Local Properties of Triangulations (Extended Abstract) *

Oswin Aichholzer†
Institute for Theoretical Computer Science
Graz University of Technology
Klosterwiesgasse 32/2, A-8010 Graz, Austria
e-mail: oaich@igi.tu-graz.ac.at

1 Introduction

In this paper we study local properties of two well known triangulations of a planar point set S, both of which are defined in a non-local way. The first one is the greedy triangulation (GT) that is defined algorithmically: it can be obtained by starting with the empty set and at each step adding the shortest compatible edge between two points of S, where a compatible edge is defined to be an edge that crosses none of the previously added edges. For this triangulation recent investigations have shown how to compute it in expected time $O(n \log n)$ [DDMW] and most recently in $O(n)$ [DRA].

One key observation for this algorithms was the so called *exclusion region* for a possible edge pq (throughout this paper let $p, q \in S$) of the triangulation: If each half-region of the exclusion region of two points p and q contains at least one point in S (other than p and q) then the edge pq cannot be in the desired triangulation of S. We will survey some different exclusion regions for the GT and derive some upper bounds for their size. With the help of this exclusion regions the computation of the GT for uniformly distributed points is possible in linear expected time.

Next we investigate *inclusion regions* for an edge pq: If the inclusion region of two points p and q does not contain any point from S (other than p and q) then the edge pq is in the considered triangulation of S. For the GT we give some examples and discuss their properties. Note that for the Delaunay triangulation exclusion and inclusion regions can be derived directly from its definition.

The other triangulation we deal with is the minimum-weight triangulation (MWT) which minimizes the total length of all edges among all possible triangulations of S. It arises in many fields, e.g. numerical analysis, numerical approximation, or in the reconstruction of surfaces from contours [WA]. Though it has been shown how to compute the MWT in time $O(n^3)$ for the special case of n-vertex polygons [Kl], there are no known efficiently computable algorithms for the MWT in the general case [GJ]. For a better understanding of the structure of the MWT it is necessary to know some local properties of this global defined triangulation. Recent results [AART] use a local edge property (the so called *light edge*, see below) to give better lower bounds for the weight of the MWT and to derive special point sets for that the MWT can be computed in polynomial time.

2 Overview of Results

2.1 Exclusion Regions

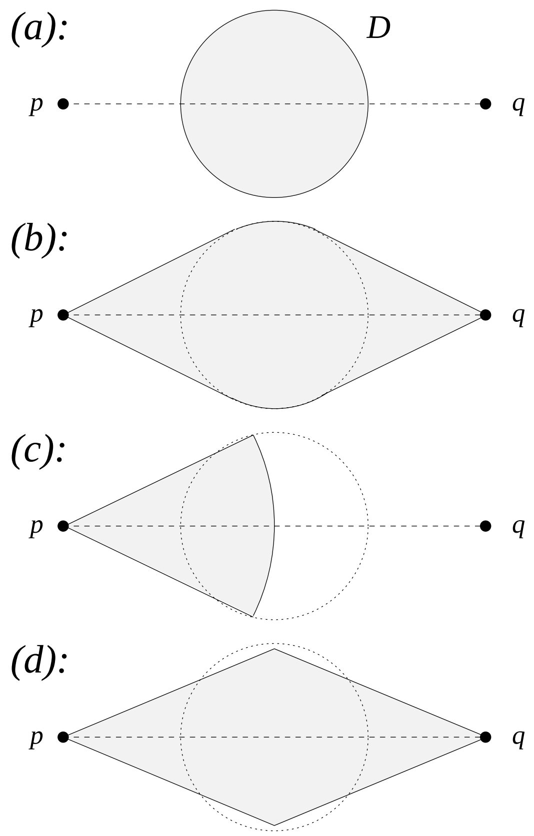

Figure 1: Exclusion regions for the greedy (a-d) and minimum-weight triangulation (d).

In the literature four different exclusion regions for a possible edge pq of the GT can be found (see Figure 1). The first was given in [DDMW] and was just a circle D of radius $|pq|/(2\sqrt{5})$ centered at the midpoint of the

*This paper benefited from discussions with Franz Aurenhammer, Scot Drysdale and Günter Rote.

