Bishellable drawings of K_n[a]

Bernardo M. Ábrego[b1], Oswin Aichholzer[c2], Silvia Fernández-Merchant[d1], Dan McQuillan[e3], Bojan Mohar[f4], Petra Mutzel[g5], Pedro Ramos[h6], R. Bruce Richter[i7], and Birgit Vogtenhuber[j2]

[1]Department of Mathematics, California State University, Northridge
[2]Institute for Software Technology, Graz University of Technology
[3]Department of Mathematics, Norwich University
[4]Department of Mathematics, Simon Fraser University
[5]Department of Computer Science, Technische Universität Dortmund
[6]Departamento de Física y Matemáticas, Universidad de Alcalá
[7]Department of Combinatorics and Optimization, University of Waterloo

Abstract

The Harary-Hill conjecture, still open after more than 50 years, asserts that the crossing number of the complete graph K_n is

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Ábrego et al. [Shellable drawings and the cylindrical crossing number of K_n. *Disc. & Comput. Geom.*, 52(4):743–753, 2014.] introduced the notion of shellability of a drawing D of K_n. They proved that if D is s-shellable for some $s \geq \lfloor\frac{n}{2}\rfloor$, then D has at least $H(n)$ crossings. This is the first combinatorial condition on a drawing that guarantees at least $H(n)$ crossings.

In this work, we generalize the concept of s-shellability to bishellability, where the former implies the latter in the sense that every s-shellable drawing is, for any $b \leq s-2$, also b-bishellable. Our main result is that $(\lfloor\frac{n}{2}\rfloor-2)$-bishellability also guarantees, with a simpler proof than for s-shellability, that a drawing has at least $H(n)$ crossings. We exhibit a drawing of K_{11} that has $H(11)$ crossings, is 3-bishellable, and is not s-shellable for any $s \geq 5$. This shows that we have properly extended the class of drawings for which the Harary-Hill Conjecture is proved.

Introduction

We consider *topological drawings* of the complete graph K_n in the plane. In these drawings vertices are drawn as points in the plane and edges as simple planar curves that contain no vertices other than their endpoints. As usual, we require that all intersections are proper crossings (no tangencies) and that two edges share only a finite number of points. A drawing is called *simple* if edges do not self-intersect and if each pair of edges shares at most one point. The number $\mathrm{cr}(D)$ of crossings in a drawing D is the sum of the number of intersection points of all unordered pairs of interiors of edges. The *crossing number* $\mathrm{cr}(G)$ is the minimum $\mathrm{cr}(D)$ over all drawings D of G. A drawing is *crossing optimal (or minimal)* if $\mathrm{cr}(D) = \mathrm{cr}(G)$.

A long-standing conjecture is that the crossing number $\mathrm{cr}(K_n)$ of the complete graph K_n is equal to

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

A very fine history of this and related problems is given by Beineke and Wilson [8]. They attribute the conjecture to Anthony Hill. As it is first published by Harary and Hill in [9], we propose the notation $H(n)$ used above to denote the conjectured value of $\mathrm{cr}(K_n)$ and attribute the conjecture to Harary-Hill.

[a]Research for this article was initiated during the *AIM Workshop on Exact Crossing Numbers* held from April 28 to May 2, 2014 in Palo Alto (California, USA). We are grateful to the American Institute of Mathematics (AIM) for their support and we thank the participants of that workshop for fruitful and inspiring discussions.

[b]Email: bernardo.abrego@csun.edu.

[c]Email: oaich@ist.tugraz.at. Research supported by ESF EUROCORES programme EuroGIGA – CRP ComPoSe, Austrian Science Fund (FWF): I648-N18

[d]Email: silvia.fernandez@csun.edu. Research supported by NSF grant DMS-1400653

[e]Email: dmcquill@norwich.edu.

[f]Email: mohar@sfu.ca.

