[Empty][colored] k-gons - Recent results on some Erdős-Szekeres type problems

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Abstract

We consider a family of problems which are based on a question posed by Erdős and Szekeres in 1935: "What is the smallest integer $g(k)$ such that any set of $g(k)$ points in the plane contains at least one convex k-gon?" In the mathematical history this has become well known as the "Happy End Problem". There are several variations of this problem: The k-gons might be required to be empty, that is, to not contain any points of the set in their interior. In addition the points can be colored, and we look for monochromatic k-gons, meaning polygons spanned by points of the same color. Beside the pure existence question we are also interested in the asymptotic behavior, for example whether there are super-linear many k-gons of some type. And finally, for several of these problems even small non-convex k-gons are of interest.

We will survey recent progress and discuss open questions for this class of problems.

1 Introduction

In 1935 Erdős and Szekeres [22] considered a problem about the existence of a number $g(k)$ such that any set S of $g(k)$ points in general position in the plane has a subset of k points that are the vertices of a convex k-gon. Later Erdős and Guy [21] stated the following more general question. "What is the least number of convex k-gons determined by any set of n points in the plane?".

Both versions turned out to be rather challenging and have attracted many researchers. Meanwhile there exists a whole family of problems based on these questions and we will survey recent results and pose open problems for some of them. More specifically we will discuss the following variants:

- **General vs. empty k-gons:** A k-gon is called empty, or for short a k-*hole*, if it does not contain any points of the set in its interior.
- **Convex vs. non-convex:** We will consider different levels of non-convexity for k-gons and k-holes using the recent notion of j-convexity.
- **Colored point sets:** If the underlying point set is colored, k-gons and k-holes are required to be monochromatic, that is, they are spanned by points of the same color.

The collection of problems in this paper is by no means exhaustive, but rather reflects the subjective preference of the author. For a history of this class of problems and an exhaustive list of references we refer the reader to the surveys [10, 39, 48] and Chapter 8 of [13].

Throughout this paper all point sets in the plane are assumed to be in *general position*, that is, no three points in the set are collinear. When a subset of a point set S is the vertex set of a polygon P, we say that P is *spanned* by points in S. Further we will only consider simple (non self-intersecting) polygons.

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2 k-gons

Around 1933 Esther Klein raised the following question which was partially answered in a classical paper by Erdős and Szekeres [22] 1935: "Is it true that for any k there is a smallest integer $g(k)$ such that any set of $g(k)$ points contains at least one convex k-gon?" In the mathematical history this problem is also known as the "Happy End Problem", since Szekeres and Klein became engaged while collaborating on this topic and married shortly afterwards [28, 13]. Klein observed that $g(4) = 5$ and Kalbfleisch et al. [33] solved the more involved case of $g(5) = 9$.

More than 70 years after the problem was posed and more then 35 years after the proof for $g(5) = 9$, the case $k = 6$ has been settled. In 2006 Szekeres and Peters [46] showed that $g(6) = 17$ by an exhaustive computer search. The approach is based on the order type of a point set – introduced by Goodman and Pollack in 1983 [26] – which assigns an orientation to each triple of points. Thus only a finite number of configurations needs to be considered when combinatorial problems on point sets are investigated, see e.g. [8] for various applications. Using several observations, Szekeres and Peters significantly reduced the number of configurations to be computed to make the problem tractable.

The well-known Erdős–Szekeres Theorem [22] states that $g(k)$ is finite for any k. The currently best bounds are

$$2^{k-2} + 1 \leq g(k) \leq \binom{2k-5}{k-2} + 1,$$

where the lower bound goes back to Erdős and Szekeres [23] and is conjectured to be tight.

Problem 2.1. [23] Prove or disprove that $g(k) = 2^{k-2} + 1$. It is known to be true for $k \leq 6$.

The upper bound of Erdős–Szekeres was $g(k) \leq \binom{2k-4}{k-2} + 1$. Subsequently Chung and Graham [14] removed the additive +1, Kleitman and Pachter [36] improved to $g(k) \leq \binom{2k-4}{k-2} + 7 - 2k$. The bound $\binom{2k-5}{k-2} + 2$ was given by G. Tóth, P. Valtr [47] in 1998, who finally reduced the +2 to +1 in 2005 to obtain the currently best upper bound [48].

