Perfect k-colored matchings and $k{+}2$-gonal tilings

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Abstract

We derive a simple bijection between geometric plane perfect matchings on $2n$ points in convex position and triangulations on $n+2$ points in convex position. We then extend this bijection to monochromatic plane perfect matchings on periodically k-colored vertices and $(k+2)$-gonal tilings of convex point sets. These structures are related to Temperley-Lieb algebras and our bijections provide explicit one-to-one relations between matchings and tilings. Moreover, for a given element of one class, the corresponding element of the other class can be computed in linear time.

1 Introduction

The Fuss-Catalan numbers $f(k,m) = \frac{1}{m}\binom{km+m}{m-1}$ are known to count the number of $k{+}2$-gonal tilings of a convex polygon of size $km+2$, they go back to Fuss-Euler (cf. [4]). Bisch and Jones introduced k-colored Temperley-Lieb algebras in [1] as a natural generalisation of Temperley-Lieb algebras. These algebras have representations by certain planar k-colored diagrams with $m(k{+}1)$ vertices on top and bottom. The dimension of such an algebra is $f(k,m)$, with a basis indexed by these diagrams. We call these diagrams plane perfect k-colored matchings or just k-colored matchings, assuming from now on that they are plane and perfect. Since the number of $k{+}2$-gonal tilings coincides with the number of k-colored matchings, these sets are in bijection. Przytycki and Sikora [4] prove this through an inductive implicit construction but do not give an explicit bijection of the structures.

Furthermore, from work of Marsh and Martin [3], one can derive an implicit correspondence between triangulations and diagrams for $k=1$. However, to our knowledge, no explicit bijection is known.

In this paper, we will give bijections between these two sets of plane graphs on sets of points in convex position. We will first address the case $k=1$ (Section 2) and then treat the general case. Our main theorems are the explicit bijections between the set of k-colored matchings and the $(k+2)$-gonal tilings (Theorems 1 and 8). A key ingredient is the characterization of valid k-colored matchings in Theorem 3. Due to lack of space, most proofs are deferred to the full version of this paper.

2 Matchings and triangulations

We will draw the matchings with two parallel rows of n vertices each, labeled v_1 to v_n and v_{n+1} to v_{2n} in clockwise order, and with non-straight edges; see Figure 1(left). We will draw the triangulations (and tilings) on $n{+}2$ points in convex position, labeled p_1 to p_{n+2} in clockwise order; see Figure 1(right). For the sake of distinguishability, throughout this paper we will refer to $p_1,\dots,p_{n+2}$ as *points* and to $v_1,\dots,v_{2n}$ as *vertices*.

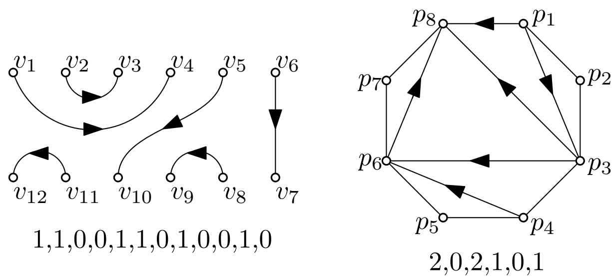

Figure 1: A perfect matching (left) and the corresponding triangulation for $n=6$ (right).

The above defined structures are undirected graphs. We next give an implicit direction to the edges of these graphs: an edge v_iv_j (p_ip_j) is directed from v_i to v_j (p_i to p_j) for $i<j$, that is, each edge is directed from the vertex / point with lower index to the vertex / point with higher index. This also defines the outdegree of every vertex / point, which we denote as b_i for each vertex v_i and as d_i for each point p_i. For technical reasons, we do not count the edges of the convex hull of a triangulation when computing the outdegree of a point p_i, with the exception of the edge p_1p_{n+2}. We call the sequence $(b_1,\dots,b_{2n})$ of the outdegrees of a matching (or the sequence $(d_1,\dots,d_n)$ of the first n outdegrees of a triangulation) its *outdegree sequence*; see again Figure 1. We first show that for both structures, this sequence is sufficient to encode the graph.

For matchings, the outdegree sequence is a 0/1-sequence with $2n$ digits, where n digits are 1 and n digits are 0. Moreover, the directions of the edges imply that an incoming edge at a vertex v_j must be outgoing for a vertex v_i with $i<j$. Thus, we have the condition $\sum_{i=1}^{k} b_i \geq k/2$ for any $1 \leq k \leq 2n$, that is, in any subsequence starting at v_1, we have

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at least as many 1s as 0s. Such sequences are called ballot sequences, see [2, p.69]. Obviously, the outdegree sequence of a matching can be computed from a given matching in $O(n)$ time. But also the reverse is true: We consider the outdegrees from b_1 to b_{2n}. We use a stack (with the usual push and pop operations) to store the indices of considered vertices that still need to be processed. Initially, the stack is empty. If $b_i = 1$, we push the index i on the stack. If $b_i = 0$, we pop the topmost index k from the stack and output the edge $v_k v_i$. In this way, always the last vertex with 'open' outgoing edge is connected to the next vertex with incoming edge, implying that the subgraph with vertices v_k to v_i is a valid plane perfect matching. A simple induction argument shows that the whole resulting graph is plane and can be reconstructed from the outdegree sequence in $O(n)$ time.

