Classifying Hyperplanes in Hypercubes

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Abstract

We consider hyperplanes spanned by vertices of the unit d-cube. We classify these hyperplanes by parallelism to coordinate axes, by symmetry of the d-cube vertices they avoid, as well as by so-called hull-honesty. (Hull-honest hyperplanes are those whose intersection figure with the d-cube coincides with the convex hull of the d-cube vertices they contain; they do not cut d-cube edges properly.) We describe relationships between these classes, and give the exact number of hull-honest hyperplanes, in general dimensions. An experimental enumeration of all spanned hyperplanes up to dimension eight showed us the intrinsic difficulty of developing a general enumeration scheme. Motivation for considering such hyperplanes stems from coding theory, from linear programming, and from the theory of machine learning.

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1 Introduction

Among the simplest high-dimensional geometric objects is the d-dimensional hypercube (d-cube), which is the vector sum of d mutually orthogonal line segments of equal length. The d-cube is a convex polytope that, when having unit edge length, may be expressed as $C^d = [0,1]^d$. Sometimes, C^d is called the measure polytope, suggested by its use as the unit of content. Despite of its simple definition, C^d has been an object of study from various different points of view. The theory of convex polytopes provides classical results concerning sections and projections of hypercubes; see Coxeter [7] and Grünbaum [12]. Purely combinatorial properties of C^d, mainly involving certain subgraphs formed by its edges and vertices (the latter are just the various d-tuples of binary digits) have been investigated extensively in coding theory and in communication theory; see, e.g., [5, 9, 10, 25]. Many easily stated questions concerning the geometry of C^d are still unsettled. A long-standing elementary conjecture on hypercube space fillings (Keller's conjecture) has been recently disproved by Lagarias and Shor [16].

In the last years, increased interest in problems involving the placement of hyperplanes in hypercubes can be observed. Motivation stems, among other areas, from coding theory [25], from linear programming [11, 23], and from the theory of machine learning [2, 14, 20, 24]. Again, various elementary questions turned out to be surprisingly difficult. Let us mention just a few of them. What is the minimum number of (non-axis-parallel) hyperplanes that cover all the 2^d vertices of C^d ? How many edges of C^d may be cut by a single hyperplane? Which hyperplane intersects C^d in a $(d-1)$-polytope of maximum volume? (Solved in 1986 by Ball [3]; see also [4].) How many and which types of hyperplanes can be spanned by vertices of C^d ?

The present paper is devoted to the last question. Let us denote with $\mathcal{H}(C^d)$ the set of all hyperplanes that can be spanned by (d affinely independent) vertices of C^d. An attempt is made to characterize and classify the hyperplanes in $\mathcal{H}(C^d)$. An obvious criterion is *parallelism*, where we distinguish between k-parallel hyperplanes (being parallel to exactly $k > 0$ coordinate axes) and skew (0-parallel) hyperplanes. A concept that turns out to be important is *symmetry*, which is fulfilled by hyperplanes that halve the set of d-cube vertices they avoid.

We show that k-parallel hyperplanes in $\mathcal{H}(C^d)$ correspond to skew ones in $\mathcal{H}(C^{d-k})$, and that skew and asymmetric hyperplanes in $\mathcal{H}(C^d)$ correspond to skew and symmetric ones in $\mathcal{H}(C^{d+1})$, in a unique manner. This tells us that the skew and symmetric type is the only really interesting one. The enumeration of all hyperplanes of this type, however, is complicated by the following unpleasant phenomenon: In each further dimension new kinds of hyperplanes appear that seem not to be accessible by using the results for lower dimensions. In fact, a theorem by Naumann [19] gives some evidence for the intrinsic complexity of $\mathcal{H}(C^d)$: Every $(d-1)$-polytope is obtainable as the intersection of a $(d-1)$-dimensional hyperplane and a hypercube of sufficiently high dimension. We were able to enumerate experimentally the sets up to $\mathcal{H}(C^8)$, but did not succeed in developing an enumeration scheme for $\mathcal{H}(C^d)$ for general d. We rediscovered an upper bound on the size of the normal vector of hyperplanes in $\mathcal{H}(C^d)$ which we needed to speed up the enumeration. Generally, we count the contributions of this paper among the first steps towards a systematic study of the set $\mathcal{H}(C^d)$.