Erdős and Guy [21] posed the following generalization: "What is the least number of convex k-gons determined by any set S of n points in the plane?" The trivial solution for the case $k = 3$ is $\binom{n}{3}$. But already for convex 4-gons this question is related to the search for the rectilinear crossing number $\bar{cr}(S)$ of S. This is the number of proper intersections in the drawing of the complete straight line graph on S. The number of convex 4-gons is equal to $\bar{cr}(S)$ and is thus minimized by sets minimizing the rectilinear crossing number, a well known, difficult problem in discrete geometry, see [13] and [21] for details. Asymptotically the number of convex k-gons is $c_k\binom{n}{k} = \Theta(n^k)$ for sufficiently large n and a constant $\frac{1}{\binom{g(k)}{k}} \leq c_k < 1$. Since c_4 equals the rectilinear crossing constant we get $0.37992 \leq c_4 \leq 0.38048$, and tight values for the number of convex 4-gons are known for $n \leq 27$ points, see e.g. [1]. Table 1 gives the minimum number of convex k-gons every n-point set S determines, for $n \leq 15$ and $k = 3, 4, 5$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
3-gon	1	4	10	20	35	56	84	120	165	220	286	364	455
4-gon	-	0	1	3	9	19	36	62	102	153	229	324	447
5-gon	-	-	0	0	0	0	1	2	7	$\geq 12, \leq 13$	$\geq 20, \leq 34$	$\geq 40, \leq 62$	$\geq 60, \leq 113$

Table 1: Minimum numbers of convex k-gons [8].

3 k-holes

A *k-hole* is an empty k-gon, that is, a k-gon which does not contain any points of the underlying set in its interior. In 1978 Erdős [19] raised the following question for convex k-holes: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?" For $k \leq 5$ exact values for $h(k)$ are known, see Table 2. As already observed by Esther Klein, every set of

k	3	4	5	6	≥ 7
$h(k)$	3	5	10	≥ 30 ≤ 1717	∞

Table 2: Bounds for $h(k)$.

5 points determines a convex 4-hole, and 10 points always contain a convex 5-hole, a fact proved by Harborth [27]. However, in 1983 and on the contrary to what was conjectured, Horton showed that there exist arbitrarily large sets of points not containing any convex 7-hole [29].

It again took almost a quarter of a century after Hortons construction to answer the existence question for 6-holes. In 2007/08 Nicolás [40] and independently Gerken [25] proved that every sufficiently large point set contains a convex 6-hole. In both approaches the result has been obtained by deriving a relation between convex 6-holes and larger convex k-gons, and by considering nested convex hull layers inside the convex k-gons. While Nicolás provided a simpler proof for $h(6) \leq g(25)$, Gerken succeeded with a rather exhaustive case analysis (57 cases) to show $h(6) \leq g(9)$. In fact he shows that every point set that spans a convex 9-gon also contains a convex 6-hole. This is best possible in the sense that there exist point sets without convex 6-holes that have a convex hull of size 8 [41].

From the bounds known for $g(k)$ it follows that any set of at least 1717 points contains a convex 6-hole. Moreover, if the conjecture $g(k) = 2^{k-2} + 1$ of Erdős and Szekeres is true, then this bound drops down to 129 points. A better bound of $h(6) \leq \max\{g(8), 400\} \leq 463$ has been claimed [37, 38], but not been properly published yet.

Valtr [50] provides a simpler and more general version of Gerken's proof, but requires more points, namely $h(6) \leq g(15)$. As for a lower bound it is known that at least 30 points are needed, that is, there exists a set of 29 points without convex 6-hole. This set was found by an extensive computer search, including heuristical point insertion and removal techniques [41].

Problem 3.1. What is the minimum cardinality $h(6)$ such that any set of at least $h(6)$ points determines a convex 6-hole? It is known that $30 \leq h(6) \leq 1717$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
3-hole	1	3	7	13	21	31	43	58	75	94	114...116	136...141	160...169
4-hole	-	0	1	3	6	10	15	23	32	42	51...55	61...71	72...90
5-hole	-	-	0	0	0	0	0	1	2	3	3...5	3...8	3...12

Table 3: Minimum number of convex k-holes [8, 15, 27].

