs. For instance, the authors of [3] and [4] determined the exact value of α_1 and ψ_1 for some specific complete graphs. In this paper we extend the notion of pseudoachromatic and achromatic indices to geometric graphs, and present upper and lower bounds for the case of complete geometric graphs.

The next section introduces geometric graphs, generalizes the different numbers and indices and provides some basic relations for them. In Section 3 we consider the complete graph where the vertex set is a set of points in convex position, and show lower and upper bounds of the indices. Finally, we generalize this considerations to point sets in general position in Section 4 and give bounds for the geometric pseudoachromatic index.

2 Preliminaries

Throughout this paper we assume that all sets of points in the plane are in general position, that is, no three points are on a common line. Let $G = (V, E)$ be a simple graph. A *geometric embedding* of G is an injective function that maps V to a set S of points in the plane, and E to a set of (possibly crossing) straight-line segments whose endpoints belong to S. A *geometric graph* G is the image of a particular geometric embedding of G. For brevity we refer to the points in S as vertices of G, and to the straight-line segments connecting two points in S as edges of G. Please note that any set of points in the plane induces a complete geometric graph. We say that two edges of G *intersect* if they have a common endpoint or they cross. Two edges are *disjoint* if they do not intersect. A coloring of the edges of G is *proper* if every pair of edges of the same color are disjoint. A coloring is *complete* if each pair of colors appears on at least one pair of intersecting edges.

The *chromatic index* $\chi_1(G)$ of G is the smallest number k for which there exists a proper coloring of the edges of G using k colors. The *achromatic index* $\alpha_1(G)$ of G is the biggest number k for which there exists a complete and proper coloring of the edges of G using k colors. The *pseudoachromatic index* $\psi_1(G)$ of G is the biggest number k for which there exists a complete coloring of the edges of G using k colors.

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We extend these definitions to graphs in the following way. Let G be a graph. The *geometric chromatic index* $\chi_g(G)$ of G is the largest value k for which a geometric graph H of G exists, such that $\chi_1(\mathsf{H}) = k$. Likewise, the *geometric achromatic index* $\alpha_g(G)$ and the *geometric pseudoachromatic index* $\psi_g(G)$ of G, are defined as the smallest value k for which a geometric graph H of G exists such that $\alpha_1(\mathsf{H}) = k$ and $\psi_1(\mathsf{H}) = k$, respectively.

From the above definitions we get for graphs

$$\chi_1(G) \leq \chi_g(G) \tag{2.1}$$

$$\chi_1(G) \leq \alpha_1(G) \leq \psi_1(G) \leq \psi_g(G) \tag{2.2}$$

$$\chi_1(G) \leq \alpha_1(G) \leq \alpha_g(G) \leq \psi_g(G) \tag{2.3}$$

and for geometric graphs we obtain

$$\chi_1(\mathsf{G}) \leq \alpha_1(\mathsf{G}) \leq \psi_1(\mathsf{G}). \tag{2.4}$$

Consider the cycle C_n of length $n \geq 3$. In this case $\chi_1(C_n)$ is equal to 2 if n is even, and is equal to 3 if n is odd. On the other hand, it is not hard to see that $\chi_g(C_n) = n - 1$ if n is even and $\chi_g(C_n) = n$ if n is odd. However, $\alpha(C_n) = \alpha_1(C_n) = \alpha_g(C_n) = \max\{k \colon k\lfloor \frac{k}{2} \rfloor \leq n\} - s(n)$, where $s(n)$ is the number of positive integer solutions to $n = x^2 + x + 1$. Also, $\psi(C_n) = \psi_1(C_n) = \psi_g(C_n) = \max\{k \colon k\lfloor \frac{k}{2} \rfloor \leq n\}$. These results can be found in [6, 11, 13].

It is known that if G is a planar graph then there always exists a geometric embedding j, where no two edges of $j(G)$ intersect, except possibly in a common endpoint [8]. Therefore, $\psi_1(G) = \psi_1(j(G)) = \psi_g(G)$ and $\alpha_1(G) = \alpha_1(j(G)) = \alpha_g(G)$. However, $\chi_1(G) = \chi_1(j(G)) \leq \chi_g(G)$ (for instance, and as we mentioned before, $\chi_1(C_4) = 2$ and $\chi_g(C_4) = 3$).

The chromatic index of a geometric graph G has been studied before. Let l be a positive integer and $I(S)$ the graph in which one vertex corresponds to one subset of S of size l, and one edge corresponds to two vertices of G whose respective convex hulls intersect. This graph was defined in [2], where the authors study its chromatic number for the case when $l = 2$. If we denote by K_n the complete geometric graph with vertex set S, then for the case $l = 2$, $\chi(I(S)) = \chi_1(\mathsf{K}_n)$. In the same paper the authors define and study the number $i(n) = \max\{\chi(I(S)) \colon S \subset \mathbb{E}^2 \text{ in general position}, |S| = n\}$. Note that for the case $l = 2$ it happens that $i(n) = \chi_g(K_n)$. Recall that by K_n we denote the complete graph on n vertices. The following theorem appears in [2].

Theorem 2.1. *For each $n \geq 3$: i) If the vertices of K_n are in convex position then $\chi_1(\mathsf{K}_n) = n$, ii) $n \leq \chi_g(K_n) \leq cn^{3/2}$ for some constant $c > 0$.*

In this paper we prove:

Theorem 2.2. *i) For each $n \neq 4$, if the vertices of K_n are in convex position then*

$$\alpha_1(\mathsf{K}_n) = \psi_1(\mathsf{K}_n) = \lfloor \tfrac{n^2+n}{4} \rfloor,$$

ii) For each $n > 18$, $0.0710n^2 - \Theta(n) \leq \psi_g(K_n) \leq 0.1781n^2 + \Theta(n^{\frac{3}{2}})$.

3 Points in convex position

In this section we prove Claim i) of Theorem 2.2. In Subsection 3.1 we present an upper bound for $\psi_1(\mathsf{G})$ for any geometric graph G; and then in Subsection 3.2 we exclusively work with point sets in convex position, and derive a tight lower bound for $\alpha_1(\mathsf{K}_n)$.

3.1 Upper bound: $\psi_1(\mathsf{G}) \leq \lfloor \frac{n^2+n}{4} \rfloor$

The following theorem was shown in [7].

Theorem 3.1. *Any geometric graph with n vertices and $n + 1$ edges, contains two disjoint edges.*

Using this theorem we obtain the following result, where the order of a graph denotes the number of its vertices.

Corollary 3.2. *Let G be a geometric graph of order n. There are at most n chromatic classes of size one in any complete coloring of G.*

This corollary immediately implies an upper bound on $\psi_1(\mathsf{G})$.