†This research has been supported by the Spezialforschungsbereich F003 "Optimierung und Kontrolle".

edge pq (Fig. 1(a)). With this exclusion region Dickerson et al. developed a generate-and-test algorithm for uniformly distributed point sets where they generate a set of only $O(n \log n)$ candidate edges and then compute the GT from this edge set in $O(n \log n)$ expected time. Basically the same idea was used by Drysdale et al. [DRA] but they used an improved exclusion region: the so called eye-shaped region, cf. Figure 1(b). This is just the same circle extended by the sectors build by the tangent lines from p and q, respectively. Using this region their algorithm generates only $O(n)$ candidate pairs and then they show how to build the GT from this subset in linear expected time. Thus the usefulness of the exclusion region for the GT is to reduce the number of possible edges by local tests which can be done very efficiently.

In [DDMW] it was conjectured that the central circle D may have a radius up to $|pq|/(2\sqrt{2})$, motivated by Figure 2.

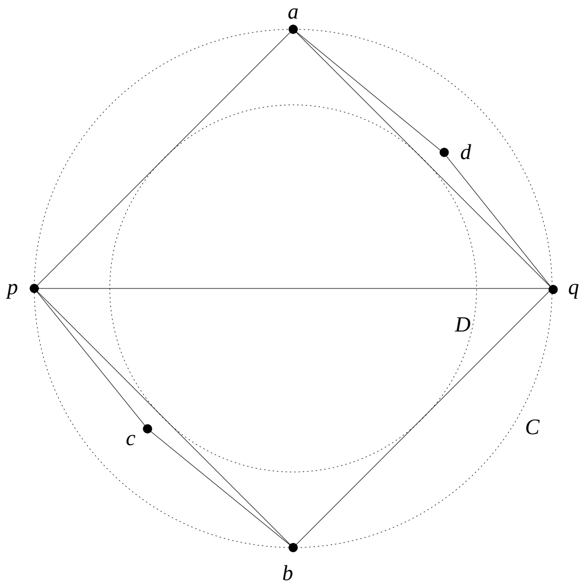

Figure 2: A simple upper bound.

Note that a and b are ϵ outside of the circle C with diameter pq and that the points c and d may be moved almost arbitrary as long as they are inside C and 'outside' the lines pb and aq, respectively. Thus the diamond provides an upper bound for any symmetric exclusion region for the GT.

The following example disproves the above conjecture: Starting with Figure 2 move the points a and b counter clockwise along C and move the points c and d closer to p and q, respectively. Note that we do this in a symmetric way. Also note that for illustration the movement in Figure 3 is much larger than it is possible for the real example (we skip the exact locations for this abstract). At last we add the points e and f which may be as closed to the line aq (pb) as desired, e.g. only ϵ away from the center of the line. Thus e (f) falls into the circle of radius $|pq|/(2\sqrt{2})$, disproving the conjecture. The new upper bound is now a circle with radius $\sqrt{2/17}\ |pq| = 0.343|pq|$ that is only about 3% better than $|pq|/(2\sqrt{2}) = 0.354|pq|$. Computing the angle qpf gives approx. $43.31°$ as an upper bound.

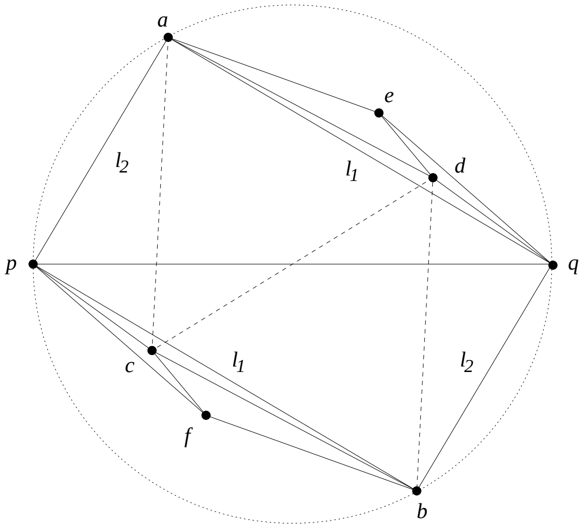

Figure 3: Disproving the conjecture of [DDMW].