Varying the problem of Erdős and Guy from Section 2 we get the following question [20]. "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?" Table 3 gives exact values for small sets, and we know by Hortons construction that $h_k(n) = 0$ for $k \geq 7$. Table 4 summarizes the currently best general lower and upper bounds for $k = 3 \dots 6$.

$$
\begin{aligned}
n^2 - O(n \log n) \leq h_3(n) &\leq 1.6196n^2 + o(n^2) \\
\tfrac{n^2}{2} - O(n) \leq h_4(n) &\leq 1.9397n^2 + o(n^2) \\
3\lfloor \tfrac{n-4}{8} \rfloor \leq h_5(n) &\leq 1.0207n^2 + o(n^2) \\
\lfloor \tfrac{n-5}{1712} \rfloor \leq h_6(n) &\leq 0.2006n^2 + o(n^2)
\end{aligned}
$$

Table 4: Lower and upper bounds on the number $h_k(n)$ of k-holes.

All upper bounds in Table 4 are taken from [11], improving over previous bounds: For 3-holes Katchalski and Meir [35] showed that for all $n \geq 3$ a lower bound is given by $\binom{n-1}{2}$ and that there exists a constant $c > 0$ such that there exist sets with at most cn^2 3-holes. Around the same time, Bárány and Füredi [9] showed that any set of n points has at least $n^2 - O(n \log n)$ 3-holes. They also gave examples with at most $2n^2$ 3-holes if n is a power of 2. Valtr [49] described a configuration of n points related to Horton sets [29] with fewer than $1.8n^2$ 3-holes and also provided examples with small numbers of convex k-holes, e.g. with at most $2.42n^2$ convex 4-holes. Later Dumitrescu [18] improved these constructions to a configuration with $\approx 1.68n^2$ 3-holes, which then consequently was further improved by Bárány and Valtr [11], obtaining the currently best bounds shown in the table. For 3-holes it is still unknown whether the constant could be smaller than 1, that is, whether there exists a family of n-element sets with fewer than n^2 3-holes.

Problem 3.2. [11] Prove or disprove that $h_3(n) \geq (1+\varepsilon)n^2$ for sufficiently large n and some fixed $\varepsilon \geq 0$.

Concerning lower bounds, we already mentioned the result from [9] for 3-holes. The lower bound for 4-holes can be found in [18, 49]. For convex 5-holes the existence of at least three convex 5-holes in every set of 12 points (cf. Table 3) leads two the lower bound of $h_5(n) \geq 3\lfloor\frac{n-4}{8}\rfloor$ [30], improving over the previous bound $h_5(n) \geq \lfloor\frac{n-4}{6}\rfloor$ of Bárány and Károlyi [10]. To obtain this bound simply sort the point set from left to right, split it into groups of 8 points, and, for each group, reuse the rightmost 4 points from the group to its left. In a similar way we obtain the lower bound $h_6(n) \geq \lfloor\frac{n-5}{1712}\rfloor$ by using the upper bound $h(6) \leq 1717$.

Problem 3.3. Show a super-linear lower bound for $h_5(n)$ and/or give a sub-quadratic upper bound for $h_5(n)$. Similar for $h_6(n)$.

Note that proving $h_5(n) \geq \varepsilon n^2$ for some $\varepsilon > 0$ is equivalent to a positive answer of Problem 3.2 by results in [44].

3.1 Non-convex k-holes

As we know that there are arbitrarily large point sets which do not contain convex k-holes of a certain size, we now relax the problem by skipping the convexity requirement. Of course we still want the holes to be as 'near-convex' as possible, as otherwise any polygonization (spanning cycle) of the point set will be an n-hole.

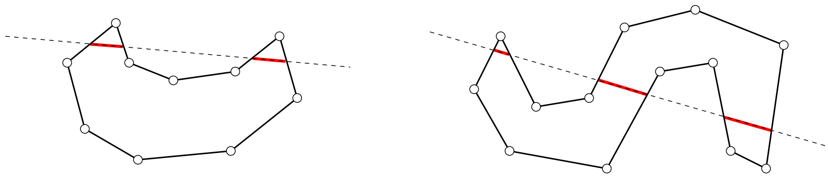

Figure 1: 2-convex polygon (left) and 3-convex (but not 2-convex) polygon (right).