The situation gets strikingly simpler if we restrict attention to hyperplanes in $\mathcal{H}(C^d)$ that do not (properly) cut d-cube edges. Those might be called *hull-honest* hyperplanes as their intersection figure with C^d coincides with the convex hull of the vertices of C^d they contain. We give an easy-to-use characterization of hull-honest hyperplanes that leads us to their exact total number, in general dimensions.

The following notation will be used throughout. A *hyperplane* H in Euclidean d-space $\mathbf{R}^d$ is fixed by a pair (v, b), with $v \in \mathbf{R}^d$, $b \in \mathbf{R}$, and $H = \{x \in \mathbf{R}^d | v \cdot x = b\}$. For brevity, we shall also write $H = (v, b)$. It follows from linear algebra (Cramer's rule) that there is always an integer solution for v provided $H \in \mathcal{H}(C^d)$. It will be assumed that v is an integer vector and as short as possible, i.e., the gcd of its entries is one. For convenience, we will not distinguish between a vector and its transposed form.

With V^d we denote the vertex set of C^d. Since C^d was defined to be the unit d-cube, $V^d = \{0, 1\}^d$, the set of all the binary strings of length d. The *Hamming distance* of two vertices $x, y \in V^d$ is the number of coordinates where x differs from y. Note that x and y are the two endpoints of an edge of C^d iff their Hamming distance is one (and occurs at the i-th bit iff the edge is parallel to the i-th axis of $\mathbf{R}^d$). By symmetry of C^d, for each vertex $x = (x_1, x_2, \ldots, x_d) \in V^d$ there is a vertex $\overline{x} = (1 - x_1, 1 - x_2, \ldots, 1 - x_d) \in V^d$ which we call the *antipode* of x in C^d.

2 Parallel and skew hyperplanes

Our set of interest, $\mathcal{H}(C^d)$, can be partitioned into classes by considering parallelism of its hyperplanes to coordinate axes. Let us call a hyperplane $H \in \mathbf{R}^d$ *k-parallel* $(0 \leq k \leq d-1)$ if H is parallel to exactly k axes of $\mathbf{R}^d$, that is, to exactly k spanning edges of C^d. 0-parallel hyperplanes are called *skew*. A trivial but useful observation is the following:

Observation 1. $H = (v, b) \in \mathcal{H}(C^d)$ is k-parallel iff exactly k entries of v vanish.

For $d > 0$ and $0 \leq k < d$, let $e_k(d)$ denote the number of k-parallel hyperplanes in $\mathcal{H}(C^d)$. It is possible to relate e_k to e_0 (the number of skew hyperplanes) in a simple way.

Observation 2. For $d > 0$ and $0 \leq k < d$ we have $e_k(d) = \binom{d}{k} e_0(d-k)$.

Proof. Let $H = (v, b) \in \mathcal{H}(C^d)$ be k-parallel. Then k out of d entries of v are zero and there are $\binom{d}{k}$ possibilities to place them. The remaining entries of v define a vector of dimension $d-k$ without zero entries which obviously is the normal vector of a skew hyperplane in $\mathcal{H}(C^{d-k})$. The number of such hyperplanes was defined to be $e_0(d-k)$. ∎

Note that we can thus restrict attention to skew hyperplanes, as each non-skew hyperplane has its skew analog in some dimension lower.

3 Symmetric hyperplanes

Next we study in which ways a hyperplane $H \in \mathcal{H}(C^d)$ may partition the vertex set V^d. To this end, we call H a (j, p, k)-*plane* if it contains $p \geq d$ vertices and splits the rest into subsets of cardinalities j and k where $j \leq k$.

One might conjecture that a valid triple (j, p, k) uniquely determines a hyperplane in $\mathcal{H}(C^d)$, up to hypercube symmetries. This is false already in dimension 5 where hyperplanes with a fixed triple (j, p, k) may have a different intersection with the d-cube. For instance, the two hyperplanes $H = ((-3, 1, 1, 1, 1), 0)$ and $H' = ((-1, -3, 1, 2, 2), 0)$ in $\mathcal{H}(C^5)$ are both (11,5,16)-planes. However, H cuts 10 edges of C^5 in their interior while H' does this 13 times. Dimension 5 is the first where this phenomenon occurs which turns out to be frequent in higher dimensions. This already reveals part of the difficulty of classifying the hyperplanes in $\mathcal{H}(C^d)$.

