All Good Drawings of Small Complete Graphs*

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Abstract

Good drawings (also known as *simple topological graphs*) are drawings of graphs such that any two edges intersect at most once. Such drawings have attracted attention as generalizations of geometric graphs, in connection with the crossing number, and as data structures in their own right. We are in particular interested in good drawings of the complete graph. In this extended abstract, we describe our techniques for generating all different weak isomorphism classes of good drawings of the complete graph for up to nine vertices. In addition, all isomorphism classes were enumerated. As an application of the obtained data, we present several existential and extremal properties of these drawings.

1 Introduction

We consider drawings of simple graphs in the plane or, equivalently, on the sphere. Vertices are represented by distinct points. Edges are drawn as Jordan arcs connecting two vertices (of that edge) and not containing any vertex except those at their endpoints. Note that we do not distinguish between the elements of the graph and their representation in the drawing. A *good drawing* is a drawing of a graph such that any two edges intersect at most once, either at a common endpoint or at a proper crossing, and no three edges cross at a common point. Good drawings have been extensively studied, and are also referred to as "topological graphs" (e.g., in [14]), "simple topological graphs" (e.g., in [9]), or simply "drawings" (e.g., in [8]). We are interested in good drawings of the complete graph K_n on n vertices.

One main motivation for considering good drawings comes from the problem of minimizing the number of crossings in drawings of K_n (where crossings are counted by the overall sum of the number of points in which each pair of edges crosses, as opposed to the number of crossing edge pairs; see [15]). Indeed, for any drawing of a graph, there exists a good drawing of the same graph with at most the same number of crossings. The Harary-Hill conjecture states that the number of crossings in any drawing of K_n is at least $H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. This has been verified for $n \leq 12$; see [18]. While it has recently been shown that the Harary-Hill conjecture holds for many classes of drawings of K_n (see [1] and references therein), it still remains open for the general case.

Two drawings are *isomorphic* if there is a homeomorphism of the sphere that transforms one drawing into the other. For good drawings, this partitions the infinite number of drawings into a finite number of isomorphism classes; Kynčl [9] showed that this number is in $2^{\Theta(n^4)}$. With applications like determining the crossing number in mind, the following coarser classification turns out to be useful. Two good drawings are *weakly isomorphic* if there is an incidence-preserving bijection between the drawings such that two edges cross in one drawing if and only if their images in the other drawing cross as well. Roughly speaking, weakly isomorphic drawings that are non-isomorphic differ in the order in which their edges intersect; see [4] for details. The number of weak isomorphism classes of K_n is in $2^{n^2\alpha(n)^{O(1)}}$ [11] and $2^{\Omega(n^2)}$ [16].

Already in 1988, Rafla [20] enumerated all weak isomorphism classes of good drawings of K_n for $n \leq 7$ by a computer program, under the (still unproven) assumption that every good drawing contains a simple (i.e., crossing-free) Hamiltonian cycle. Gronau and Harborth [5] enumerated all non-isomorphic good drawings for $n=6$. Here, we describe our construction of all weak isomorphism classes and the enumeration of all isomorphism classes of good drawings of K_n for $n \leq 9$. The resulting data has been used to obtain exact values for various extremal and existential problems on good drawings of K_n, both for $n \leq 9$ and, via extension of relevant instances, for more vertices. Similar data has been successfully used for combina-

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torially different configurations of points [2], to obtain counterexamples, induction bases, or, in general, a better intuition for various problems.

In contrast to, e.g., [20], our generation of all weak isomorphism classes is based on rotation systems. In Section 2, we give the basic theoretical background on rotation systems and sketch techniques that reduced the required computational effort. In Section 3, we describe the enumeration of all non-isomorphic drawings of each weak isomorphism class. Applications and the outcome of several computations on the data are given in Section 4. Parts of this work have been presented in the master's thesis [17] of Pammer.

2 Rotation Systems

Rotation systems were devised as tools for investigating embeddings of graphs on higher-genus surfaces [6]. Let D be an (arbitrary) drawing of a graph $G(V, E)$. The *rotation* $\rho_D(v)$ (or $\rho(v)$ when D is clear from the context) of a vertex v in D is the clockwise cyclic order of edges incident to v, given as a sequence (that is to be interpreted circularly) of the second vertices of all edges at v. (Note that if $G = K_n$ then $\rho(v)$ is a cyclic permutation of $V \setminus \{v\}$). The *rotation system* (abbrev. RS) of D is the set of rotations of all vertices of D and is denoted by $\mathcal{R}(D)$. We consider two rotation systems to be equivalent if one can be obtained from the other by relabeling and optional inversion of all rotations. Further, we call a rotation system *realizable* if it is the rotation system of a good drawing of a complete graph. The following two results imply that for complete graphs, the rotation system uniquely determines the weak isomorphism class of a good drawing (see also [9]), a property that is central to our work.

