Classifying Hyperplanes in Hypercubes (Extended Abstract)

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1 Introduction

Among the simplest high-dimensional geometric objects is the d-dimensional hypercube (d-cube) $C^d = [0,1]^d$. Dispite of its simple definition, C^d has been an object of study from various different points of view. The theory of convex polytopes provides classical results concerning sections and projections of hypercubes; see Coxeter [3] and Grünbaum [6]. Purely combinatorial properties of C^d, mainly involving certain subgraphs formed by its edges and vertices (the latter are just the various d-tuples of binary digits) have been investigated extensively in coding theory and in communication theory; see, e.g., [4, 2, 11, 5]. Many easily stated questions concerning the geometry of C^d are still unsettled. A long-standing elementary conjecture on hypercube space fillings (Keller's conjecture) has been recently disproved [7].

In the last years, increased interest in problems involving the placement of hyperplanes in hypercubes can be observed. Motivation stems, among other areas, from coding theory [11] and from the theory of machine learning [1, 9, 10]. Again, several elementary questions turn out to be surprisingly difficult: What is the minimum number of (non-axis-parallel) hyperplanes that cover all the 2^d vertices of C^d ? How many edges of C^d may be cut by a single hyperplane? How many and which types of hyperplanes can be spanned by vertices of C^d ?

The present paper is devoted to the third question. Let us denote with $\mathcal{H}(C^d)$ the set of hyperplanes that can be spanned by (d affinely independent) vertices of C^d. An attempt is made to characterize and classify the hyperplanes in $\mathcal{H}(C^d)$. An obvious criterion is *parallelism*, where we distinguish between k-parallel hyperplanes (being parallel to exactly $k > 0$ coordinate axes) and skew (0-parallel) hyperplanes. A concept that turns out to be important is *symmetry*, which is fulfilled by hyperplanes that halve the set of d-cube vertices they avoid.

We show that k-parallel hyperplanes in $\mathcal{H}(C^d)$ correspond to skew ones in $\mathcal{H}(C^{d-k})$, and that skew and non-symmetric hyperplanes in $\mathcal{H}(C^d)$ correspond to skew and symmetric ones in $\mathcal{H}(C^{d+1})$, in a unique manner. This tells us that the skew and symmetric type is the only really interesting one. The enumeration of all hyperplanes of this type, however, is complicated by the following unpleasant phenomenon: In each further dimension new kinds of hyperplanes appear that seem to be not accessible by using the results for lower

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dimensions. In fact, a theorem by Naumann [8] gives some evidence for the intrinsic complexity of $\mathcal{H}(C^d)$: Every $(d-1)$-polytope is obtainable as the intersection of a $(d-1)$-dimensional hyperplane and a hypercube of sufficiently high dimension. We were able to enumerate experimentally the sets up to $\mathcal{H}(C^8)$, but did not succeed in developing an enumeration scheme for $\mathcal{H}(C^d)$ for general d. An upper bound on the size of the normal vector of hyperpanes in $\mathcal{H}(C^d)$ is offered. Generally, we see our contributions as the first steps towards a systematic study of the set $\mathcal{H}(C^d)$.
The situation gets strikingly simpler if we restrict attention to hyperplanes in $\mathcal{H}(C^d)$ that do not (properly) cut d-cube edges. Those might be called *hull-honest* hyperplanes as their intersection figure with C^d coincides with the convex hull of the vertices of C^d they contain. We give an easy-to-use characterization of hull-honest hyperplanes that leads us to their exact total number, in general dimensions.

2 Overview of Results

Our set of interest, $\mathcal{H}(C^d)$, can be partitioned into classes by considering parallelism of its hyperplanes to coordinate axes. Let us call a hyperplane $H \in \mathcal{H}(C^d)$ *k-parallel* $(0 \leq k \leq d-1)$ if H is parallel to exactly k axes of $\mathbf{R}^d$, that is, to exactly k spanning edges of C^d. 0-parallel hyperplanes are called *skew*. For $d > 0$ and $0 \leq k < d$, let $e_k(d)$ denote the number of k-parallel hyperplanes in $\mathcal{H}(C^d)$. It is possible to relate e_k to e_0 (the number of skew hyperplanes) in a simple way.