A *symmetric* hyperplane in $\mathbf{R}^d$ is one which passes through the center $(\frac{1}{2}, \ldots, \frac{1}{2})$ of C^d, that is, a hyperplane $(v, b) \in \mathbf{R}^d$ with $\sum_{i=1}^{d} v_i = 2b$. For symmetric hyperplanes H, every two antipodal vertices $x, \overline{x} \in V^d$ either both lie in H or they lie on opposite sides of H. Symmetric (j, p, k)-planes thus have $j = k$, that is, the vertices of the d-cube which are avoided by the hyperplane are partitioned into equal-sized subsets.

There is a bijection between asymmetric skew (j, p, k)-planes in $\mathcal{H}(C^d)$ and symmetric skew $(j+k, 2p, j+k)$-planes in $\mathcal{H}(C^{d+1})$.

Theorem 1. The number of asymmetric skew hyperplanes in $\mathcal{H}(C^d)$ that contain exactly p vertices of C^d is equal to the number of symmetric skew hyperplanes in $\mathcal{H}(C^{d+1})$ with exactly $2p$ vertices of C^{d+1}.

Proof. There is an obvious bijection between hyperplanes $(v, b) \in \mathbf{R}^d$ and symmetric hyperplanes $(w, b) \in \mathbf{R}^{d+1}$, given by $w_i = v_i$ for $i = 1, \ldots, d$ and $w_{d+1} = 2b - \sum_{i=1}^{d} v_i$. Note that, under this correspondence, (v, b) is asymmetric and skew if and only if (w, b) is symmetric and skew. Furthermore, it is easy to check that a vertex x of C^d lies in (v, b) if and only if the antipodal vertex pair $(x, 0)$ and $(\overline{x}, 1)$ of

C^{d+1} lies in (w, b). Hence (v, b) is an asymmetric and skew hyperplane in $\mathcal{H}(C^d)$ just if (w, b) is a symmetric and skew hyperplane in $\mathcal{H}(C^{d+1})$. The number of vertices of C^{d+1} that lie in (w, b) is exactly twice the number of vertices of C^d that lie in (v, b). ∎

The merit of Theorem 1 is that attention can be restricted to skew symmetric hyperplanes for further classification. Note also that the number of vertices covered by a symmetric hyperplane $H \in \mathcal{H}(C^d)$ is at least $2(d-1)$.

4 Hull-honest hyperplanes

A completely different way to classify the hyperplanes in $\mathcal{H}(C^d)$ is by considering their surface of intersection with C^d. To this end, $H \in \mathcal{H}(C^d)$ is called a *hull-honest* hyperplane if the convex hull of $H \cap V^d$ coincides with the polytope $H \cap C^d$. Expressed in different terms, H cuts edges of the d-cube only at their vertices but not in their interior. In fact, in dimensions two (square) and three (cube) each spanned line (plane) is hull-honest. But starting with dimension four, spanned hyperplanes can properly intersect edges of the d-cube, and the convex hull of all the vertices lying on the hyperplane is no longer the surface of intersection of hyperplane and d-cube. Hull-honest hyperplanes seem to be the only type in $\mathcal{H}(C^d)$ which has been studied previously, motivated by the intersection polytopes they generate [7, 6]. We now give an easy-to-use criterion to characterize hull-honest hyperplanes, and we also give the exact number of such hyperplanes, in general dimensions.

Theorem 2. Let H be a hyperplane in $\mathcal{H}(C^d)$ and let $v = (v_1, \ldots, v_d)$ be the (shortest integer) normal vector of H. Then H is hull-honest iff $v \in \{-1, 0, 1\}^d$.

Proof. The assertion is trivial for $d = 1$, so let $d \geq 2$ below.

Now assume that H is hull-honest. We first prove $v \in \{-1, 0, 1\}^d$ under the assumption that H contains the origin.