Theorem 3.3. *Let G be a geometric graph of order n. The pseudoachromatic index $\psi_1(\mathsf{G})$ of G is at most $\lfloor \frac{n^2+n}{4} \rfloor$.*

Proof. We proceed by contradiction. Assume there exists a geometric graph G for which a complete coloring using $\lfloor \frac{n^2+n}{4} \rfloor + 1$ colors exist. This coloring must have at most $\binom{n}{2} - \left(\lfloor \frac{n^2+n}{4} \rfloor + 1 \right)$ chromatic classes of cardinality larger than one. Thus, there are at least $\lfloor \frac{n^2+n}{4} \rfloor + 1 - \left(\binom{n}{2} - \lfloor \frac{n^2+n}{4} \rfloor - 1 \right)$ chromatic classes of size one, that is:

$$1 - \left(\binom{n+1}{2} - 2 \left\lfloor \frac{\binom{n+1}{2}}{2} \right\rfloor \right) + n + 1 = \begin{cases} n+1 & \text{if } \binom{n+1}{2} \text{ is odd,} \\ n+2 & \text{if } \binom{n+1}{2} \text{ is even.} \end{cases} \quad (3.1)$$

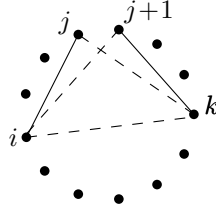This contradicts Corollary 3.2 and therefore the theorem follows. $\qquad\square$

Figure 1: Example for $n = 13$. Edges of the halving pair $(e_{i,j}, e_{j+1,k})$ are shown solid, dashed edges represent the possible halving edges, of which $e_{i,k}$ is a halving edge (the witness) in the shown example. The convex hull of a pair $(e_{i,j}, e_{j+1,k})$ is always a quadrilateral.

3.2   Tight lower bound: $\alpha_1(\mathsf{G}) \geq \lfloor \frac{n^2+n}{4} \rfloor$

In this subsection we prove that the bound presented in Theorem 3.3 is tight. To derive the lower bound we use a complete geometric graph induced by a set of points in convex position. We call this type of graph a *complete convex geometric graph*. The crossing pattern of the edge set of a complete convex geometric graph depends only on the number of vertices, and not on their particular position. Without loss of generality we therefore assume that the point set of the graph corresponds to the vertices of a regular polygon. In the remainder of this section we exclusively work with this type of graphs.

To simplify the proof of the main statement of this section, in the following we will define different sets of edges and prove some important properties of these sets.

Let G be a complete convex geometric graph of order n, and let $\{1, \ldots, n\}$ be the vertices of the graph listed in clockwise order. For the remainder of this subsection it is important to bear in mind that all sums are taken modulo n; for the sake of simplicity we will avoid writing this explicitly. We denote by $e_{i,j}$ the edge between the vertices i and j. We call an edge $e_{i,j}$ a *halving edge* if in both of the two open semi-planes defined by the line containing $e_{i,j}$, there are at least $\lfloor \frac{n-2}{2} \rfloor$ points of G. Using this concept we obtain the following definition.

Definition 3.4. Let $i, j, k \in \{1, \ldots, n\}$, such that $e_{i,j}$ and $e_{j+1,k}$ do not intersect. We call a pair of edges $(e_{i,j}, e_{j+1,k})$ a *halving pair of edges* (*halving pair*, for short) if at least one of $e_{i,j+1}$, $e_{i,k}$, or $e_{j,k}$ is a halving edge. This halving edge is called the *witness* of the halving pair.

See Figure 1 for an example of a halving pair $(e_{i,j}, e_{j+1,k})$, with $e_{i,k}$ as witness. Note that a halving pair may have more than one witness.

We say that an edge e intersects a pair of edges (f, g) if e intersects at least one of f or g. We say that two pairs of edges intersect if there is an edge in the first pair which intersects the second pair.

Lemma 3.5. *Let* G *be a complete convex geometric graph of order* n. *i) Each two halving edges intersect. ii) Any halving edge intersects any halving pair of edges. iii) Any two halving pairs intersect.*

Proof. To prove Claim i) assume that there are two halving edges which do not intersect. These edges divide the set of vertices of G into two disjoint sets of size at least $\lfloor \frac{n-2}{2} \rfloor$ and one set of size at least 4 (the vertices of the two halving edges). Then, the total number of vertices is:

$$2 \left\lfloor \frac{n-2}{2} \right\rfloor + 4 = \begin{cases} n+1 & \text{if n is odd} \\ n+2 & \text{if n is even} \end{cases} \tag{3.2}$$

This is a contradiction, which proves Claim i).

To prove Claims ii) and iii) observe that the convex hull of each halving pair $(e_{i,j}, e_{j+1,k})$ defines a quadrilateral $(i, j, j+1, k)$, see Figure 1. The halving edge witnessing the halving pair is contained in the corresponding convex hull: it is either the edge $e_{i,k}$, or one of the diagonals of the quadrilateral.

It is easy to see, that if either $e_{i,k}$ or one of the diagonals is intersected by an edge f, then f also intersects at least one edge of the pair $(e_{i,j}, e_{j+1,k})$.

Using this observation we prove the remaining two cases by contradiction: Assume there exists a halving edge and a halving pair which do not intersect, or two halving pairs which do not intersect. Then their corresponding halving edges (witnesses) do not intersect either, because they are contained in the quadrilaterals. This contradicts Claim i), and thus proves Claim ii) and iii). $\square$

Definition 3.6. Let G be a complete convex geometric graph of even order n. We call an edge $e_{i,j}$ an *almost-halving edge* if $e_{i,j+1}$ is a halving edge.

Please observe that this definition and the following lemma are only stated (and valid) for even n; therefore if $e_{i,j}$ is an almost-halving edge, $e_{i-1,j}$ is a halving edge.

Lemma 3.7. *Let* G *be a complete convex geometric graph of even order* n. *Let* f *be an almost-halving edge,* e *a halving edge, and* E *a halving pair. i)* f *and* e *intersect, ii)* f *and* E *intersect.*

Proof. We prove Claim i) by contradiction. If e and f do not intersect, then they divide the set of vertices of G into three sets: one of size at least $\frac{n-2}{2}$, one of size at least $\frac{n-2}{2} - 1$, and one of size at least 4. In total the number of vertices is (at least):

$$2 \left(\frac{n-2}{2} \right) - 1 + 4 = n + 1 \tag{3.3}$$

This is a contradiction, which proves Claim i). To prove Claim ii) we use Claim i): the halving edge witnessing E must intersect f. On the other hand such a halving edge is inside the convex hull of E, see Figure 1. From these two observations it follows that E and f intersect. $\square$
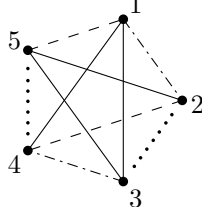
Figure 2: Proof of Theorem 3.8 for $n = 5$: $\alpha_1(\mathsf{K}_5) = 7$. Edges $e_{1,3}, e_{3,5}, e_{4,1}, e_{5,2}$ (solid) are colored with colors 1 to 4, respectively. Edges $e_{1,2}, e_{3,4}$ are colored with color 5; and edges $e_{2,3}, e_{4,5}$ are colored with color 6. Finally, edges $e_{2,4}, e_{1,5}$ are colored with color 7. Each pair of chromatic classes intersect, and each pair of edges of the same color are disjoint.