For triangulations, first note that the outdegrees of p_{n+1} and p_{n+2} are 0. Thus we do not lose information when restricting the outdegree sequence of a triangulation to $(d_1, \ldots, d_n)$. Similar as before, the directions of edges imply that for any valid outdegree sequence, it holds that $\sum_{i=1}^{k} d_{n+1-i} \leq \sum_{i=1}^{k} 1 = k$ for any $1 \leq k \leq n$. This sum is precisely the maximum number of edges which can be outgoing from the 'last' k points p_{n+1-k} to p_n. Recall that we do not consider the edges of the convex hull, except for $p_1 p_{n+2}$, and thus the number of edges which contribute to the outdegree sequence is exactly $n-2$. As before, it is straightforward to compute the outdegree sequence from a given triangulation in $O(n)$ time. For the reverse process, we again use a stack to store the indices of considered points that still need to be processed. We initialize the stack with push$(n+2)$ and push$(n+1)$ and output all the (non-counted) edges $p_i p_{i+1}$ for $1 \leq i \leq n+1$. Then we consider the outdegrees in reversed order, that is, from d_n to d_1. For each degree d_i we perform two steps. (1) d_i times, we pop the topmost index from the stack and after each pop we output the edge $p_i p_k$, where k is the (new) topmost index on the stack. (2) We push i on the stack. This process constructs the triangulation from back to front. When processing p_i, all points from p_{i+1} to p_{n+2} that are still 'visible' from p_i are in this order on the stack. Thus, drawing the edges in the described way generates a planar triangulation. At the end of the process, the stack contains exactly the two indices $n+2$ and 1, which can be ignored.

So far we have shown that there exist one-to-one relations between outdegree sequences on the one side and matchings respectively triangulations on the other side. We now present a bijective transform between outdegree sequences of matchings and those of triangulations.

For a given outdegree sequence $B = (b_1, \ldots, b_{2n})$ of a perfect matching, we compute the outdegree d_i for the corresponding point of the triangulation as the number of 1s between the $(i-1)$-st 0 and the i-th 0 in B for $i > 1$, and set d_1 to the number of 1s before the first 0 in B.

For the reverse transformation, we process the outdegree sequence $(d_1, \ldots, d_n)$ of a triangulation from d_1 to d_n and set the entries of B in order from b_1 to b_n in the following way: For each entry d_i we first set the next d_i consecutive elements (possibly none) of B to 1; then we set the next element of B to 0.

It is an easy excercise to see that the two transformations are inverse to each other, and that they form a bijection between valid outdegree sequences of triangulations and outdegree sequences of matchings. Moreover, each transformation can be performed in $O(n)$ time. Figure 2 shows all corresponding perfect matchings, triangulations, and outdegree sequences for $n = 3$.

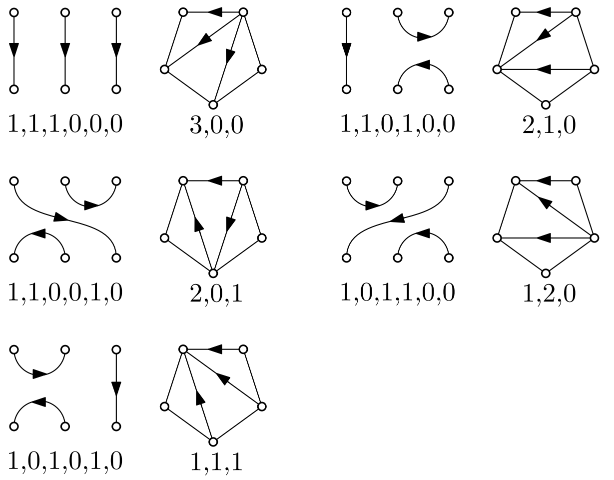


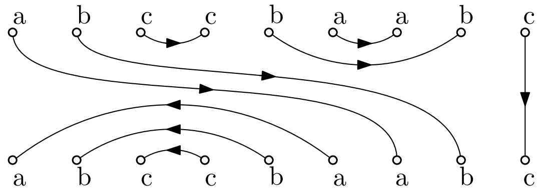
Figure 2: All perfect matchings, triangulations, and outdegree sequences for $n = 3$.