Several ways of measuring the convexity or non-convexity of a given polygon have been proposed in the literature. See [4] for a short overview, and a generalization of convexity which will prove useful in our context: A polygon P is called *j-convex* if there exists no straight line that intersects P in more than j connected components. Thus 1-convexity refers to convexity in its standard meaning, and among non-convex polygons 2-convex ones can be considered to be as convex as possible. Figure 1 shows examples of 2-convex and 3-convex polygons. Typical examples for 2-convex polygons are also pseudo-triangles. These are simple polygons with precisely three convex vertices (so-called corners) with internal angles less than π; see the recent survey [45] for details and applications.

Obviously 4-gons and 5-gons are always 2-convex, while a 6-gon might be not. Moreover for any set of up to 9 points there always exists a 2-convex polygonization [3], which implies that 2-convex k-holes exist for $k \leq 9$ for any $n \geq k$. Utilizing convex chains of logarithmic length, whose existence follow from the exponential upper bound for $g(k)$, it is easy to see that every set of n points determines a 2-convex hole of size $\Omega(\log n)$. Moreover there exist sets of n points such that the largest spanned 2-convex polygon has size $O(\log^2 n)$ [4].

Problem 3.4. [4] Prove or disprove that for any n there are sets of n points in $\mathcal{R}^2$ in general position where every 2-convex k-hole is of size at most $k = O(\log n)$.

Problem 3.5. [4] Prove or disprove that any set of n points in $\mathcal{R}^2$ in general position spans a 2-convex k-gon of size $k = \Omega(\log^2 n)$.

Another variation is to ask for the existence of 2-convex k-gons of a special type. For example, what is the minimum cardinality $p(k)$ such that any point set of size $p(k)$ spans either a convex k-gon, or a pseudo-triangle of size at least k? It is not hard to obtain $p(k)$ for $k \leq 5$ and it is known that $p(6) = 12$ and $21 \leq p(7) \leq 23$ [2] and obviously $g(\frac{k-3}{3}) \leq p(k) \leq g(k)$.

Problem 3.6. [2] Determine $p(k)$. What if we consider k-holes and empty pseudo-triangles?

3.2 4-gons and 4-holes

Considering 2-convex k-holes is an interesting question even for small, constant values of k. For small point sets Table 5 shows the minimum number of convex 4-holes, the maximum number of non-convex 4-holes, and the minimum and maximum number of 2-convex 4-holes, and, for easy comparison, the number of 4-tuples (which is identical to the maximum number of convex 4-holes).

n	convex	non-convex	2-convex		$\binom{n}{4}$
	min	max	min	max	
3	0	0	0	0	0
4	0	3	1	3	1
5	1	8	5	9	5
6	3	18	15	22	15
7	6	36	35	43	35
8	10	64	66	77	70
9	15	100	102	126	126
10	23	150	147	210	210
11	32	216	203	330	330

Table 5: 2-convex 4-holes

For $n = 1...7$ it can be seen that the minimum number of 2-convex 4-holes is $\binom{n}{4}$, while it seems that the maximum number of 2-convex 4-holes is $\binom{n}{4}$ for $n \geq 9$. That is, convex sets minimize the cardinality of 2-convex 4-holes for $n \leq 7$ but seem to maximize it for $n \geq 9$.

Problem 3.7. Show that point sets in convex position maximize the cardinality of 2-convex 4-holes for $n \geq 9$ points.

Problem 3.8. Which family of point sets (a) minimizes the number of convex 4-holes? (b) maximizes the number of non-convex 4-holes? (c) minimizes the number of 2-convex 4-holes?