We claim that if v_i and v_j are non-zero entries of v of opposite sign then $|v_j| = |v_i|$. Suppose for contradiction that $|v_j| > |v_i|$. Then the adjacent vertices $x = (0, \ldots, 0_i, \ldots, 1_j, \ldots, 0)$ and $y = (0, \ldots, 1_i, \ldots, 1_j, \ldots, 0)$ lie on opposite sides of H and thus H intersects the edge between them.

Thus $|v_j| = |v_i|$ whenever v_i and v_j have opposite sign. Since not all non-zero entries of v have the same sign (since H contains the origin but also other vertices), we conclude that all non-zero entries of v must have the same absolute value A. But $v_1, \ldots, v_d$ have gcd 1, so we have $A = 1$, and $v \in \{-1, 0, 1\}^d$.

To cover hull-honest hyperplanes $H = (v, b)$ that avoid the origin, we choose some vertex x of C^d in H and consider $\mathbf{R}^d$ as being spanned by the d-cube edges emanating from x. This transformation effects only the signs of the entries of v, leaving their absolute values unchanged.

For the if part, let $H = (v, b)$ have $v \in \{-1, 0, 1\}^d$. Suppose H is not hull-honest. Then there is at least one edge of C^d which is intersected by H. The vertices of this edge, x and y, lie on opposite sides of H, hence $v \cdot x > b$ and $v \cdot y < b$. Since x, y, and v are all integer, we have $v \cdot x \geq b + 1$ and $v \cdot y \leq b - 1$ which implies $v \cdot x \geq v \cdot y + 2$. On the other hand, x and y have Hamming distance one. Say they differ at position i. This gives $|v \cdot x - v \cdot y| = |v_i|$, hence $|v_i| \geq 2$. This contradiction proves H hull-honest. ∎

For skew hyperplanes no entry of a normal vector is zero. Skew hull-honest hyperplanes therefore have always a normal vector with entries from $\{-1, 1\}$. We will now use this fact to derive their exact number in general dimensions. Intuitively speaking, we look at the 2^{d-1} main diagonales of the d-cube and show that there are precisely $d - 1$ hull-honest hyperplanes perpendicular to a diagonal.

Theorem 3. Let v be a vector in $\{-1, 1\}^d$, $d \geq 2$, and b an integer. Then the hyperplane (v, b) is spanned by a set of vertices of C^d if and only if b satisfies $n(v) - d < b < n(v)$, where $n(v)$ is the number of 1's in v.

Before proving the theorem, we use it to count the number of skew hull-honest hyperplanes in $\mathcal{H}(C^d)$.

Corollary 1 The number of skew hull-honest hyperplanes in $\mathcal{H}(C^d)$ is given by

$$h_0(d) = \begin{cases} 2 & d = 1, \\ (d-1)2^{d-1} & d \geq 2. \end{cases}$$

Proof. The case $d = 1$ is obvious, so let $d \geq 2$. By Theorems 2 and 3, we only need to count the number of distinct hyperplanes of the form (v, b) with $v \in \{-1, 1\}^d$ and $n(v) - d < b < n(v)$. For each $v \in \{-1, 1\}^d$, there are exactly $d - 1$ integers b within these bounds. Thus there are $2^d(d-1)$ pairs (v, b) satisfying the requirements. Each skew hull-honest hyperplane is counted twice as (v, b) and $(-v, -b)$ give the same hyperplane. ∎

Proof of Theorem 3. Note first that, if b lies outside the given range, then (v, b) contains at most one vertex of C^d, and so the only if part of the theorem follows.

For the if part, define, for $v \in \{-1, 1\}^d$, the vertex $x(v)$ of C^d such that $x_i(v) = \frac{1}{2}(1 + v_i)$. Let $X(v, k)$ denote the set of all vertices of C^d that differ from $x(v)$ in exactly k coordinates, i.e., whose Hamming distance from $x(v)$ is exactly k. Observe that $X(v, k)$ is contained in the hyperplane $(v, n(v) - k)$. Furthermore, it is an easy exercise to show that for k between 1 and $d-1$, the vertices in $X(v,k)$ span the hyperplane. Thus for $n(v)-d<b<n(v)$ the hyperplane (v,b) is spanned by the vertices in $X(v,n(v)-b)$. ∎

Note that a skew and hull-honest hyperplanes $(v,b) \in \mathcal{H}(C^d)$ thus contains precisely $\binom{d}{n(v)-b}$ vertices of C^d. (In fact, the corresponding intersection figures turn out to be well-known polytopes [7] and to have interesting properties [6, 15].) Moreover, up to hypercube symmetries, there is a unique symmetric, skew, and hull-honest hyperplane in every even dimension and no such one in odd dimensions.