[g]Email: petra.mutzel@tu-dortmund.de. Research supported by the ESF EUROCORES programme EuroGIGA – CRP GraDR, 10-EuroGIGA-OP-003 (DFG)

[h]Email: pedro.ramos@uah.es. Research supported by MINECO project MTM2014-54207 and ESF EUROCORES programme EuroGIGA – CRP ComPoSe, MICINN Project EUI-EURC-2011-4306

[i]Email: brichter@math.uwaterloo.ca. Research supported by NSERC

[j]Email: bvogt@ist.tugraz.at. Research supported by ESF EUROCORES programme EuroGIGA – CRP ComPoSe, Austrian Science Fund (FWF): I648-N18

Recently, an important line of research has been started by Ábrego et al. [2], who restricted the allowed drawing styles and proved that for these drawings the conjecture is true. In [2], they consider 2-page book drawings of K_n and later in [3, 7] the technique was extended to *monotone* drawings of K_n where all the vertices have different x-coordinates and the edges are x-monotone curves. In [4], Ábrego et al. generalize those previous results giving the first general combinatorial condition on a drawing D of K_n that guarantees that D has at least $H(n)$ crossings. In their paper, which also prompted this work, they introduced the notion of *shellability* of a drawing of K_n.

In this work we define a more general version of shellability that we call bishellability, and which is implied by shellability. The main benefit of our approach is the simplification of the principal concept. This allows a significantly simpler and more intuitive proof for the fact that bishellable drawings satisfy the Harary-Hill Conjecture. Moreover, bishellability reflects better the required properties. We are convinced that this is a further step to gain more insight into the structure of crossing minimal drawings, with the ultimate goal to prove the Harary-Hill Conjecture.

1 k-edges and crossings

The relation between the number of crossings in a rectilinear (or pseudo-linear) drawing of K_n and the number of its k-edges was first described by Lovász et al. [10] and, independently, by Ábrego and Fernández [6]. Ábrego et al. [2] generalized the notion of k-edges to topological (simple) drawings, as follows. Fix a drawing D of K_n and let uv be a directed edge of D. Let w be a vertex of D other than u or v. We denote by uvw the oriented, closed curve defined by concatenating the (oriented) edges uv,vw and wu. Note that uvw is a simple closed curve that can be oriented in the standard way. We say that w is on the left (respectively, right) side of uv if uvw is oriented counterclockwise (respectively, clockwise). Finally, we say that the edge uv is a k-edge of D if it has exactly k points of D on one side (left or right).

The relation between k-edges and crossings turns out to be useful to give a lower bound for the number of crossings of rectilinear drawings when $(\leq k)$-edges are considered. Specifically, if we denote by $E_k(D)$ the number of k-edges of D and define

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_j(D)$$

then a lower bound for $E_{\leq k}(D)$ traslates inmediately into a lower bound for $\operatorname{cr}(D)$ [6, 10].

In [2] it is shown that the lower bound $E_{\leq k} \geq 3\binom{k+2}{2}$ (that is true for rectilinear drawings and would imply $\operatorname{cr}(D) \geq H(n)$), is not true for topological drawings. A further step is done considering

$$E_{\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{j=0}^{k}\sum_{i=0}^{j} E_i(D)$$

and showing that

$$\begin{aligned}\operatorname{cr}(D) = 2\sum_{k=0}^{\lfloor n/2\rfloor-2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor \\ - \frac{1}{2}(1+(-1)^n)E_{\leq\leq\lfloor n/2\rfloor-2}(D).\end{aligned} \tag{1}$$

Using this equality, a straightforward calculation translates the lower bound $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$ into $\operatorname{cr}(D) \geq H(n)$.

2 Bishellable drawings

The lower bound $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$ was shown in [2] for 2-*page* drawings and then, in [3, 7] the technique was extended to *monotone* drawings.

Shellable drawings are introduced in [4] as the first combinatorial condition on a drawing that guarantees at least $H(n)$ crossings. For the definition, we recall that, for a drawing D of K_n, a *face* of D is a component of $\mathbb{R}^2 \setminus D[K_n]$. Finally, if V is a subset of vertices in the drawing, $D - V$ denotes the drawing obtained when vertices of V and all edges incident to them are deleted from D.