Note that if for a point set S we consider 4-gons instead of 4-holes, then the number of 2-convex 4-gons is related to the rectilinear crossing number $\bar{cr}(S)$ of S mentioned in Section 2. The number of convex 4-gons equals $\bar{cr}(S)$ while the number of non-convex 4-gons is $3\left(\binom{n}{4} - \bar{cr}(S)\right)$. So the number of 2-convex 4-gons is $3\binom{n}{4} - 2\bar{cr}(S)$, and therefore minimized for point sets in convex position and maximized for sets minimizing the rectilinear crossing number.

4 Monochromatic k-holes

In this section we consider variations of the above problems where the points of the given set S belong to different classes – usually described as *colors*. We say that a (not necessarily convex) k-gon or k-hole is monochromatic if all its vertices have the same color. This colorful family of problems was introduced in 2003 by Devillers et al. [16]. They showed for example that any bichromatic set of n points determines at least $\lceil \frac{n}{4} \rceil - 2$ monochromatic 3-holes with pairwise disjoint interiors, which is tight. In fact all these 3-holes are of the same color.

It is natural to wonder whether results similar to the just mentioned one are possible when there are more than two colors: What is the minimum number of colors such that there exist sets of n points which we can color in a way so that they do not determine any convex k-hole? In [16] (Theorem 3.3) this question has been settled by showing that already for three colors there exists a coloring of the Horton set so that it does not span any monochromatic 3-hole.

Note that the according question for k-gons can be reduced to the uncolored version. It is not hard to see that any c-colored set of at least $c \cdot (g(k) - 1) + 1$ points contains at least one monochromatic k-gon [16]. Variations of the problem, where the vertices of the k-gon/k-hole can be colored differently (polychromatic), or where all vertices must have different colors (heterochromatic), also exist. See [16] for details.

Another result Devillers et al. provide in [16] is that for $k \geq 5$ and any n there are bichromatic sets of n points without convex monochromatic k-holes. The proof is based on the 3-coloring of the Horton set mentioned above. So the most interesting remaining cases in this family of problems are monochromatic 3-holes and 4-holes in bichromatic sets, which we will consider in the next two sections. A possible relaxation of the problem is to allow j-convex holes, see also Section 4.2.

Problem 4.1. For which combinations of $j \geq 1$ and $k \geq 4$ does any sufficiently large bichromatic point set in $\mathbb{R}^2$ in general position determine a j-convex k-hole?

Note that from the results in [16] it is clear that there are point sets with more than two colors where no j-convex k-holes exist for any $j \geq 1$ and $k \geq 3$. See also [13] and [34] for a number of related problems on colored point sets.

4.1 Monochromatic 3-holes

It is easy to see that any bichromatic set of at least 10 points contains a monochromatic 3-hole: From Section 3 we know that any (uncolored) set of 10 points contains a convex 5-hole. Hence, however these 5 points are colored, we will obtain a monochromatic 3-hole. In fact it can be argued that any bichromatic set of $n \geq 7$ points contains at least $\lfloor \frac{n-7}{2} \rfloor$ monochromatic 3-holes. Therefore, already bichromatic point sets of size 9 are sufficient to determine a monochromatic 3-hole. From the point set order type data base [8] it is also known that there are bichromatic sets of 9 and 10 points which only contain a unique monochromatic 3-hole. Moreover there are sets of 8 points without monochromatic 3-holes, see Figure 2.

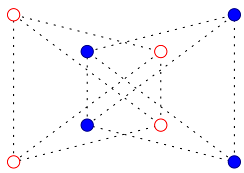

Figure 2: Bichromatic point set without monochromatic 3-hole.

As we can split any large set of points into groups of constant cardinality the above observation already implies that there is a linear number of monochromatic 3-holes. As mentioned above, a stronger result has been shown by Devillers et al. [16], namely that any bichromatic set of n points determines at least $\lceil \frac{n}{4} \rceil - 2$ monochromatic 3-holes with pairwise disjoint interiors, which is tight.

So the question arises whether there exist super-linear many monochromatic 3-holes, where it is of course not required that the 3-holes are disjoint. In contrast to the race for the best constant in the uncolored case described in Section 2, there are only recent results for the asymptotic number of monochromatic 3-holes in bichromatic sets, and no tight bounds are known yet. The first super-linear bound of $\Omega(n^{5/4})$ was given in [5]. The proof is based on a so-called discrepancy lemma, which implies that if there is a significant difference in the cardinality of the two color classes then the number of monochromatic 3-holes is sufficiently large. Combining this with special triangulations of the point set obtained by using Dilworth's Theorem [17], the bound follows.