For $d \geq 1$, let $h(d)$ denote the total number of hull-honest hyperplanes in $\mathcal{H}(C^d)$, and let $h_k(d)$ count those of them which are k-parallel $(0 \leq k \leq d-1)$.

Theorem 4.

$$h_k(d) = \begin{cases} \binom{d}{k}(d-k-1)2^{d-k-1} & \text{for } 0 \leq k \leq d-2, \\ 2d & \text{for } k = d-1. \end{cases}$$

$$h(d) = \frac{3^d(2d-3)}{6} + 2d + \frac{1}{2}.$$

Proof. According to Observation 2 on the relation between skew and k-parallel hyperplanes we have $e_k(d) = \binom{d}{k} e_0(d-k)$. The proof of Observation 2 shows that hull-honesty is not effected by the construction of k-parallel hyperplanes as we only add zeros to the normal vector of the original skew hyperplane. Therefore the relation also applies to hull-honest hyperplanes: $h_k(d) = \binom{d}{k} h_0(d-k)$. By using the result of Corollary 1 for $h_0(d)$ we get the formula for $h_k(d)$ as claimed above. Finally, summing over all values of k gives the total number of hull-honest hyperplanes in $\mathcal{H}(C^d)$,

$$h(d) \quad = \quad \sum_{k=0}^{d-2} \left[\binom{d}{k} (d-k-1) 2^{d-k-1} \right] + \underbrace{2d}_{k=d-1} .$$

This simplifies to $h(d) = \frac{3^d(2d-3)}{6} + 2d + \frac{1}{2}$ by routine manipulation using the binomial theorem. ∎

5 Exhaustive enumeration of hyperplanes

To get an idea of the behaviour of the numbers and types of hyperplanes in $\mathcal{H}(C^d)$ for dimensions $d \geq 4$ we did some numerical investigations which are briefly reported below. For a more complete description and discussion see Aichholzer [1].

In order to make the enumeration of $\mathcal{H}(C^d)$ efficient we considered the following question: For a hyperplane $(v, b) \in \mathcal{H}(C^d)$, how large can the magnitude of the entries of v be? (Recall that we assumed v to be integer and as short as possible.) A similar problem was considered in the threshold circuit literature many years ago, and the upper bound $d^{d/2}$ is well known. Actually, the bound there is slightly weaker, $(d+1)^{(d+1)/2}$, since it matters from which side the determining points of a threshold hyperplane come from.

We found the improved bound below which, as we have recently learned, was already given by Muroga et al. [18] and can be found in more detail in the book of Muroga [17]. Actually, this improvement was rediscovered several times (cf. the survey papers of Orponen [21] and Parberry [22]), most recently by Håstad [14].

Fact 1. Let A be a $(0,1)$-matrix of size $d \times d$. Then $|\det(A)| \leq 2\left(\frac{d+1}{4}\right)^{\frac{d+1}{2}}$ which is sharp for $d = 2^k - 1$, $k \in \mathbf{N}$.

Theorem 5. Let $H = (v, b) \in \mathcal{H}(C^d)$ and $v = (v_1, \ldots, v_d)$.

$$|v_i| \leq 2^{-(d-1)} d^{\frac{d}{2}} \quad \text{for } i = 1, \ldots, d,$$

$$|b| \leq 2^{-d}(d+1)^{\frac{d+1}{2}}.$$

For threshold circuits the bound for the weights v_i and the threshold level b is the same, whereas we can bound v_i stronger than b, for the reason pointed out above.

The bound in Fact 1 is achieved via a transformation from Hadamard matrices of dimension $d+1$. A *Hadamard matrix* M is an $n \times n$ matrix with entries from $\{-1, 1\}$ which satisfies $MM^T = M^T M = nI_n$. In [13] it is shown that there always exists a Hadamard matrix when $n = 2^k$, and its existence is conjectured for every $n = 4k$. (This conjecture has been verified for $n < 428$; see [8, 13]). Depending on d mod 4, some better bounds on the determinant of $(0,1)$-matrices of order d and therefore for Theorem 5 are known; see [8] and references therein.