Definition 1 ([4]) *For a positive integer s, a planar drawing D of K_n is* s-shellable *if there is a sequence $v_1, v_2, \ldots, v_s$ of distinct vertices of D so that, relative to a reference face F, for all integers r, t with $1 \leq r < t \leq s$, the vertices v_r and v_t are both incident with the face of $D - \{v_1, \ldots, v_{r-1}, v_{t+1}, \ldots, v_s\}$ containing F.*

One of the disadvantages of the notion of shellability is that s-shellable does not imply $(s-1)$-shellable. This is because the sequence $v_1, v_2, \ldots, v_s$ of vertices needs to be circular to get from the reference face F and back to F again, and a long circular sequence does not imply a shorter circular sequence.

We introduce a more general variant of shellability that we call *bishellability.*

Definition 2 *For a non-negative integer s, a drawing D of K_n is* s-bishellable *if there exist sequences $a_0, a_1, \ldots, a_s$ and $b_s, b_{s-1}, \ldots, b_1, b_0$, each sequence consisting of distinct vertices of K_n, so that, with respect to a reference face F:*

1. *for each $i = 0, 1, 2, \ldots, s$, the vertex a_i is incident with the face of $D - \{a_0, a_1, \ldots, a_{i-1}\}$ that contains F;*

2. *for each $i = 0, 1, 2, \ldots, s$, the vertex b_i is incident with the face of $D - \{b_0, b_1, \ldots, b_{i-1}\}$ that contains F; and*

3. *for each $i = 0, 1, \ldots, s$, $\{a_0, a_1, \ldots a_i\} \cap \{b_{s-i}, b_{s-i-1}, \ldots, b_0\} = \emptyset$.*

We remark that if $a_0, a_1, \ldots, a_s$ and $b_s, b_{s-1}, \ldots, b_0$ show that D is s-bishellable, then the same sequences without a_s and b_s show that D is $(s-1)$-bishellable. Moreover, the vertices a_0 and b_0 must lie on the boundary of the common face F.

Also, if D is s-shellable, with witnessing sequence $v_1, v_2, \ldots, v_s$, then D is (s–2)-bishellable with witnessing sequences $a_0, a_1, \ldots, a_{s-2}$ and $b_{s-2}, b_{s-3}, \ldots, b_0$ defined by $a_i = v_{i+1}$ and $b_i = v_{s-i}$.

With this new definition, the key lemma from [4] is replaced by the following version (the full proof can be found in [1]):

Lemma 3 *If a drawing D of K_n is k-bishellable and $0 \le k \le \lfloor \frac{n}{2} \rfloor - 2$, then*

$$E_{\le\le k}(D) \ge 3\binom{k+3}{3}.$$

Proof. (Sketch) Bishellability allows us to simplify the approach in [4]. We proceed by induction on k. The base case of $k = 0$ is trivial, as the face F is incident with at least three edges and each of these is a 0-edge. Thus,

$$\sum_{i=0}^{0}(0+1-i)E_0(D) = E_0(D) \ge 3 = 3\binom{0+3}{3},$$

as required.

For the induction step, let $a_0, a_1, \ldots, a_k$ and $b_k, b_{k-1}, \ldots, b_0$ be sequences witnessing k-bishellability. Consider the drawing $D - a_0$. Then $a_1, \ldots, a_k$, $b_{k-1}, \ldots, b_0$ show it is $(k-1)$-bishellable and, since $k - 1 \le (\lfloor \frac{n}{2} \rfloor - 2) - 1 \le \lfloor \frac{n-1}{2} \rfloor - 2$, the induction implies that

$$\sum_{i=0}^{k-1}((k-1)+1-i)E_i(D-a_0) \ge 3\binom{(k-1)+3}{3}.$$

Rewritten, this is

$$\sum_{i=0}^{k-1}(k-i)E_i(D-a_0) \ge 3\binom{k+2}{3}.$$

Consider an edge e in $D - a_0$. If e is an i-edge with $i \le \lfloor \frac{n-1}{2} \rfloor - 2$, then it is either an i-edge or an $(i+1)$-edge of D, depending on whether a_0 joins the majority or minority part of the R's and L's with respect to e in $D - a_0$. We call those that are i-edges in both $D - a_0$ and D *invariant.*