Theorem 1 (Pach, Tóth [16]) *The rotation system of a good drawing of the complete graph determines the pairs of crossing edges.*

Theorem 2 (Gioan [4]) *The set of crossing pairs of edges determines the equivalence class of the rotation system of a good drawing of the complete graph.*

Note that this is, in general, only true for complete graphs: Determining the crossing number of a (general) graph with a predefined rotation system is NP-complete [19]. A result similar to the above ones is also known for isomorphism classes:

Theorem 3 (Kynčl [9]) *Two good drawings are isomorphic iff there exists a bijection between their vertices such that (i) they are weakly isomorphic, (ii) for each edge, the order of crossings along its image is the same, and (iii) for each crossing the radial order of the edge parts emanating to the four involved vertices is the same (or inverted for all crossings).*

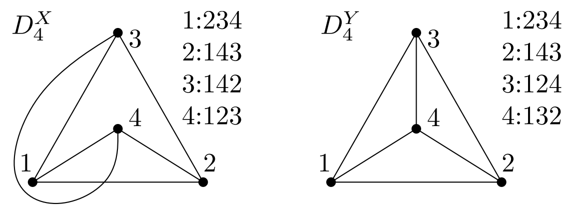

Figure 1: The two different drawings of K_4, with their rotation systems.

K_4 has only two (weak) isomorphism classes, see Figure 1. We denote them by D_4^X and D_4^Y. The basic observation leading to Theorem 1 is that the sub-drawing induced by four points has a rotation system equivalent to $\mathcal{R}(D_4^X)$ if the four points are involved in a crossing, and a rotation system equivalent to $\mathcal{R}(D_4^Y)$ otherwise. (Therefore, Property (iii) in Theorem 3 is also determined by the rotation system for drawings of the complete graph.) The other direction (Theorem 2) is slightly more involved and requires considering also 5-tuples. Unless stated otherwise, we will consider only good drawings of complete graphs (and their rotation systems). We have:

Observation 1 *When given the rotation around three vertices in a drawing of K_4, the relative position of these three vertices in the rotation around a fourth vertex v is determined.*

We generate all rotation systems of size n by *extending* the ones of size $n-1$ in the following way. In the sequence representing the rotation around every vertex, we place the new vertex v_n in all possible ways. Each choice also determines parts of $\rho(v_n)$ by Observation 1. The relative order of two vertices might be different when considering different 4-tuples (which indicates that the choice is invalid) and therefore all 4-tuples containing v_n have to be checked. Hence, we obtain a set of rotation systems where each rotation system restricted to any four vertices is either the one of D_4^X or D_4^Y. We call such a rotation system *consistent*. Still, there exist non-realizable consistent rotation systems. For K_5, there are five (weak) isomorphism classes, and two non-realizable consistent rotation systems. For $n \geq 6$, there are more isomorphism classes than weak isomorphism classes. We describe our approach for checking realizability, which is also used for enumerating all isomorphism classes, in the next section.

To ensure that no two equivalent rotation systems are stored, we guess a vertex that is given the label 1. Then we guess a second vertex to label all vertices from 2 to n, either counterclockwise or clockwise around the first one. This way, we obtain $2n(n-1)$ different labelings. Each labeling gives a matrix consisting of the n rotations. We use the lexicographically smallest one for storing the rotation system. Hence, duplicates can be filtered easily.

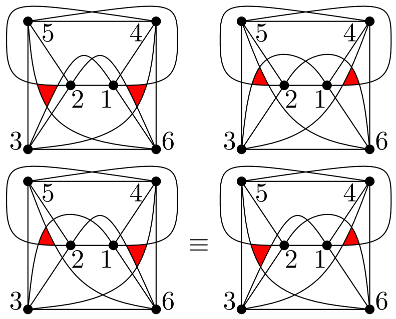

Figure 2: Four drawings of the same rotation system. The two at the bottom are also isomorphic (consider the labeling horizontally mirrored), the others are not.

3 Realizability and Enumeration

It remains to decide realizability of a rotation system and, in case of a positive decision, to count the number of its isomorphism classes. Deciding whether a rotation system of K_n can be realized as a good drawing can be done in polynomial time [10]. Since we also want to enumerate all non-isomorphic drawings of each rotation system, we use a less sophisticated approach that, using properties of the rotation system, works fast for small instances. The basic idea is to use a backtracking algorithm to incrementally build a good drawing, which is represented as a doubly-connected edge list. This algorithm can be used for both checking realizability and for obtaining all realizations of a rotation system.