The bound on v in Theorem 5 is not tight for dimensions $d = 2^k$, as one might conjecture from Fact 1. We can give a counterexample for dimension 8: The bound on order 7 matrix determinants is 32 and has to be tight by Fact 1. This is also the bound on normal vector entries in dimension 8 that results from Theorem 5. But by exhaustive enumerative computation of $\mathcal{H}(C^8)$ the largest normal vector entry was 18. This is somewhat surprising because testing all those hyperplanes also means testing all linear systems of order 7 with coefficients in $\{0, 1\}$ and with unique solutions. By Cramer's rule each solution coefficient is the ratio of two integers, each having an absolute value ≤ 18 (and not 32, to which the corresponding determinants could increase). Given the importance of linear systems with coefficients in $\{0, 1\}$ in many fields, the gap between maximal solution coefficients and maximal determinant values is worth mentioning. See [8] for some related work in the context of determinant functions.

The table below displays the upper bound of Theorem 5 versus the determinant of $(0,1)$-matrices and the size of normal vectors, respectively, as far as we computed them.

Dimension d	1	2	3	4	5	6	7	8	9	10
$2 \cdot (\frac{d+1}{4})^{\frac{d+1}{2}}$	1.00	1.30	2.00	3.49	6.75	14.18	32.00	76.89	195.3	521.6
$\max \lvert\det(\{0,1\}^{d\times d})\rvert$	1	1	2	3	5	9	32	56	144	320
enumerative max. vector entry	1	1	1	2	3	5	9	18	42	≥ 98 ≤ 144

Using the above bounds we computed all spanned hyperplanes in dimensions up to 8. Their numbers turned out to increase superexponentially with d (roughly 2^{d^2}), as one would expect from the trivial upper bound $\binom{2^d}{d}$. (Each d-tuple chosen from the 2^d vertices of C^d potentially gives rise to a hyperplane.) An exhaustive enumeration for dimensions higher than 8 currently seems to be out of reach. For $\mathcal{H}(C^8)$ our program, though exploiting theoretical results reported in the present paper and in [1], needed about 12 days on a DEC 5000/240, and for $\mathcal{H}(C^9)$ we estimate a running time of about 35 years.

The experimental results, on the one hand, led us to observe structural properties of $\mathcal{H}(C^d)$ which could be made rigorous, but on the other showed us the intrinsic difficulty of classifying d-cube hyperplanes. In each further dimension we encountered new types of hyperplanes which could not be brought into connection with hyperplanes in lower dimensions. Although we only needed to deal with skew symmetric hyperplanes this did not really help, since the skew type prevails over all other types, and symmetry gains only one dimension. For instance, in $\mathcal{H}(C^8)$ about 99.3 percent of all hyperplanes are skew.

Most hyperplanes are spanned by very few vertices. In $\mathcal{H}(C^8)$ about 30 percent cover only the minimum of 8 vertices. A correlation between the number of vertices covered by a hyperplane, its degree of parallelism, and the size of its normal vector can be observed: The smaller the normal vector and the higher the degree of parallelism, the more vertices may be covered. Saks [24] showed that the maximum number of vertices covered by a skew hyperplane is $\binom{d}{\lfloor \frac{d}{2} \rfloor}$. Section 4 on hull-honest hyperplanes tells us that this bound is attainable. It is not difficult to deduce that a k-parallel hyperplane thus can cover not more than $2^k \binom{d-k}{\lfloor \frac{d-k}{2} \rfloor}$ vertices. Note that $(d-1)$-parallel hyperplanes cover exactly half of the 2^d vertices of C^d, the maximum possible.

Some detailed enumerative results are listed in the table below.

Dimension d	Number of hyperplanes in $\mathcal{H}(C^d)$	Number of skew hyperplanes $e_0(d)$	Number of different types of skew hyperplanes	Number of skew hull-honest hyperplanes $h_0(d)$
1	2	2	1	2
2	6	2	1	2
3	20	8	1	8
4	140	88	3	24
5	3254	2704	8	64
6	252434	234688	35	160
7	71343208	69640192	219	384
8	86246755608	85682904704	1293	896
9	?	?	?	2048
10	?	?	?	4608

Table of numbers of different kinds of hyperplanes in the d-cube.