We need two more concepts from the literature. A *straight-line thrackle* [12] of G is a subset of edges of G with the property that any two distinct edges intersect (they have a common endpoint or they cross). A straight-line thrackle is *maximal* if it is not a proper subset of any other thrackle. Theorem 3.1 implies that the size of any straight-line thrackle of G is at most n. In the following we always refer to a straight-line thrackle as thrackle, since we are only working with geometric embeddings of graphs.

Given a set $J \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, a *circulant graph* $C_n(J)$ of G is defined as the graph with vertex set equal to $V(G)^1$ and $E(C_n(J)) = \{e_{i,j} \in E(G) \colon j - i \equiv k \mod n, \text{ or } j - i \equiv -k \mod n, k \in J\}$. In this paper we use this concept for geometric graphs, in the natural way; see Figure 3 (left) for an example of a circulant geometric graph $C_n(J)$ with $J = \{\lfloor \frac{n}{2} \rfloor - 1\}$ and $n = 13$.
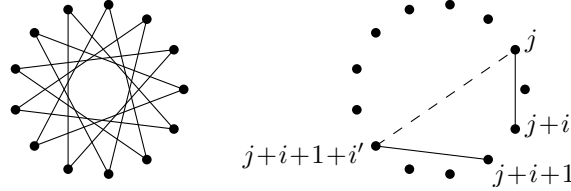

Figure 3: Examples for $n = 13$. Left: A circulant graph $C_n\left(\left\{\lfloor \frac{n}{2} \rfloor - 1\right\}\right)$. Right: A pair of edges (solid) with same color from $E(C_n(\{i, i'\}))$, with $i = 2$ and some fixed j. The witness of the halving pair is shown dashed.

The following theorem provides the lower bound on the achromatic index.

Theorem 3.8. *Let* G *be a complete convex geometric graph of order* $n \neq 4$.

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[1]This definition is different from that usually given, in which the vertex set of the graph is $\mathbb{Z}_n$. However, as it is not hard to see that this definition is equivalent to the usual one, and to keep our arguments as simple as possible, we opt for this choice.

The achromatic index of G satisfies the following bound:

$$\alpha_1(\mathsf{G}) \geq \lfloor \tfrac{n^2+n}{4} \rfloor.$$

Proof. The theorem follows easily for $n \leq 3$; we prove the case $n = 5$ in Figure 2. For $n > 5$, consider the following partition of the set of edges of G:

$$E(\mathsf{G}) = E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \bigcup E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup_{i \in I} E(C_n(\{i, \lfloor \tfrac{n}{2} \rfloor - 1 - i\})) \quad (3.4)$$

where $I = \{1, \ldots, \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor\}$.

Observe that the first term is a circulant graph of halving edges and thus, by Lemma 3.5, its set of edges defines a thrackle. This thrackle is maximal (containing n edges) if n is odd but it is not maximal (containing only $\frac{n}{2}$ edges) if n is even.

Note further, that for fixed i the third term is either the union of two circulant graphs of size n, or one circulant graph of size n (only in the case when $i = \lfloor \frac{n}{2} \rfloor - 1 - i$).

If n is odd, then the edge set of G is partitioned into $\frac{n-1}{2}$ circulant graphs, each of them of size n. If n is even, then the edge set of G is partitioned into $\frac{n}{2} - 1$ circulant graphs, each of them of size n, plus one circulant graph of size $\frac{n}{2}$. Using partition 3.4, we give a coloring on the edges of G, and prove that this coloring is proper and complete.

We start by coloring all circulant graphs in the third term of the partition, except for $i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$.

In the following we set $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$ and therefore refer to $C_n(\{i, \lfloor \frac{n}{2} \rfloor - 1 - i\})$ as $C_n(\{i, i'\})$. For every $i \in I \setminus \left\{ \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor \right\}$ we assign colors to $C_n(\{i, i'\})$ using the following function.

$$f_i \colon E(C_n(\{i, i'\})) \longrightarrow \{(i-1)n + 1, \ldots, (i-1)n + n\} \text{ such that:}$$

$$\begin{aligned}
e_{j,j+i} &\mapsto (i-1)n + j, \text{ and} \\
e_{j+i+1,j+i+1+i'} &\mapsto (i-1)n + j.
\end{aligned}$$

for $j \in \{1, \ldots, n\}$. See Figure 3 (right) for an example with $i = 2$.

The first rule colors the edges of $C_n(\{i\})$, while the second rule colors the edges of $C_n(\{i'\})$. For fixed i and j both rules assign the same color. Therefore, the chromatic classes are pairs of edges, one edge $(e_{j,j+i})$ from $C_n(\{i\})$ and one edge $(e_{j+i+1,j+i+1+i'})$ from $C_n(\{i'\})$. Observe, that all these pairs are halving pairs $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ of G, because the edge $e_{j,j+i+1+i'} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ is halving.

Hence, the partial coloring so far is complete (by Lemma 3.5) and proper (because the two edges in each color class do not intersect).

The number of colors we have used so far is $N_1 = n \left(\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor - 1 \right)$.

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So far, a subset of edges of the third term of the partition 3.4 is colored. This leaves the following parts uncolored:

$$E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \bigcup E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \tfrac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \tfrac{n}{2} \rfloor - 1 - i$.

These remaining circulant graphs differ for n even or n odd. Further, the two cases $i = i'$ and $i \neq i'$ need to be distinguished (for the remainder of the third term). This basically results in the four cases $n \equiv x \mod 4$, for $x \in \{0, 1, 2, 3\}$.

In a nutshell, to color the remaining edges, first the thrackle, $E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\}))$, will be colored (if n is even together with one half of $E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})))$. Then (the remaining half of) the circulant graph $C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})$ together with $C_n(\{i, i'\})$ ($i = \lfloor \tfrac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \tfrac{n}{2} \rfloor - 1 - i$) is colored. In each step we will prove, that the (partial) coloring is proper and complete.

1. Case $n > 5$ is odd. To color the maximal thrackle, $E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\}))$, we assign colors to its edges using the function

$$f \colon E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \longrightarrow \{N_1 + 1, \ldots, N_1 + n\} \text{ such that:}$$

$$e_{j, j + \lfloor \frac{n}{2} \rfloor} \mapsto N_1 + j,$$

for each $j \in \{1, \ldots, n\}$. Observe that $E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\}))$ is a set of n halving edges. See Figure 4 (left) for an example of such a thrackle.
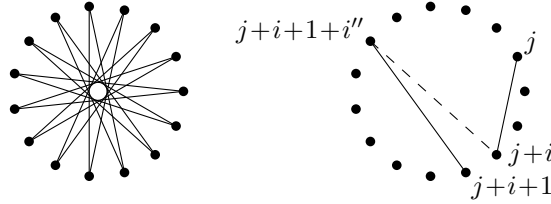

Figure 4: Examples for $n = 15$. Left: A circulant graph $C_n\left(\{\lfloor \tfrac{n}{2} \rfloor\}\right)$ of halving edges if n is odd. Right: Halving pair (solid) with color $N_2 + j$ from $E(C_n(\{i, i''\}))$, with $n \equiv 3 \mod 4$ and some fixed j. The witness of the halving pair is shown dashed.