By refining the techniques used in [5], this result has been improved by Pach and Tóth [43] to the currently best known bound of $\Omega(n^{4/3})$ monochromatic 3-holes. Unfortunately, the conjecture in [5] that any bichromatic set of n points spans a quadratic number of monochromatic 3-holes is still unsettled.

Problem 4.2. [5] Prove or disprove that any bichromatic set of n points in $\mathbb{R}^2$ in general position determines $\Omega(n^2)$ monochromatic 3-holes.

An affirmative answer would follow from showing that $h_5(n) \geq \varepsilon n^2$ for some $\varepsilon > 0$, cf. Problem 3.3.

4.2 Monochromatic 4-holes

From the above discussion it follows that for the existence question of monochromatic k-holes the most interesting remaining case is the existence of monochromatic 4-holes in bichromatic point sets.

Figure 3(a) shows a set with 18 points which does not contain a convex monochromatic 4-hole, and larger examples with 20 [12], 30 [24], 32 [51] and most recently 36 [31] points (Fig 3(b)) have been found. However, all larger examples do contain non-convex monochromatic 4-holes, while the one in Figure 3(a) does not.

Problem 4.3. Find large examples of bichromatic point sets which do not contain (convex) monochromatic 4-holes.

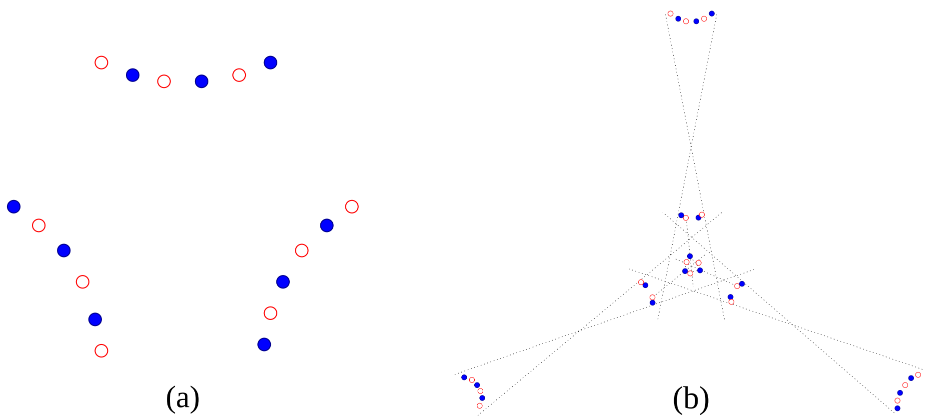

Figure 3: (a) 18 points without 2-convex monochromatic 4-holes. (b) 36 points without convex monochromatic 4-holes (sketch, see [31] for details).

Notice that every (uncolored) point set that admits a convex 7-hole will contain a convex monochromatic 4-hole for any bicoloration, because at least four of the vertices of the heptagon will have the same color. However, it has been shown that that for $n \geq 64$ any bichromatic Horton set contains convex monochromatic 4-holes [16], and thus the authors of this paper conjecture that for sufficiently large n any bichromatic point set contains at least one convex monochromatic 4-hole.

Until recently this conjecture has not been settled even for 2-convex monochromatic 4-holes, that is, 4-holes which are not required to be convex. This weaker version of the problem arose [32, 42] as no progress for the original question had been obtained. It was considered to be an important step towards solving the initial problem. The current situation is that in [7] this relaxed version of the conjecture has been shown to be true. If the cardinality of the bichromatic point set S is sufficiently large, there always exists a 2-convex monochromatic 4-gon spanned by S. In [7] the given lower bound on the cardinality was $n \geq 5044$. Using observations on vertex degree parity constraints for triangulations of S this bound has most recently been lowered to 2760 points [6].

Problem 4.4. [16] Prove or disprove that any sufficiently large bichromatic set of points in $\mathcal{R}^2$ in general position determines at least one convex monochromatic 4-hole.

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