Note that as for the diamond the positions of e and f are not fixed, thus they provide a stronger bound along the line aq (pb) for symmetric exclusion regions.

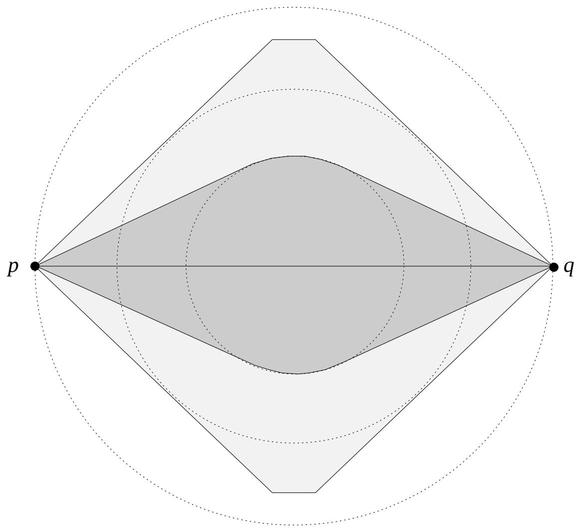

Figure 4: Summary for eye-shaped regions.

The dark shaded region of Figure 4 shows the largest known exclusion region so far while the light shaded region gives the possible area, bounded by the presented and other examples. The inner dotted circles have a radius of $|pq|/(2\sqrt{5})$ and $\sqrt{2/17}\ |pq|$, thus the angles on the left side are $26.57°$ and $43.31°$.

Some questions are: If we assume the exclusion region to be symmetrical to both axes, is there a unique exclusion region with max. area? Does it have to be convex? What is the shape (are the possible shapes) of the exclusion region(s)?

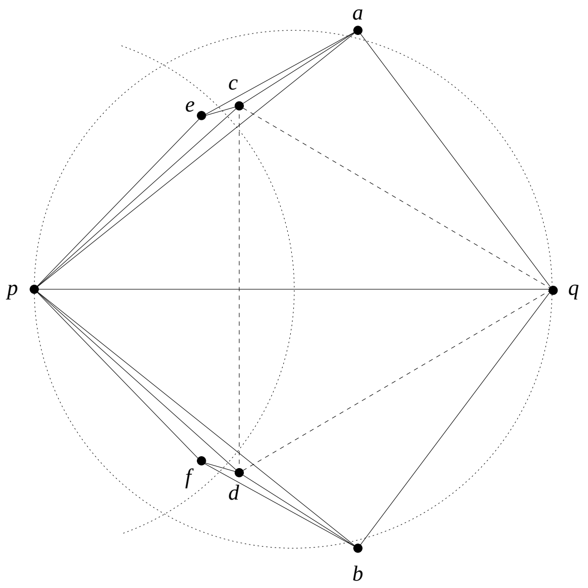

Figure 5: The circular sector has $< 43.87°$.

Another possible exclusion region is a circular sector centered at p with radius $|pq|/2$ which is just a part of the eye-shaped region (cf. Fig. 1(c) and Fig. 5). Actually Drysdale et al. [DRA] only use this region instead of the eye-shaped region: For each point $p \in S$ they arrange 30 circular sectors around p and test the possible edges emanating from p. They prove that a constant number of tests per point is sufficient, thus in linear expected time one can produce all $O(n)$ candidate edges.

Since the previous examples do not bound the circular sector we give a new construction, using the same ideas as for Figure 3. We start with a very similar example to Figure 2 but place all critical points in the left half. For the abstract we skip the detailed description but note that to improve the example we move the points a and b horizontal to the right, see Figure 5.

Computing the angle apq we have 43.87°, which is slightly greater than the upper bound for the eye-shaped region. This difference gives rise to the question if there is a difference between the max. possible angle for the eye-shaped region and the 'half-sided' circular sector.