The coloring so far is proper, because each new chromatic class has size one. Further, each chromatic class so far consists of either a halving edge or a halving pair. Hence, by Lemma 3.5, the coloring is also complete.

It is easy to see that we are using $N_2 = N_1 + n = n \lfloor \tfrac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ colors so far.

The remaining uncolored edges are:

$$E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

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where $i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$. These two circulant graphs will be colored together. Let $i'' = \lfloor \frac{n}{2} \rfloor - 1$. As n is odd, $C_n(\{i''\})$ consists of n edges. The size of $E(C_n(\{i, i'\}))$ depends on the two cases $i = i'$ and $i \neq i'$.

(a) $i = i'$: As n is odd, $n \equiv 3 \mod 4$. The circulant graph $C_n(\{i, i'\}) = C_n(\{i\})$ is of size n. Thus, $2n$ edges remain uncolored.

We assign n colors to the $2n$ edges of $C_n(\{i, i''\})$ as follows:

$$f_i \colon E(C_n(\{i, i''\})) \longrightarrow \{N_2 + 1, \ldots, N_2 + n\}, \text{ such that}$$

$$
\begin{aligned}
e_{j, j+i} &\mapsto N_2 + j, \\
e_{j+i+1, j+i+1+i''} &\mapsto N_2 + j
\end{aligned}
$$

for $j \in \{1, \ldots, n\}$.

Each new chromatic class consists of a pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i''})$ of edges. See Figure 4 (right) for an example of such a pair. Because the edge $e_{j+i,j+i+1+i''} = e_{j+i,j+i+\lfloor \frac{n}{2} \rfloor}$ is a halving edge, the pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i''})$ is a halving pair. Therefore, all edges are colored and each chromatic class consists of either a halving edge or a halving pair. By Lemma 3.5 the coloring is complete and proper (as the edges of halving pairs are disjoint).

The total number of colors used is $N_3 = N_2 + n = n(\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor + 1)$, that is $N_3 = \lfloor \frac{n^2 + n}{4} \rfloor$ colors, as $n \equiv 3 \mod 4$ in this case.

(b) $i \neq i'$: As n is odd, $n \equiv 1 \mod 4$. The circulant graph $C_n(\{i, i'\})$ is of size $2n$. Thus, $3n$ edges remain uncolored. We assign n colors to the $2n$ edges of $C_n(\{i, i'\})$ and $\lfloor \frac{n}{2} \rfloor$ colors to the n edges of $C_n(\{i''\})$ as follows:

$$f_i \colon E(C_n(\{i, i', i''\})) \longrightarrow \{N_2 + 1, \ldots, N_2 + n + \lfloor \frac{n}{2} \rfloor\}, \text{ such that}$$

$$
\begin{aligned}
e_{j,j+i} &\mapsto N_2 + j, \\
e_{j+i+1, j+i+1+i'} &\mapsto N_2 + j,
\end{aligned}
$$

for $j \in \{1, \ldots, n\}$, and

$$
\begin{aligned}
e_{j,j+i''} &\mapsto N_2 + n + j, \\
e_{j+i''+1, j+i''+1+i''} &\mapsto N_2 + n + j,
\end{aligned}
$$

for $j \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. See Figure 5 (left and middle) for examples.

Each new chromatic class consists of a pair of edges. These pairs are either $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ or $(e_{j,j+i''}, e_{j+i''+1,j+i''+1+i''})$ combined from the edges of $C_n(\{i, i'\})$ or $C_n(\{i''\})$, respectively. The
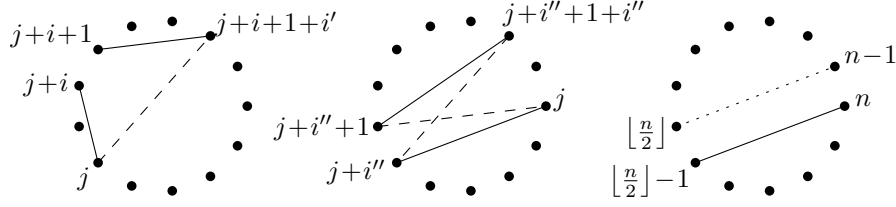
Figure 5: Examples with $n = 13$, for n is odd and $i \neq i'$: $n \equiv 1 \mod 4$. Left: Halving pair with color $N_2 + j$ from $E(C_n(\{i, i'\}))$. Middle: Halving pair with color $N_2 + n + j$ from $E(C_n(\{i''\}))$. Both for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed. Right: The single remaining edge $e_{n,\lfloor \frac{n}{2} \rfloor - 1}$ (solid) is combined with the halving edge $e_{\lfloor \frac{n}{2} \rfloor, n-1}$ (dotted), colored with color $N_1 + \lfloor \frac{n}{2} \rfloor$.

pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ is a halving pair with the halving edge $e_{j,j+i+1+i'} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ as witness, and $(e_{j,j+i''}, e_{j+i''+1,j+i''+1+i''})$ is a halving pair with the halving edges $e_{j,j+i''+1} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ and $e_{j+i'',j+i''+1+i''} = e_{j+i'',j+i''+\lfloor \frac{n}{2} \rfloor}$ as witnesses. Each chromatic class so far consists of either a halving edge or a halving pair. Hence, the coloring is complete (by Lemma 3.5) and proper (as the edges of halving pairs are disjoint).

Note, that a single edge, $e_{n,\lfloor \frac{n}{2} \rfloor - 1}$ of $C_n(\{i''\})$, remains uncolored. We add this edge to the chromatic class (with color $N_1 + \lfloor \frac{n}{2} \rfloor$) containing the halving edge $e_{\lfloor \frac{n}{2} \rfloor, n-1} = e_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor}$. See Figure 5 (right). Observe, that $e_{n,\lfloor \frac{n}{2} \rfloor - 1}$ and $e_{\lfloor \frac{n}{2} \rfloor, n-1}$ are disjoint, thus the coloring remains proper. Further, adding an edge to an existing chromatic class of a complete coloring, maintains the completeness of the coloring.

As all edges are colored, the total number of colors used is $N_3 = N_2 + n + \lfloor \frac{n}{2} \rfloor = n(\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor + 1) + \lfloor \frac{n}{2} \rfloor$, that is $N_3 = \left\lfloor \frac{n^2 + n}{4} \right\rfloor$, as $n \equiv 1 \mod 4$ in this case.

2. Case $n > 5$ is even. Recall that only N_1 chromatic classes exist so far, each containing a halving pair of edges. The thrackle $E(C_n(\{\lfloor \frac{n}{2} \rfloor\})) = E(C_n(\{\frac{n}{2}\}))$ is not maximal in this case. See Figure 6 (left). To get a maximal thrackle we add half the edges of $C_n(\{\frac{n}{2} - 1\})$ to $C_n(\{\frac{n}{2}\})$. Note that $E(C_n(\{\frac{n}{2} - 1\}))$ is the set of almost-halving edges in the case of n is even.