Similar to recognizing equivalent rotation systems, we use a lexicographically smallest labeling to check isomorphism. However, finding a "fingerprint" for the isomorphism class is more complicated. Consider Theorem 3 (i) and (ii). The first part of the fingerprint is the lexicographically smallest rotation system. The labeling of the vertices defines an order on them, which, in turn, gives a lexicographic order on the edges. For each edge e, from smallest to largest, we list the indices of the edges that cross e, in the order when going from the smaller to the larger vertex. However, there are, in general, several lexicographically smallest labelings for a rotation system, which could give different sequences for the edge crossings. Hence, for a given good drawing, we have to check all such labelings of the rotation system to obtain the lexicographically smallest sequence of edge crossings. An example is given in Figure 2, showing four drawings of one rotation system, of which two are isomorphic.

4 Applications

The numbers of (weak) isomorphism classes are given in Table 1.

n	realizable RS	non-iso. drawings	non-iso. drawings per RS
3	1	1	$1 \dots 1$
4	2	2	$1 \dots 1$
5	5	5	$1 \dots 1$
6	102	121	$1 \dots 3$
7	11 556	46 999	$1 \dots 57$
8	5 370 725	502 090 394	$1 \dots 46\ 571$
9	7 198 391 729		$1 \dots > 2.3 \times 10^{10}$

Table 1: The numbers of weak isomorphism classes and non-isomorphic good drawings of K_n, in total and per RS.

4.1 Simple Hamiltonian Cycles

For $n = 7$, Rafla's numbers [20] match ours, confirming the conjecture that every good drawing has a simple (i.e., crossing-free) Hamiltonian cycle for $n \leq 7$. In addition, we verified the conjecture for $n \leq 9$.

4.2 Maximum Number of Crossings

As every drawing of K_4 has at most one crossing, there are at most $\binom{n}{4}$ crossings in a good drawing. But in contrast to complete geometric graphs (where only the set with all points in convex position attains this bound), there exist many weak isomorphism classes with this maximum number of crossings. We call them *max-crossing* drawings. Harborth and Mengersen [8] already considered max-crossing drawings, enumerating all 15 non-isomorphic ones for K_6. Kynčl [9] gives a lower bound of $2^{n-5}\frac{(n-3)!}{n}$ for the number of max-crossing realizable RS, but no upper bounds better than that for all realizable RS are known. Table 2 gives the numbers obtained from our data. Observe that all max-crossing realizable RS can be obtained by extending only max-crossing realizable RS. Therefore, we can go beyond the $n = 9$ barrier by extending only such systems. Note the slight difference to the question in [11, Problem 2], asking for the number of max-crossing consistent RS.

It is known that every good drawing of K_n contains a max-crossing sub-drawing of size $\Omega(\log^{1/8} n)$ [14] (in fact, the bound is given for two particular max-crossing graphs). Table 2 also lists the number of realizable RS with no max-crossing 5-tuple, showing that no such RS of size larger than 12 exists.

4.3 Crossing Number of K_{13}

The crossing number of K_{13} is known to be between 219 [12] and 225 [18]. For odd n, the crossing number has the same parity as $H(n)$ [13]. For a drawing D of K_n with $\mathrm{cr}(D)$ crossings there exists a vertex v s.t. $\mathrm{cr}(D \setminus v) \leq \left\lfloor \frac{n-4}{n}\mathrm{cr}(D) \right\rfloor$ [18]. This allows us to obtain the exact value for $\mathrm{cr}(K_{13})$ by only extending rotation systems with few crossings. By this we were able

n	max-crossing realiz. RS	max-crossing drawings	realiz. RS without 5-crossing 5-tuple
4	1	1	2
5	2	2	3
6	10	15	33
7	115	1 477	606
8	2 657	8 373 474	19 195
9	82 957		449 188
10	3 226 173		4 208 379
11			4 162 266
12			32 290
13			0

Table 2: The number of realizable rotation systems with the maximum number of crossings and the number of sets with no 5-tuple with 5 crossings.

to show that $\mathrm{cr}(K_{13}) \in \{223, 225\}$. For obtaining all rotation systems where K_{13} has at most 223 crossings (if they exist), it is sufficient to extend all rotation systems of K_9 with at most 38 crossings, with intermediate RS for $n = 10, 11, 12$ of at most 64, 102, and 154 crossings. The computations are ongoing.

4.4 Empty Triangles

In a good drawing D, a 3-cycle spanned by three edges of D is called an *empty triangle* if the interior of one of its sides does not contain any vertices of D. Let $\Delta(n)$ be the minimum number of empty triangles over all good drawings of K_n. Harborth [7] showed $2 \leq \Delta(n) \leq 2n - 4$, asking whether this upper bound is tight. The currently best known lower bound of n is given in [3], where also tightness of the upper bound is stated for $n \leq 8$. Using our data, we could extend the positive answer to Harborth's question for $n = 9$.

5 Conclusion

We described the generation of all weakly isomorphic good drawings of K_n for $n \leq 9$. The obtained data allowed us to investigate several open existential and extremal problems for such drawings. We expect the data to be helpful for settling further questions in this area, like the crossing number of K_{13} or the question of which RS maximize the number of non-isomorphic drawings.

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