It is not hard to see that that the coefficient of a non-invariant i-edge in the sum for $D - a_0$ is the same as in the D-sum, while an invariant i-edge contributes $k - i$ to the $(D - a_0)$-sum and $k + 1 - i$ to the D-sum. Therefore, we have

$$\sum_{i=0}^{k}(k+1-i)E_i(D) \ge \sum_{i=0}^{k-1}(k-i)E_i(D-a_0) + I_D + S_0,$$

where I_D is the number of invariant edges and S_0 the number of edges incident with a_0.

In order to finish the proof we need to show that there are at least $\binom{k+2}{2}$ invariant edges and the contribution of the edges incident with a_0 is at least $2\binom{k+2}{2}$. □

From Lemma 3 and Equation (1) we get:

Theorem 4 *If D is an $(\lfloor \frac{n}{2} \rfloor - 2)$-bishellable drawing of K_n, then $\mathrm{cr}(D) \ge H(n)$.*

There are two main remarks to be made here. First, in the previous version we had to ask for shellability for some $s \ge \lfloor \frac{n}{2} \rfloor$ while with bishellability only a fixed value is required. Second, even though the same principal ideas are used in the two proofs, the proof of Lemma 3 involves a significantly simpler induction than its version in [4]. Both advantages are due to the monotonicity of the new concept.

3 Shellable and bishellable drawings

The main point of this work is to simplify the notion of shellability and to simplify the proof that shellable drawings of K_n have at least $H(n)$ crossings. However, there is also some interest in understanding the distinctions between shellable, bishellable, and general drawings. To simplify the discussion, we define a drawing D of K_n to be *shellable* if it is s-shellable for some $s \ge \lfloor \frac{n}{2} \rfloor$, and *bishellable* if it is $(\lfloor \frac{n}{2} \rfloor - 2)$-bishellable. That is, shellable and bishellable drawings have at least $H(n)$ crossings. Furthermore, we call a drawing of K_n *Harary-Hill optimal* if it has $H(n)$ crossings. We use this notation to keep in mind that drawings with $H(n)$ crossings are only conjectured to be optimal.

Two Harary-Hill optimal drawings of K_n with $n \ge 11$ odd are given in [4]. One especially relevant to us has every edge crossed at least once. Figure 1 gives the drawing for K_{11}. In particular, no face of this drawing is incident with two vertices, so this cannot be s-bishellable for any $s \ge 0$. Thus, there are Harary-Hill optimal drawings that are neither shellable nor bishellable.

On the other hand, a tin can drawing of K_n has a cycle of length $\lceil n/2 \rceil$ having no edges crossed. Such a cycle shows that the drawing is $\lceil n/2 \rceil$-shellable. Thus, it is shellable and so also bishellable. In particular, it has at least $H(n)$ crossings.

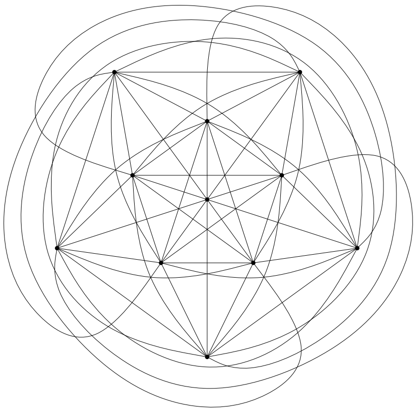

Figure 1: Non-bishellable, Harary-Hill optimal drawing of K_{11} where all edges are crossed [5].

To distinguish between shellable and bishellable drawings is more subtle. Consider the gadget in Figure 2. A sequence of vertices of length at least $\lfloor n/2 \rfloor$ witnessing shellability would have to have vertices on both sides of both the blue and green simple closed curves through v. In particular, v must be an internal vertex of the sequence. However, if both regions A and B have enough vertices, each of A and B has an end of the sequence. Since the ends of the sequence must be incident with the same face of the drawing, this is evidently impossible.