Let the thrackle $E(C_n'(\{\frac{n}{2} - 1\})) = \{e_{1,\frac{n}{2}}, \ldots, e_{\frac{n}{2},n-1}\}$ and the thrackle $E(C_n''(\{\frac{n}{2} - 1\})) = \{e_{\frac{n}{2}+1,n}, \ldots, e_{n,\frac{n}{2}-1}\}$ define the two halves of $C_n(\{\frac{n}{2} - 1\})$ with $\frac{n}{2}$ almost-halving edges each. See Figure 6 (middle and right). It is easy to see that $E(C_n'(\{\frac{n}{2} - 1\}))$ is a thrackle (all its edges intersect each other). Further, by Lemma 3.7, each almost-halving edge intersects each halving edge. Thus, $E(C_n(\{\frac{n}{2}\}) \cup C_n'(\{\frac{n}{2} - 1\}))$ is a maximal thrackle
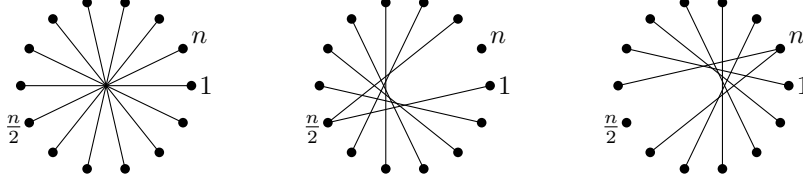
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Figure 6: Examples with $n = 14$, for the case when n is even. Left: The thrackle, $E(C_n(\{\frac{n}{2}\}))$, of the $\frac{n}{2}$ halving edges. Middle: The thrackle, $E(C'_n(\{\frac{n}{2} - 1\}))$, of the first $\frac{n}{2}$ almost-halving edges of $E(C_n(\{\frac{n}{2} - 1\}))$. Right: The thrackle, $E(C''_n(\{\frac{n}{2} - 1\}))$, of the second $\frac{n}{2}$ almost-halving edges of $E(C_n(\{\frac{n}{2} - 1\}))$.

of size n. The following function assigns one color to each edge of this maximal thrackle.

$$f\colon E(C_n(\{\frac{n}{2}\}) \cup C'_n(\{\frac{n}{2} - 1\})) \longrightarrow \{N_1 + 1, \dots, N_1 + n\}, \text{ such that}$$

$$\begin{aligned}
e_{j,j+\frac{n}{2}} &\mapsto N_1 + j, \\
e_{j,j+\frac{n}{2}-1} &\mapsto N_1 + \frac{n}{2} + j
\end{aligned}$$

for each $j \in \{1, \dots, \frac{n}{2}\}$.

The coloring so far is proper, because each new chromatic class has size one. Further, each chromatic class consists of either a halving edge, a halving pair, or an almost-halving edge. The almost-halving edges used so far form a thrackle and thus, intersect each other. Hence, by Lemmas 3.5 and 3.7, the coloring is also complete. It is easy to see that we are using $N_2 = N_1 + n = n\lfloor \frac{n-2}{4} \rfloor$ colors so far.

The remaining uncolored edges are

$$E(C''_n(\{\lfloor \frac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \frac{n-2}{4} \rfloor$ and $i' = \frac{n}{2} - 1 - i$ (as n is even). These two circulant graphs will be colored together. For brevity, let $i'' = \frac{n}{2} - 1$ and $C''_n(\{i''\})$ be the set of the remaining $\frac{n}{2}$ almost-halving edges. The size of $E(C_n(\{i, i'\}))$ depends on the two cases $i = i'$ and $i \neq i'$.

(a) $i = i'$: As n is even, $n \equiv 2 \pmod 4$. The circulant graph $C_n(\{i, i'\}) = C_n(\{i\})$ is of size n. Thus, $n + \frac{n}{2}$ edges remain uncolored. We assign $\frac{n}{2} + \lfloor \frac{n}{4} \rfloor$ colors to the $n + \frac{n}{2}$ edges of $C_n(\{i\}) \cup C''_n(\{i''\})$ as follows:

$$f_i\colon E(C_n(\{i\}) \cup C''_n(\{i''\})) \longrightarrow \{N_2 + 1, \dots, N_2 + \frac{n}{2} + \lfloor \frac{n}{4} \rfloor\}, \text{ such that}$$
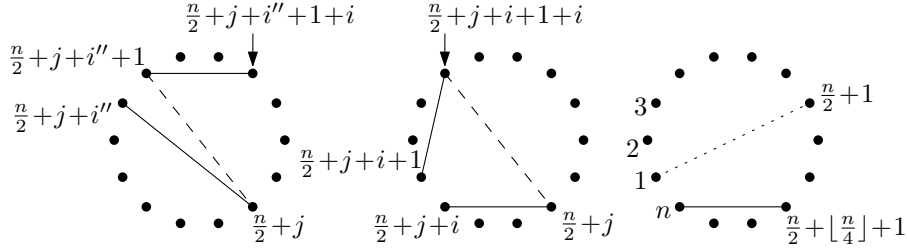
Figure 7: Examples with $n = 14$, for the case when n is even and $i = i'$: $n \equiv 2 \mod 4$. Left: Halving pair with color $N_2 + j$. Middle: Halving pair with color $N_2 + \frac{n}{2} + j$. Both for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed. Right: The single remaining edge $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ (solid) is combined with the halving edge $(e_{1,\frac{n}{2}+1})$ (dotted), colored with color $N_1 + 1$.

$$
\begin{aligned}
e_{\frac{n}{2}+j,\frac{n}{2}+j+i''} &\mapsto N_2 + j, \\
e_{\frac{n}{2}+j+i''+1,\frac{n}{2}+j+i''+1+i} &\mapsto N_2 + j
\end{aligned}
$$

for $j \in \{1, \ldots, \frac{n}{2}\}$, and

$$
\begin{aligned}
e_{\frac{n}{2}+j,\frac{n}{2}+j+i} &\mapsto N_2 + \tfrac{n}{2} + j, \\
e_{\frac{n}{2}+j+i+1,\frac{n}{2}+j+i+1+i} &\mapsto N_2 + \tfrac{n}{2} + j
\end{aligned}
$$

for $j \in \{1, \ldots, \lfloor\frac{n}{4}\rfloor\}$. See Figure 7 (left and middle) for examples.

Each new chromatic class consists of a halving pair of edges from $E(C_n(\{i\}) \cup C_n''(\{i''\}))$, either $(e_{\frac{n}{2}+j,\frac{n}{2}+j+i''}, e_{\frac{n}{2}+j+i''+1,\frac{n}{2}+j+i''+1+i})$ with the halving edge $e_{\frac{n}{2}+j,\frac{n}{2}+j+i''+1} = e_{\frac{n}{2}+j,\frac{n}{2}+j+\frac{n}{2}}$ as witness, or $(e_{\frac{n}{2}+j,\frac{n}{2}+j+i}, e_{\frac{n}{2}+j+i+1,\frac{n}{2}+j+i+1+i})$ with, again, the halving edge $e_{\frac{n}{2}+j,\frac{n}{2}+j+i+1+i} = e_{\frac{n}{2}+j,\frac{n}{2}+j+\frac{n}{2}}$ as witness. Each chromatic class so far consists of either a halving edge, a halving pair, or one of the $\frac{n}{2}$ almost-halving edges that form a thrackle. Hence, the coloring is complete (by Lemmas 3.5 and 3.7) and proper (as the edges of halving pairs are disjoint).