(1) For $d > 0$ and $0 \leq k < d$ we have $e_k(d) = \binom{d}{k} e_0(d-k)$.

Note that we can thus restrict attention to skew hyperplanes, as each non-skew hyperplane has its skew analog in some dimension lower.
Next we study in which ways a hyperplane $H \in \mathcal{H}(C^d)$ may partition the vertex set V^d of C^d. To this end, we call H a *(j,p,k)-plane* if it contains $p \geq d$ vertices and splits the rest into subsets of cardinalities j and k where $j \leq k$. A (j,p,k)-plane is called *symmetric* if $j = k$, i.e., if the vertices of the d-cube which are avoided by the hyperplane are partitioned into equal-sized subsets. One might conjecture that a valid triple (j,p,k) uniquely determines a hyperplane in $\mathcal{H}(C^d)$, up to hypercube symmetries. This is false already in dimension 5: For instance, the two hyperplanes $H = ((-3,1,1,1,1),0)$ and $H' = ((-1,-3,1,2,2),0)$ in $\mathcal{H}(C^5)$ are both (11,5,16)-planes. However, H cuts 10 edges of C^5 in their interior while H' does this 13 times. Dimension 5 is the first where this phenomenon occurs which turns out to be frequent in higher dimensions. This already reveals part of the difficulty of classifying the hyperplanes in $\mathcal{H}(C^d)$. An important property is the following:

(2) There exists a one-to-one correspondence between skew non-symmetric hyperplanes with p vertices in $\mathcal{H}(C^d)$ and skew symmetric hyperplanes with $2p$ vertices in $\mathcal{H}(C^{d+1})$.

The merit of (2) is that attention can be restricted to skew symmetric hyperplanes for further classification. Note also that the number of vertices covered by a symmetric hyperplane $H \in \mathcal{H}(C^d)$ is at least $2(d-1)$.
A completly different way to classify the hyperplanes in $\mathcal{H}(C^d)$ is by considering their

surface of intersection with C^d. To this end, $H \in \mathcal{H}(C^d)$ is called a *hull-honest* hyperplane if the convex hull of $H \cap V^d$ coincides with the polytope $H \cap C^d$. Expressed in different terms, H cuts edges of the d-cube only at their vertices but not in their interior. In fact, in dimensions two (square) and three (cube) each spanned line (plane) is hull-honest. But starting with dimension four, spanned hyperplanes can properly intersect edges of the d-cube, and the convex hull of all the vertices lying on the hyperplane is no longer the surface of intersection of hyperplane and d-cube. Hull-honest hyperplanes seem to be the only type in $\mathcal{H}(C^d)$ which has been studied previously, motivated by the intersection polytopes they generate [3]. We now give an easy-to-use criterion to characterize hull-honest hyperplanes, and we also give the exact number of such hyperplanes, in general dimensions.

(3) Let $H \in \mathcal{H}(C^d)$ be a hyperplane through the origin (which is no loss of generality) and let $v = (v_1, \ldots, v_d)$ be the (shortest integer) normal vector of H. Then H is hull-honest iff $v_i \in \{-1, 0, 1\}$ for $i = 1, \ldots, d$.

(4) For $d \geq 2$ and $1 \leq p \leq d-1$ there always exists a skew hull-honest hyperplane $H = (v, 0) \in \mathcal{H}(C^d)$ with exactly p arbitrarily chosen entries of v being $+1$.

(3) and (4) implies that, for a given number of positive entries in the normal vector, there is a unique skew and hull-honest hyperplane in $\mathcal{H}(C^d)$, up to hypercube symmetries. (In fact, the corresponding intersection figures turn out to be well-known polytopes [3].) In particular, there is a unique symmetric, skew, and hull-honest hyperplane in every even dimension and no such one in odd dimensions. More detailly, we have:

(5) Let $H = (v, 0) \in \mathcal{H}(C^d)$, $d \geq 2$, be skew and hull-honest and let p, $1 \leq p \leq d-1$, entries of v be $+1$. There are $\binom{d}{p}$ such hyperplanes H and each of them contains exactly $\binom{d}{p}$ vertices of C^d.