Theorem 1 *There exists a bijection between geometric plane perfect matchings on $2n$ points in convex position and geometric triangulations on $n+2$ points in convex position. Further, for an element of one structure, the corresponding element of the other structure can be computed in linear time.*

3 k-colored matchings

In this section we add colors to the vertices of the perfect matchings and require the matching edges to be monochromatic. For $k \geq 2$, let $c_1, \ldots, c_k$ be the k colors. We color the vertices in a bitonic way, that is, in the order $c_1, c_2, \ldots, c_{k-1}, c_k, c_k, c_{k-1}, \ldots, c_2, c_1, c_1, c_2, \ldots$ and so on. In a *perfect k-colored matching*, all matching edges connect vertices of the same color, and hence n is a multiple of k; see Figure 3 for an example of a k-colored matching with $k = 3$ colors and $n = 9$.

Clearly, any k-colored matching fulfills all conditions of the previous section. But not every match-



$1, 1, 1|0, 1, 1|0, 0, 1|0, 0, 0|1, 1, 1|0, 0, 0$

Figure 3: Perfect k-colored matching for $k = 3$ colors and $n = 9$ and its outdegree sequence.

ing obtained in the previous section is a k-colored matching and hence not every outdegree sequence of a matching is an outdegree sequence of a valid k-colored matching. Thus we now derive additional properties to determine which outdegree sequences of matchings correspond to k-colored matchings.

We denote k consecutive vertices $v_i, \dots, v_{i+k-1}$ that are colored with either $c_1, \dots, c_k$ or $c_k, \dots, c_1$ as a *block*. In total we have $2n/k$ such blocks and they form a partition of $2n$ vertices. Observe that within a block, there cannot be a vertex with an incoming edge after a vertex with an outgoing edge, as this would cause a bichromatic edge. Hence, in a k-colored matching, the outdegree sequence of any block has to be of the form $|0, \dots, 0, 1, \dots, 1|$ (where it can consist entirely of 0 or 1 entries). For better readability, we sometimes mark block boundaries in an outdegree sequence with vertical lines. We say that an outdegree sequence (and the matching) fulfilling this property has a *valid block structure*.

Lemma 2 *Let M be a perfect matching with valid block structure that is not a k-colored matching. Then there exists an edge v_sv_e in M with the following properties:*

(i) The vertices v_s and v_e lie in different blocks, say $v_s \in S$ and $v_e \in E$.

(ii) The subsequence from v_{s+1} to v_{e-1} contains no bichromatic matching edge.

(iii) The number of blocks between S and E is odd.

(iv) Let v_s be the i-th vertex in S. Then v_e is the $(i + 1)$-st vertex in E.

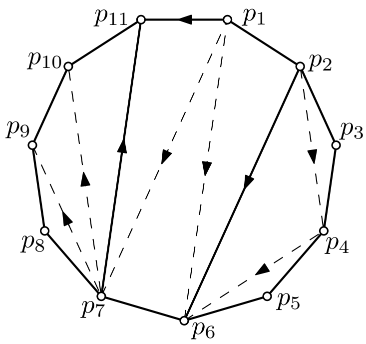
Together with the previous observations, Lemma 2 implies the following theorem.

Theorem 3 *A matching is a k-colored matching if and only if it has a valid block structure and does not contain an edge as described in Lemma 2.*

Remark: For a given outdegree sequence we can check in linear time if it is an outdegree sequence of a k-colored matching by using the reconstruction algorithm described in Section 2.

4 t-gonal tilings

For any $t \geq 3$, a *t-gonal tiling* T on $n + 2$ points in convex position, labeled p_1 to p_{n+2} in clockwise order, is a plane graph where every bounded face is a t-gon and the vertices along the unbounded face are $p_1, p_2, \dots, p_{n+2}$ in this order; see Figure 4 for an example. For the special case of $t = 3$, T is a triangulation. In the next section, we will show that the k-colored matchings on $2n$ vertices of the previous section correspond to $k+2$-gonal tilings of $n + 2$ points in convex position, where $n = km$ for some integer $m > 0$. This is a generalization of the fact that matchings (i.e., $k = 1$) correspond to triangulations. To this end we first derive several properties of t-gonal tilings of convex sets.



3,2,0,1,0,0,3,0,0

Figure 4: 5-gonal tiling corresponding to the 3-colored matching of Figure 3 and the outdegree sequence of its k-color valid triangulation.

The *dual graph* of a t-gonal tiling T has a vertex for each bounded face T and two vertices are connected by an edge if the corresponding faces share a common edge in T (every pair of bounded faces shares at most one edge). An *ear* of T is a t-gon which shares all but one edge with the unbounded face and can thus be cut off of T (along this edge) so that the remaining part is a valid t-gonal tiling of $n+2-(t-2) = n+4-t$ points.