Note, that a single edge, $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ of $C_n(\{i\})$, remains uncolored. We add this edge to the chromatic class (with color $N_1 + 1$) containing the halving edge $(e_{1,\frac{n}{2}+1})$. See Figure 7 (right). Observe, that $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ and $(e_{1,\frac{n}{2}+1})$ are disjoint. Thus, the coloring remains proper. Further, adding an edge to an existing chromatic class of a complete coloring, maintains the completeness of the coloring.

As all edges are colored, the total number of colors used is $N_3 = N_2 + \frac{n}{2} + \lfloor\frac{n}{4}\rfloor = n\lfloor\frac{n-2}{4}\rfloor + \frac{n}{2} + \lfloor\frac{n}{4}\rfloor$, that is $N_3 = \left\lfloor\frac{n^2+n}{4}\right\rfloor$, as $n \equiv 2 \mod 4$ in this case.
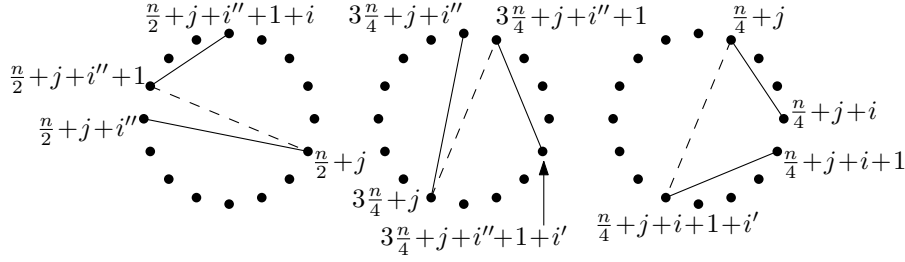
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Figure 8: Examples with $n = 16$, for the case when n is even and $i \neq i'$: $n \equiv 0$ mod 4. Left: Halving pair with color $N_2 + j$. Middle: Halving pair with color $N_2 + \frac{n}{4} + j$. Right: Halving pair with color $N_2 + \frac{n}{2} + j$. All for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed.

(b) $i \neq i'$: As n is even, $n \equiv 0 \mod 4$. The circulant graph $C_n(\{i, i'\})$ is of size $2n$. Thus, $2n + \frac{n}{2}$ edges remain uncolored. We assign $\frac{n}{2} + 3\frac{n}{4}$ colors to the $2n + \frac{n}{2}$ edges of $C_n(\{i, i'\}) \cup C_n''(\{i''\})$ as follows:

$f_i \colon E(C_n(\{i, i'\}) \cup C_n''(\{i''\})) \longrightarrow \{N_2 + 1, \dots, N_2 + \frac{n}{2} + 3\frac{n}{4}\}$, such that

$$e_{\frac{n}{2}+j, \frac{n}{2}+j+i''} \mapsto N_2 + j,$$
$$e_{\frac{n}{2}+j+i''+1, \frac{n}{2}+j+i''+1+i} \mapsto N_2 + j,$$

$$e_{3\frac{n}{4}+j, 3\frac{n}{4}+j+i''} \mapsto N_2 + \frac{n}{4} + j,$$
$$e_{3\frac{n}{4}+j+i''+1, 3\frac{n}{4}+j+i''+1+i'} \mapsto N_2 + \frac{n}{4} + j,$$

for each $j \in \{1, \dots, \frac{n}{4}\}$, and

$$e_{\frac{n}{4}+j, \frac{n}{4}+j+i} \mapsto N_2 + \frac{n}{2} + j,$$
$$e_{\frac{n}{4}+j+i+1, \frac{n}{4}+j+i+1+i'} \mapsto N_2 + \frac{n}{2} + j$$

for each $j \in \{1, \dots, 3\frac{n}{4}\}$. See Figure 8 for an example of these three different types of pairs of edges.

Each new chromatic class consists of a halving pair of edges from $C_n(\{i, i'\}) \cup C_n''(\{i''\})$, either $(e_{\frac{n}{2}+j, \frac{n}{2}+j+i''}, e_{\frac{n}{2}+j+i''+1, \frac{n}{2}+j+i''+1+i})$ with the halving edge $e_{\frac{n}{2}+j, \frac{n}{2}+j+i''+1} = e_{\frac{n}{2}+j, \frac{n}{2}+j+\frac{n}{2}}$ as witness (Figure 8 (left)), $(e_{3\frac{n}{4}+j, 3\frac{n}{4}+j+i''}, e_{3\frac{n}{4}+j+i''+1, 3\frac{n}{4}+j+i''+1+i'})$ with the halving edge $e_{3\frac{n}{4}+j, 3\frac{n}{4}+j+i''+1} = e_{3\frac{n}{4}+j, 3\frac{n}{4}+j+\frac{n}{2}}$ as witness (Figure 8 (middle)), or $(e_{\frac{n}{4}+j, \frac{n}{4}+j+i}, e_{\frac{n}{4}+j+i+1, \frac{n}{4}+j+i+1+i'})$ with, again, the halving edge $e_{\frac{n}{4}+j, \frac{n}{4}+j+i+1+i'} = e_{\frac{n}{4}+j, \frac{n}{4}+j+\frac{n}{2}}$ as witness (Figure 8 (right)). Each chromatic class so far consists of either a halving edge, a halving pair, or one of the $\frac{n}{2}$ almost-halving edges that form a thrackle.

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Hence, the coloring is complete (by Lemmas 3.5 and 3.7) and proper (as the edges of halving pairs are disjoint).

As all edges are colored, the total number of colors used is $N_3 = N_2 + \frac{n}{2} + 3\frac{n}{4} = n\lfloor\frac{n-2}{4}\rfloor + \frac{n}{2} + 3\frac{n}{4}$, that is $N_3 = \left\lfloor \frac{n^2+n}{4} \right\rfloor$, as $n \equiv 0$ mod 4 in this case.

$\square$

Proof of Theorem 2.2 i). For $n \neq 4$, by Theorem 3.8 we get that $\left\lfloor \frac{n^2+n}{4} \right\rfloor \leq \alpha_1(\mathsf{G})$, and by Theorem 3.3 and Equation 2.4 we conclude that $\alpha_1(\mathsf{G}) = \psi_1(\mathsf{G}) = \left\lfloor \frac{n^2+n}{4} \right\rfloor$.