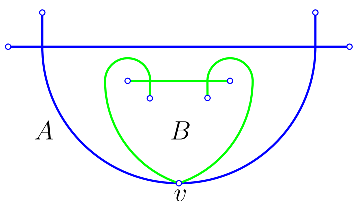

Figure 2: Gadget with two walls and one gate v blocking a shelling cycle.

For the drawing in Figure 3, the general values $n_1 = \lfloor (n-11)/3 \rfloor$, $n_2 = \lfloor (n-27)/3 \rfloor$, and $n_3 = \lfloor (n-13)/3 \rfloor$ provide an instance for each $n \geq 27$ that is bishellable but not shellable. There are many other particular instances that are also non-shellable. On the other hand, the drawing in Figure 3 is $(\lfloor n/2 \rfloor - 2)$-bishellable: take one sequence starting in the middle of the central set and go to the left, while the other starts in the middle of the central set and goes to the right.

4 Open problems

We close with two open questions:

- Is there a concept similar to bishellability that does not require the starting vertices of the sequences to share a cell, but still implies some bound for E_k or some related quantity?
- Can you find a combinatorial characterization of Harary-Hill optimal drawings?

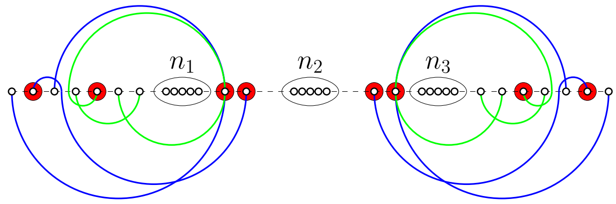

Figure 3: Base for non s-shellable with $s \geq \lfloor n/2 \rfloor$ but $(\lfloor n/2 \rfloor - 2)$-bishellable drawings.

References

[1] Bernardo M. brego, Oswin Aichholzer, Silvia Fernndez-Merchant, Dan McQuillan, Bojan Mohar, Petra Mutzel, Pedro Ramos, R. Bruce Richter, and Birgit Vogtenhuber. Bishellable drawings of k_n, 2015.

[2] Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. The 2-page crossing number of K_n. *Discrete & Computational Geometry*, 49(4):747–777, 2013.

[3] Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. More on the crossing number of K_n: Monotone drawings. *Electronic Notes in Discrete Mathematics*, 44:411–414, 2013.

[4] Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Gelasio Salazar. Shellable drawings and the cylindrical crossing number of K_n. *Discrete & Computational Geometry*, 52(4):743–753, 2014.

[5] Bernardo M. Ábrego, Oswin Aichholzer, Silvia Fernández-Merchant, Pedro Ramos, and Birgit Vogtenhuber. Non-shellable drawings of K_n with few crossings. In *CCCG 2014, Halifax, Nova Scotia, Canada*, 2014.

[6] Bernardo M. Ábrego and Silvia Fernández-Merchant. A lower bound for the rectilinear crossing number. *Graphs and Combinatorics*, 21(3):293–300, 2005.

[7] Martin Balko, Radoslav Fulek, and Jan Kynčl. Crossing numbers and combinatorial characterization of monotone drawings of K_n. *Discrete & Computational Geometry*, 53(1):107-143, 2015.

[8] Lowell Beineke and Robin Wilson. The early history of the brick factory problem. *Math. Intelligencer*, 32(2):41–48, 2010.

[9] Frank Harary and Anthony Hill. On the number of crossings in a complete graph. In *Proc. Edinburgh Math. Soc.*, volume 13, pages 333–338, 1963.

[10] L. Lovász, K. Vesztergombi, U. Wagner, and E. Welzl. Convex quadrilaterals and k-sets. Towards a theory of geometric graphs. In J. Pach, editor, *AMS Contemp. Math. Series*, volume 342, pages 139–148, 2004.