Another interesting exclusion region was studied by Das et al. [DJ], namely the diamond with angle apq of $\pi/8$ (see Figure 1(d)). For the GT this region is included in the eye-shaped region but interesting enough it is also an exclusion region for the MWT. Unfortunately it seems that up to now there are no algorithms using this fact for the MWT.

2.2 Inclusion Regions

As for exclusion regions there are several different inclusion regions for the GT. Figure 6 shows four of the 'more regular' regions. Note that all regions have the square with side length $|pq|$ in common, either symmetrical (a-c) or on one side (d). Despite the tricky proofs for exclusion regions it is almost obvious how to construct this inclusion regions for the GT. The simplest region (a) is just a circle of radius $\sqrt{2}|pq|/2$ centered at the middle of the edge pq.

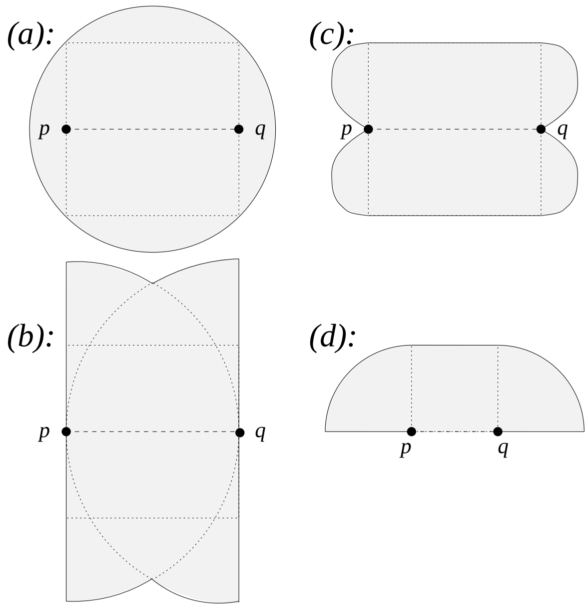

Figure 6: Inclusion regions for the greedy triangulation.

For region (b) we suppose the left and right side of the square to build the boundary and thus we have to add the two half circles of radius $|pq|$ centered at p and q, respectively. The most complicated region is (c) where we suppose the upper and lower edges of the square to bound it. To get a correct inclusion region we have to add four small regions on the left and right side. This 'extensions' can be described by $x = \pm\left(\frac{1}{2} + \frac{|y|\sqrt{3-4y^2-4|y|}}{2|y|+1}\right)|pq|$ for $y \in [-|pq|/2...|pq|/2]$, assuming that the origin is at the midpoint of the edge pq. We conjecture this example to be the minimum area inclusion region for the GT. For algorithmic purposes it is possible and might be more comfortable to approximate the additional area by four half circles of radius $|pq|/4$ and appropriate centers.

Region (d) is just a simple example for a non symmetric region. Note that from this examples it is easy to derive an infinite number of inclusion regions, either symmetrically or not. But there is one common type of edge in all of this regions: the light edge. Let us call an edge pq *light* if there is no other edge between two points of the point set S that crosses pq and is shorter than pq. Note that by definition the light edges form a subset of the greedy edges. Although, in general, not all light edges occur in a minimum-weight triangulation recent results [Ke, YXY] imply that certain subsets of light edges always have to be there. Keil [Ke] shows that the MWT always includes the so-called $\sqrt{2}$-skeleton of P. This structure consists of all edges pq such that the two circles through both p and q and with diameter $\sqrt{2}\,|pq|$ are empty of points in S; see Figure 7. It is not

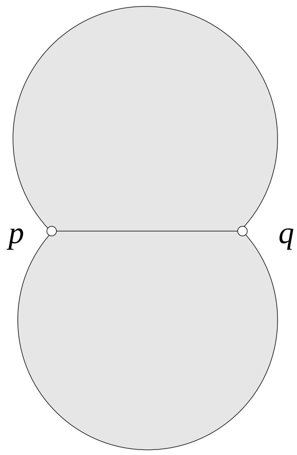

Figure 7: Keil's empty region defining the $\sqrt{2}$-skeleton.

difficult to verify that this condition forces $|pq|$ to be light. The $\sqrt{2}$-skeleton is a subset of the Gabriel graph (1-skeleton) which itself is known to be not part of the minimum-weight triangulation.