$\square$

Theorem 2.2 i) excludes the case $n = 4$, this is because K_4 is the only complete convex geometric graph for which α_1 and ψ_1 are different. We now prove that $\alpha_1(\mathsf{K}_4) = 4$ and $\psi_1(\mathsf{K}_4) = 5$. By Theorem 3.3 we have that $\psi_1(\mathsf{K}_4) \leq 5$, and by Figure 9 (left) we can conclude that $\psi_1(\mathsf{K}_4) = 5$. Now, by Figure 9 (right) we have $\alpha_1(\mathsf{K}_4) \geq 4$. Suppose that $\alpha_1(\mathsf{K}_4) = 5$, then in the coloring of K_4 with 5 colors, there must be four color classes of size one, and one color class of size two. The class of size two must be composed by two opposite edges of K_4, this implies that the remaining two opposite edges belong to classes of size one; but this is a contradiction because the coloring must be complete. Therefore, it follows that $\alpha_1(\mathsf{K}_4) = 4$.
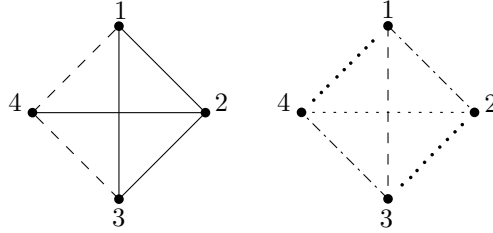

Figure 9: Left: $\psi_1(\mathsf{K}_4) = 5$. One color class of size two $\{e_{3,4}, e_{4,1}\}$; and four color classes of size one $\{e_{1,2}\},\{e_{2,3}\},\{e_{1,3}\},\{e_{2,4}\}$. Right: $\alpha_1(\mathsf{K}_4) = 4$. Two color classes of size two: $\{e_{1,2}, e_{3,4}\}$, $\{e_{2,3}, e_{4,1}\}$; and two color classes of size one $\{e_{1,3}\}$, $\{e_{4,2}\}$.

4   On $\psi_g(K_n)$

In this section we consider point sets in general position in the plane, and present lower and upper bounds for the geometric pseudoachromatic index. Recall that the geometric pseudoachromatic index of a graph G is defined as:

$$\psi_g(G) = \min\{\psi_1(\mathsf{G}) \colon \mathsf{G} \text{ is a geometric graph of } G\}.$$

4.1 Upper bound for $\psi_g(K_n)$

Let $G = (V, E)$ be a graph, and let G be a geometric representation of G. Consider two intersecting edges in G, the intersection might occur either at a common interior point (crossing), or at a common end point (at a vertex).

On one hand, if we consider all edges of G and denote by m the total number of intersections that occur at vertices, m is precisely the number of edges in the line graph of G. That is, if $deg(v)$ is the degree of $v \in V$, then $m = \sum_{v \in V} \binom{deg(v)}{2}$.

On the other hand, the *rectilinear crossing number of* G, denoted by $\overline{cr}(\mathsf{G})$, is defined as the number of edge crossings that occur in G. Given a graph G, the *rectilinear crossing number of* G is the minimum number of crossings over all possible geometric embeddings of G; notationally

$$\overline{cr}(G) = \min\{\overline{cr}(\mathsf{G}) \colon \mathsf{G} \text{ is a geometric graph of } G\}.$$

It seems natural that there should be a relationship between the rectilinear crossing number of a graph, and its geometric achromatic and pseudoachromatic indices. In the following lines we establish bounds for $\psi_g(G)$ as a function of m and $\overline{cr}(G)$.

Lemma 4.1. *Let* G *be a geometric graph of order n, denote by* $\overline{cr}(\mathsf{G})$ *the number of edge crossings in* G, *and by m the total number of edge intersections occurring at vertices of* G. *Then:*

$$\psi_1(\mathsf{G}) \leq \left\lfloor \frac{1 + \sqrt{1 + 8(m + \overline{cr}(\mathsf{G}))}}{2} \right\rfloor.$$

Proof. The total number of edge intersections is $m + \overline{cr}(\mathsf{G})$. Then, $m + \overline{cr}(\mathsf{G}) \geq \binom{\psi_1(\mathsf{G})}{2}$ so that $\psi_1(\mathsf{G})(\psi_1(\mathsf{G}) - 1) \leq 2(m + \overline{cr}(\mathsf{G}))$. Solving this inequality we get $\psi_1(\mathsf{G}) \leq \left\lfloor \frac{1 + \sqrt{1 + 8(m + \overline{cr}(\mathsf{G}))}}{2} \right\rfloor$. $\square$

Using the above Lemma, we can establish the following result.

Theorem 4.2. *Let* $G = (V, E)$ *be a graph of order n, with* $m = \sum_{v \in V} \binom{deg(v)}{2}$. *Denote by* $\overline{cr}(G)$ *its rectilinear crossing number. Then*

$$\psi_g(G) \leq \left\lfloor \frac{1 + \sqrt{1 + 8(m + \overline{cr}(G)}}{2} \right\rfloor.$$

Proof. Let G_0 be a geometric representation of G such that $\overline{cr}(\mathsf{G}_0) = \overline{cr}(G)$; that is, G_0 is a geometric graph of G with minimum number of crossings. As a

consequence of Lemma 4.1 we have the following:

$$
\begin{aligned}
\psi_g(G) \;&=\; \min\{\psi_1(\mathsf{G})\colon \mathsf{G} \text{ is a geometric graph of } G\} \le \psi_1(\mathsf{G_0}) \\
&\le\; \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G_0}))}}{2} \right\rfloor = \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(G))}}{2} \right\rfloor
\end{aligned}
$$

$\square$

Establishing bounds for $\overline{cr}(K_n)$ is a well studied problem in the literature. If we use these results, we can give a better bound for $\psi_g(K_n)$; note that in this case $m = n\binom{n-1}{2}$. The following result was shown in [1].

Theorem 4.3. $\overline{cr}(K_n) \le c\binom{n}{4} + \Theta(n^3)$ *for* $c = 0.380488$.

Using this theorem we obtain:

Theorem 4.4. *Let K_n be the complete graph of order n. The geometric pseudoachromatic index of K_n has the following upper bound:*

$$
\psi_g(K_n) \le 0.1781 n^2 + \Theta(n^{\frac{3}{2}}).
$$

Proof. Since $m = \sum\limits_{v \in V(K_n)} \binom{deg(v)}{2} = n\binom{n-1}{2}$, by Theorem 4.2 and 4.3,

$$
\begin{aligned}
\psi_g(K_n) \;&\le\; \frac{1}{2}\sqrt{8\overline{cr}(K_n)) + \Theta(n^3)} + \Theta(1) = \frac{1}{2}\sqrt{8\frac{c}{4!}n^4 + \Theta(n^3)} + \Theta(1) \\
&=\; \sqrt{\frac{c}{12}n^2 + \Theta(n^{\frac{3}{2}})} \le 0.1781 n^2 + \Theta(n^{\frac{3}{2}}).
\end{aligned}
$$

$\square$

4.2   Lower bound for $\psi_g(K_n)$

In this section we present a lower bound for $\psi_g(K_n)$. First let us state a result which will be used later; this result was shown in [5].