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References

[1] O. Aichholzer, Hyperebenen in Hyperkuben - eine Klassifizierung und Quantifizierung, Diplomarbeit, IGI-TU Graz, 1992

[2] N. Alon and Z. Füredi, Covering the Cube by Affine Hyperplanes, European Journal of Combinatorics 14, pp 79-83, 1993

[3] K. Ball, Cube Slicing in R^n, Proc. Amer. Math. Soc. 97, pp 465-473, 1986

[4] K. Ball, Volumes of Sections of Cubes and Related Problems, Lect. Notes Math. 1376, pp 251-260, 1989

[5] E.R. Berlekamp, *Algebraic Coding Theory*, McGraw Hill, New York, 1968

[6] D. Chakerian, D. Logothetti, Cube Slices, Pictorial Triangles, and Probability, Mathematics Magazine 64, pp. 219-241, 1991

[7] H.S.M. Coxeter, *Regular Polytopes*, Dover Publications, New York, 1963/73

[8] R. Craigen, The Range of the Determinant Function on the Set of $n \times n$ $(0, 1)$-Matrices, Journal of Comb. Math. and Comb. Comp. 8, pp 161-171, 1990

[9] E.N. Gilbert, Gray Codes and Paths on the n-Cube, Bell Systems Tech. J. 37, pp 1-12, 1958

[10] R.L. Graham and H.O. Pollak, On the Addressing Problem for Loop Switching, Bell Systems Tech. J. 50, pp 2495-2519, 1971

[11] M. Grötschel and M.W. Padberg, Polyhedral Aspects of the Travelling Salesman Problem, in: E.L. Lawler, J.K. Lenstra, and A.H.G. Rinnooy Kan (eds.), *The Travelling Salesman Problem*, John Wiley and Sons, London, New York, 1985

[12] B. Grünbaum, *Convex Polytopes*, Interscience, New York, 1967

[13] M. Hall Jr., Combinatorial Constructions, Studies in Combinatorics (G.-C. Rota ed.), MAA Studies in Mathematics 17, pp 218-253, 1978

[14] J. Håstad, On the Size of Weights for Threshold Gates, to appear in SIAM Journal of Discrete Mathematics

[15] L.S. Joel, D.R. Shier and M.L. Stein, Planes, Cubes and Center-Representable Polytopes, Am. Math. Monthly 84, pp 360-363, 1977

[16] J.C. Lagarias and P.W. Shor, Keller's Cube-Tiling Conjecture is False in High Dimensions, DIMACS Tech. Rep. 92-13, 1992

[17] S. Muroga, Threshold Logic and its Applications, Wiley-Interscience, New York, 1971

[18] S. Muroga, I. Toda and S. Takasu, Theory of majority decision elements, J. Franklin Inst. 271, pp 376-418, May 1961

[19] H. Naumann, Beliebige konvexe Polytope als Schnitte und Projektionen höherdimensionaler Würfel, Simplices und Masspolytope, Mathematische Zeitschrift 65, pp 91-103, 1956.

[20] P.E. O'Neil, Hyperplane Cuts of an n-Cube, Discrete Mathematics 1, pp 193-195, 1971/72

[21] P. Orponen, Computational Complexity of Neural Networks: A Survey, Nordic Journal of Computing 1, pp 94-110, 1994

[22] I. Parberry, Circuit Complexity and Neural Networks, The MIT Press, Cambridge, MA, 1994

[23] G. Reinelt, *The Linear Ordering Problem: Algorithms and Applications*, Research and Exposition in Mathematics 8, Heldermann Verlag, Berlin, 1985

[24] M.E. Saks, Slicing the Hypercube, in: *Surveys in Combinatorics*, K. Walker (ed.), London Mathematical Society Lecture Note Series 187, Cambridge University Press, pp. 211-256, 1993

[25] N.J.A. Sloane, A Short Course on Error Correcting Codes, CISM Courses and Lectures 188, Springer, Wien, New York, 1975