Yang et al. [YXY] proved that a condition stronger than lightness implies inclusion of pq in MWT: For any edge xy crossing pq, $|pq| \leq \min\{|px|, |py|, |qx|, |qy|\}$. By the quadrilateral inequality, xy then has to be the longest of the six edges involved. Hence pq is a light edge, and we obtain the region in Figure 8 whose emptiness implies $pq \in MWT$. With this inclusion regions it is e.g. possible to give lower bounds on the weight of the MWT of S.

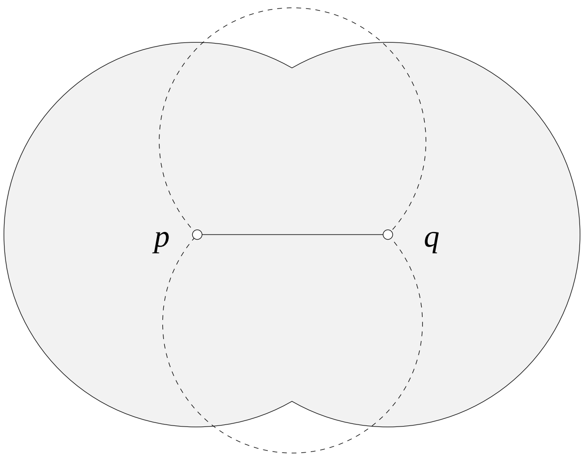

Figure 8: Empty region of Yang et al.

Very recent results give further important properties of light edges for the MWT. In [AART] the following is shown: The weight of the MWT of a point set S is bounded below by the sum of the length (weight) of all light edges of S. Note that, despite the fact that not all light edges occur in a minimum-weight triangulation in general, summing up the light edges provides a valid lower bound. This bound covers the bounds given by the above inclusion regions as the subsets of the MWT considered there are built of light edges only. Moreover they show: If a planar point set S admits a light triangulation (i.e. a triangulation consisting only of light edges) then this is the minimum-weight triangulation for S.

Thus beside new lower bounds for the MWT we have now a local edge-property which allows us to compute the MWT for special point sets in polynomial time.

References

[AART] O. Aichholzer, F. Aurenhammer, G. Rote, M. Taschwer, *Triangulations Intersect Nicely*, to appear in Proc. 11th Ann. Symp. Computational Geometry (1995).

[DDMW] M. Dickerson, R. L. Drysdale, S. McElfresh, and E. Welzl, *Fast greedy triangulation algorithms*, Proc. 10th Ann. Symp. Computational Geometry (1994), 211–220.

[DJ] G. Das and D. Joseph, "Which Triangulations Approximate the Complete Graph?", *Optimal Algorithms, Int. Symp. Proc.*, LNCS **401**, Springer-Verlag, Berlin 1989, pp. 168–192.

[DRA] R. L. Drysdale, G. Rote, O. Aichholzer, *A simple linear time greedy triangulation algorithm for uniformly distributed points*, Manuscript, 1994.

[GJ] M. Garey and D. Johnson, *Computers and Intractability. A Guide to the Theory of NP-completeness*, Freeman, 1979.

[Ke] M. Keil, *Computing a subgraph of the minimum weight triangulation*, Computational Geometry: Theory and Applications **4** (1994), 13–26.

[Kl] G. Klincsek, "Minimal triangulations of polygonal domains." *Ann. Discrete Math.* **9** 121–123.

[WA] Y. F. Wang and J. K. Aggarwal, "Surface reconstruction and representation of 3-D scenes." *Pattern Recognition* **19** (1986) 197–207.

[YXY] B.-T. Yang, Y.-F. Xu, and Z.-Y. You, *A chain decomposition algorithm for the proof of a property on minimum weight triangulations*, In: Algorithms and Computation, Proc. Conf., Beijing 1994, LNCS 834, Springer-Verlag, 1994, pp. 423–427.