As the dual graph of any t-gonal tiling T is a tree, as every tree has at least two leaves (where the minimal case is obtained by a path), and as a leaf in the dual graph of T corresponds to an ear in T, we have the following observation.

Observation 1 *Every t-gonal tiling with at least $2t-2$ points has at least two ears. At least one of these ears is not incident to the edge p_1p_{n+2}.*

Lemma 4 *Any triangulation $\mathcal{T}$ on $n + 2$ points in convex position contains at most one t-gonal tiling as a subgraph.*

A proof by induction, using Observation 1 can be found in the full version of this paper.

Obviously, if a triangulation $\mathcal{T}$ on $n+2$ points contains a t-gonal tiling T as a subgraph, then n is divisible by $t-2$. Further, as T has at least two ears, $\mathcal{T}$ contains at least two edges that cut off a triangulated t-gon from $\mathcal{T}$. We call such a t-gon that can be split off from a triangulation $\mathcal{T}$ a *t-ear* of $\mathcal{T}$, and the edge along which the t-ear can be split off the *ear-edge* (of the t-ear). Note that for $t > 3$, not every triangulation contains t-ears.

Let $\mathcal{T}$ be a triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Let B be the outdegree sequence of the corresponding matching. If $s < n+2$, then in B, the t-ear corresponds to a subsequence W of B of length $2t-3$ that starts with a 1 (for $p_r p_s$), ends with two 0s (as the last point p_{s-1} of the ear cannot have outgoing edges), and has $t-1$ 0s and $t-2$ 1s in total. If $s = n+2$, then in B, the last 0 (the one 'after' p_{n+1}) is not existing. Then the according sequence is $W = (b_{2n-2t+5}, \ldots, b_{2n})$, which must be a ballot sequence.

5 k-colored matchings and $k+2$-gonal tilings

We say that a triangulation on $n+2$ points in convex position is *k-color valid* if it corresponds to a k-colored matching as defined in Section 3. The outdegree sequence of such a triangulation is then also called k-color valid. A $k+2$-gonal tiling of $n+2$ points is called k-color valid if it can be completed to (i.e., is a subgraph of) a k-color valid triangulation. In the following, let $t = k+2$.

Observation 2 *Let $\mathcal{T}$ be a k-color valid triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Let the first entry of the subsequence W of B that corresponds to this t-ear be the i-th entry within its block, for $1 \leq i \leq k$. If $s = n+2$ then $i = 1$ and $W = (|1, \ldots, 1|0, \ldots, 0|) = (|1^k|0^k|)$. Otherwise, $W = (1, \ldots, 1|0, \ldots, 0, 1, \ldots, 1|0, \ldots, 0) = (1^{k-i+1}|0^{k-i+1}, 1^{i-1}|0^i)$.*

The following three lemmas can be derived using Observation 2. The proof of Lemma 5 also shows that the extension is uniquely determined.

Lemma 5 *Any k-color valid t-gonal tiling T on $n+2$ points can be extended by an ear at any edge $e = p_r p_{r+1}$, $1 \leq r \leq n+1$, so that the resulting t-gonal tiling on $n+k$ points is k-color valid.*

Lemma 6 *Let $\mathcal{T}$ be a k-color valid triangulation that contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$. Then the triangulation $\mathcal{T}'$ that results from removing the t-ear from $\mathcal{T}$ is again k-color valid.*

Lemma 7 *Let $\mathcal{T}$ be a k-color valid triangulation. Then $\mathcal{T}$ contains a t-ear with ear-edge $p_r p_s$ for some $r \geq 1$ and $s = r+t-1 \leq n+2$.*

Combining Lemmas 4 – 7 and Observations 1 – 2, we obtain our main result.

Theorem 8 *There exists a bijection between geometric plane perfect k-colored matchings on $2n$ points in convex position and t-gonal tilings on $n+2$ points in convex position. Further, for an element of one structure, the corresponding element of the other structure can be computed in linear time.*

6 Future Work

The Temperley-Lieb algebras arising from matchings on $2n$ vertices can be generated by n distinguished elements: An element I (consisting of n propagating lines $v_j v_{2n-j+1}$, $1 \leq j \leq n$, from top to bottom) and $n-1$ elements U_i, $1 \leq i < n$, consisting of a pair of lines $v_i v_{i+1}$ and $v_{2n-i} v_{2n-i+1}$ plus the remaining $n-2$ propagating lines.

It is natural to search for a characterization of these generators in terms of triangulations (and for the generators for the k-colored Temperley-Lieb algebras in terms of $k+2$-gonal tilings). We plan to use our explicit bijections to study the effect of edge flips in triangulations respectively in tilings on the corresponding matchings and to find out how the actions of generators of the (k-colored) Temperley-Lieb algebra can be interpreted in terms of flips in triangulations respectively in tilings. Preliminary results have already been obtained.

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