(6) The number of skew hull-honest hyperplanes in $\mathcal{H}(C^d)$ is given by

$$h_0(d) = \begin{cases} 2 & d = 1, \\ (d-1)2^{d-1} & d \geq 2. \end{cases}$$

(7) The total number of hull-honest hyperplanes in $\mathcal{H}(C^d)$ is given by

$$h(d) = \frac{3^d(2d-3)}{6} + 2d + \frac{1}{2}.$$

We also derived an upper bound on the size of the normal vector v of an arbitrary hyperplane in $\mathcal{H}(C^d)$. Beside the theoretical interest, this bound proved useful in the experimental enumeration of $\mathcal{H}(C^d)$. We assumed v to be an integer vector and as short as possible. By symmetry of the d-cube, we can restrict attention to hyperplanes which pass through the origin.

(8) Let $H = (v, 0) \in \mathcal{H}(C^d)$. The entries v_i of v are bounded by

$$|v_i| \leq 2\left(\frac{d}{4}\right)^{\frac{d}{2}} \text{ for } i = 1, \ldots, d.$$

It can be shown that the absolute value of the sum of v's entries has the same bound as a single entry.

To get an idea of the behaviour of the numbers and types of hyperplanes in $\mathcal{H}(C^d)$ for dimensions $d \geq 4$ we did some numerical investigations. We computed all these hyperplanes in dimensions up to 8. Their numbers turned out to increase superexponentially with d (roughly 2^{d^2}), as one would expect from the trivial upper bound $\binom{2^d}{d}$. (Each d-tuple chosen from the 2^d vertices of C^d potentially gives rise to a hyperplane.) An exhaustive enumeration for dimensions higher than 8 currently seems to be out of reach. For $\mathcal{H}(C^8)$ our program, though exploiting theoretical results reported in the present paper, needed about 12 days on a DEC 5000/240, and for $\mathcal{H}(C^9)$ we estimate a running time of about 35 years.

The experimental results, on the one hand, led us to observe structural properties of $\mathcal{H}(C^d)$ which could be made rigerous, but on the other showed us the intrinsic difficulty of classifying d-cube hyperplanes. In each further dimension we encountered new types of hyperplanes which could not be brought into connection with hyperplanes in lower dimensions. Although we only needed to deal with skew symmetric hyperplanes this did not really help, since their growth is superexponential, too, and they prevail over all other kinds of hyperplanes. For instance, in $\mathcal{H}(C^8)$ about 99.3 percent of all hyperplanes are skew.

Most hyperplanes are spanned by very few vertices. In $\mathcal{H}(C^8)$ about 30 percent cover only the minimum of 8 vertices. A correlation between the number of vertices covered by a hyperplane, its degree of parallelism, and the size of its normal vector can be observed: The smaller the normal vector and the higher the degree of parallelism, the more vertices may be covered. Saks [10] showed that the maximum number of vertices covered by a skew hyperplane is $\binom{d}{\lfloor \frac{d}{2} \rfloor}$. Fact (5) on skew hull-honest hyperplanes tells us that this bound is attainable. It is not difficult to deduce that a k-parallel hyperplane can cover not more than $2^k\binom{d-k}{\lfloor \frac{d-k}{2} \rfloor}$ vertices. Note that $(d-1)$-parallel hyperplanes cover exactly half of the vertices of C^d, the maximum possible.

Dimension d	Number of hyperplanes in $\mathcal{H}(C^d)$	Number of skew hyperplanes $e_0(d)$	Number of different types of skew hyperplanes	Number of skew hull-honest hyperplanes $h_0(d)$
1	2	2	1	2
2	6	2	1	2
3	20	8	1	8
4	140	88	3	24
5	3254	2704	8	64
6	252434	234688	35	160
7	71343208	69640192	219	384
8	86246755608	85682904704	1293	896
9	?	?	?	2048
10	?	?	?	4608

Table of numbers of different kinds of hyperplanes in the d-cube.

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