Theorem 4.5. *Let S be a set of n points in general position in the plane. There are three concurrent lines that divide the plane into six parts each containing at least $\frac{n}{6} - 1$ points of S in its interior.*

In order to exhibit the desired coloring, first we divide the plane into seven regions and then use this partition of the plane to construct a partition of the edges of the complete geometric graph of order $n > 18$. We utilize a specific configuration $\mathcal{L}$ of lines, defined as follows; see also Figure 10 for a drawing of the configuration. Let S be a set of $n = 13m + 6 + r$ points in general position in the plane ($r < 13$). Choose horizontal lines ℓ_1, ℓ_2, and ℓ_3 (listed top-down) so that when writing A', B' for the set of points between ℓ_1 & ℓ_2, and ℓ_2 & ℓ_3, respectively, we have $|A'| = 12m + 6$ and $|B'| = m + r$. Let ℓ_4, ℓ_5, ℓ_6 be

concurrent lines that divide the set A' into 6 parts, each containing at least $2m$ points in its interior; the existence of these lines is guaranteed by Theorem 4.5. Let p be the point of intersection of the three lines. For each one of the six subsets of points induced by the partition, we take a subset of size exactly $2m$. Let A, B, C, D, E, F be such subsets, listed in clockwise order. Take $G \subseteq B'$ such that $|G| = m$.
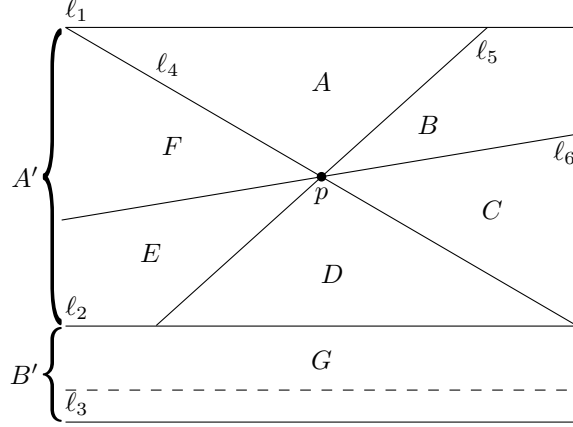

Figure 10: The line configuration $\mathcal{L}$.

Using these sets of points, first we construct three sets of subgraphs of K_n. Then, we assign one color to each of them and show that such a coloring is complete. Let $A = \{a_1, \ldots, a_{2m}\}, B = \{b_1, \ldots, b_{2m}\}, C = \{c_1, \ldots, c_{2m}\}, D = \{d_1, \ldots, d_{2m}\}, E = \{e_1, \ldots, e_{2m}\}, F = \{f_1, \ldots, f_{2m}\}$; and $G = \{g_1, \ldots, g_m\}$.

For $i, j \in \{1, \ldots, 2m\}$, consider the following sets of subgraphs of K_n:

- The subgraphs $\mathsf{X}_{i,j}$ with vertex set $\{a_i, b_j, d_i, e_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{a_i b_j, b_j d_i, d_i e_j, e_j a_i\} \cup \begin{cases} \{a_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{d_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd .} \end{cases}$$

  Note that each $\mathsf{X}_{i,j}$ is a quadrilateral plus one edge. We call each quadrilateral $\mathsf{X}'_{i,j} \leq \mathsf{X}_{i,j}$, induced by vertices a_i, b_j, d_i, e_j.

- The subgraphs $\mathsf{Y}_{i,j}$ with vertex set $\{b_i, c_j, e_i, f_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{b_i c_j, c_j e_i, e_i f_j, f_j b_i\} \cup \begin{cases} \{b_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{e_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd .} \end{cases}$$

Let $\mathsf{Y}'_{i,j} \leq \mathsf{Y}_{i,j}$ be the quadrilateral induced by vertices b_i, c_j, e_i, f_j.

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- The subgraphs $\mathsf{Z}_{i,j}$ with vertex set $\{c_i, d_j, f_i, a_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{c_i d_j, d_j f_i, f_i a_j, a_j c_i\} \cup \begin{cases} \{c_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{f_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd .} \end{cases}$$

Let $\mathsf{Z}'_{i,j} \leq \mathsf{Z}_{i,j}$ be the quadrilateral induced by vertices c_i, d_j, f_i, a_j.

Please note that the set $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$, with $i, j \in \{1, \dots, 2m\}$ contains exactly $12m^2$ subgraphs. Also note that each subgraph $\mathsf{X}'_{i,j}, \mathsf{Y}'_{i,j}$ and $\mathsf{Z}'_{i,j}$, is a (not necessarily convex) quadrilateral. The following lemma shows that p is inside each of these quadrilaterals.

Lemma 4.6. *Let p be the point of intersection of the three lines in Theorem 4.5. Then p is inside each of the polygons induced by the graphs $\mathsf{X}'_{i,j}, \mathsf{Y}'_{i,j}$ and $\mathsf{Z}'_{i,j}$, defined above.*

Proof. Consider the polygon P induced by $\mathsf{X}'_{i,j}$ and recall that the vertices of P are $\{a_i, b_j, d_i, e_j\}$. The line ℓ_4 separates the subsets A and B from the subsets D and E. Thus, ℓ_4 separates the edge $a_i b_j$ from the edge $d_i e_j$. The line ℓ_5 separates the subsets A and E from the subsets B and D. Thus, ℓ_5 intersects the edges $a_i b_j$ and $d_i e_j$, of P. Consider the segment of ℓ_5 defined by its intersection point with $a_i b_j$, and by its intersection point with $d_i e_j$; call this segment s. As ℓ_4 lies between $a_i b_j$ and $d_i e_j$, the point of intersection of ℓ_5 with ℓ_4 (which is the point p), lies in the interior of s. Furthermore, as ℓ_5 intersects P exactly twice, s is in the interior of P and thus, p is inside P.

Analogously, p is inside the polygons induced by $\mathsf{Y}'_{i,j}$ and $\mathsf{Z}'_{i,j}$. $\qquad \square$

Lemma 4.7. *For each pair of graphs from the set $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$ there exists a pair of edges, one of each graph, which intersect.*

Proof. We prove by contradiction. Assume that Q and R are two different graphs in $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$ that do not intersect. By Lemma 4.6, p is inside both polygons, P_{Q} and P_{R}, induced by Q and R, respectively. Since, by assumption, Q and R do not intersect, P_{Q} and P_{R} do not intersect either, and one has to be contained inside the other (as both contain p). Without loss of generality let P_{Q} be inside P_{R}. One edge of Q is connecting P_{Q} (in the interior of P_{R}) with a vertex in G (in the exterior of P_{R}) and therefore intersecting an edge of R. This is a contradiction to the assumption, and the theorem follows. $\qquad \square$

We can now end the proof of our main theorem.

Proof of Theorem 2.2 ii). For every geometric graph K_n of K_n with $n > 18$, one can construct the configuration $\mathcal{L}$. Using $\mathcal{L}$ we choose the edge disjoint graphs $\mathsf{G}_{i,j}$ ($\mathsf{G}_{i,j} \in \{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$). By construction K_n contains $\frac{12}{169} n^2 - \Theta(n)$ of these graphs, and we assign a different color to each of them. By Lemma 4.7 each two of these subgraphs intersect, therefore $0.0710 n^2 - \Theta(n) \leq \psi_g(K_n)$. $\qquad \square$

Acknowledgments

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